University of Washington | Fall 2003
Department of Economics | Eric Zivot

Economics 483

Final Exam
Suggested Solutions

This is a closed book and closed note exam. However, you are allowed one page of handwritten notes. Answer all questions and write all answers in a blue book. Total points = 100.

I. Matrix Algebra and Portfolio Math (20 points)

Let R_i denote the continuously compounded return on asset i ($i = 1, \ldots, N$) with $E[R_i] = \mu_i$, $\text{var}(R_i) = \sigma_i^2$ and $\text{cov}(R_i, R_j) = \sigma_{ij}$. Define the ($N \times 1$) vectors $\boldsymbol{R} = (R_1, \ldots, R_N)'$, $\mu = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{m} = (m_1, \ldots, m_N)'$, $\boldsymbol{x} = (x_1, \ldots, x_N)'$, $\boldsymbol{t} = (t_1, \ldots, t_N)'$, $\mathbf{1} = (1, \ldots, 1)'$ and the ($N \times N$) covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

Using simple matrix algebra, answer the following questions.

a. Write down the optimization problem used to determine the global minimum variance portfolio assuming short sales are allowed. Let $\boldsymbol{m}$ denote the vector of portfolio weights in the global minimum variance portfolio.

$$\min_{m} \ \boldsymbol{m'\Sigma m} \ \text{s.t.} \ \boldsymbol{m'1} = 1$$

b. Write down the optimization problem used to determine the global minimum variance portfolio assuming short sales are *not* allowed. Again, let $\boldsymbol{m}$ denote the vector of portfolio weights in the global minimum variance portfolio.

$$\min_{m} \ \boldsymbol{m'\Sigma m} \ \text{s.t.} \ \boldsymbol{m'1} = 1, \ m_i \geq 0 \ i = 1, \ldots, N$$

c. Write down the optimization problem used to determine an efficient portfolio with target return equal to μ_0 assuming short sales are allowed. Let $\boldsymbol{x}$ denote the vector of portfolio weights in the efficient portfolio.

$$\min_{x} \ \boldsymbol{x'\Sigma x} \ \text{s.t.} \ \boldsymbol{x'1} = 1 \text{ and } \boldsymbol{x'\mu} = \mu_0$$

d. Write down the optimization problem used to determine the tangency portfolio, assuming short sales are allowed and the risk free rate is give by r_f. Let $\boldsymbol{t}$ denote the vector of portfolio weights in the tangency portfolio.

$$\max_{t} \ \frac{\boldsymbol{t'\mu} - r_f}{(\boldsymbol{t'\Sigma t})^{1/2}} \ \text{s.t.} \ \boldsymbol{t'1} = 1$$

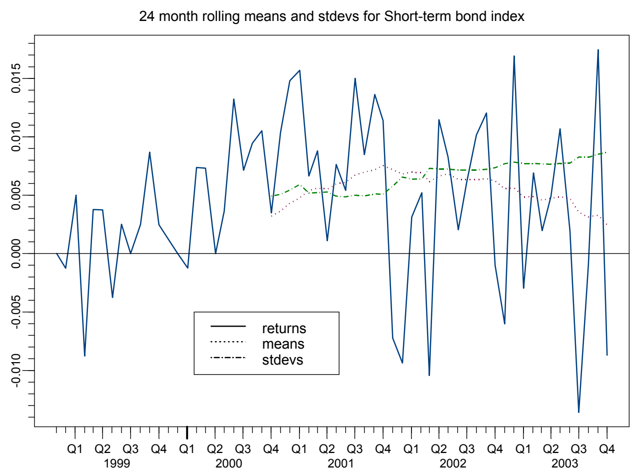
II. Stability of parameters in the CER model (20 points)

Recall the constant expected return model

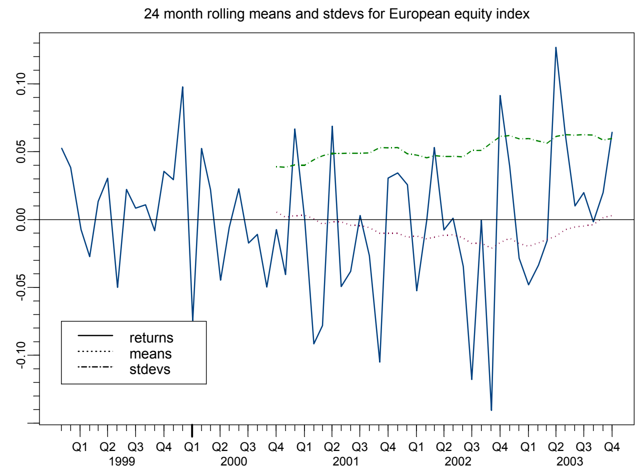
$$R_{it} = \mu_i + \varepsilon_{it},\ t = 1,\ldots,T$$
$$\varepsilon_{it} \sim iid\ N(0,\sigma_i^2)$$
$$\operatorname{cov}(\varepsilon_{it},\varepsilon_{jt}) = \sigma_{ij}$$

A key assumption of this model is that the parameters $\mu_i,\ \sigma_i^2,\ \sigma_{ij}$ are constant over time. In this question you will evaluate this constant parameter assumption for returns on the Vanguard short-term bond index (vbisx) and the Vanguard European equity index (veurx)

a. The following graphs show 24-month rolling means and standard deviations for the returns on the two mutual funds. Using these graphs, do you think that the assumption that $\mu_i,\ \sigma_i^2$ are constant over time holds for the two funds? Briefly explain your answer.

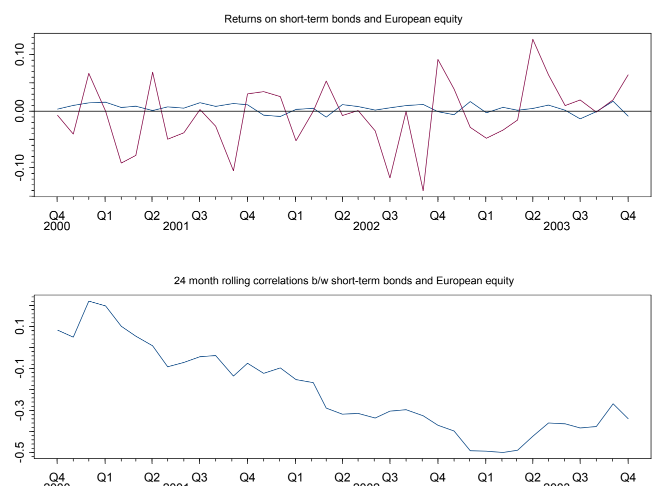


The rolling means range from about 0.003 to 0.007 (per month), which is not much variation. The mean at the end of the sample is at its minimum. On an annual basis this translates to about 3.6% to 8.4%. This is to be expected since interest rates have been very stable over the last 5 years. The rolling standard deviations increase slowing over the sample, starting at about 0.005 and ending at about 0.008 (6% to 9.6%). Overall, however, the rolling means and standard deviations are fairly stable, which supports the assumption of the CER model.



For the European equity returns, the rolling means start out slightly positive then dip to negative values and only become positive again at the end of the sample. The negative returns reflect the decline in equity markets that were generally experienced during the last two years as a result of the tech crash and the recession. The rolling standard deviations increases slightly over the sample (from about 4% per month to about 6%). The switch in the means from positive to negative seems to represent a non-constant mean. The assumption of a constant mean and standard deviation does not seem to be reasonable for the European equity returns.

b. The following graphs show 24-month rolling correlations between the returns on the short-term bond index and the European equity index. Using these graphs, do you think that the assumption that σ_{ij} is constant over time holds for the two funds? Briefly explain your answer.



The rolling correlations start out slightly positive, around 0.1, and steadily decline to a negative value of about -0.3 at the end of the sample. This result is due to the fact that the mean returns on the European equity went from positive to negative during the sample period, whereas the *short-term bond returns had a stable positive mean. Thus, the assumption of a constant covariance between bond returns and equity returns does not seem to be supported by the data.*

III. Empirical Analysis of the single index model (40 points)

The following represents S-PLUS linear regression output from estimating the single index model for the Vanguard short-term bond index (vbisx) and the Vanguard European Equity index (veurx) using monthly continuously compounded return data over the period November 1998 – October 2003. In the regressions, the market index is the Vanguard extended market index (vexmx).

```
> vbisx.fit = lm(vbisx~vexmx)
> summary(vbisx.fit,cor=F)

Call: lm(formula = vbisx ~ vexmx)
Residuals:
      Min        1Q    Median       3Q     Max
 -0.01654 -0.004081 -0.000257 0.004462 0.01301

Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept)  0.0042  0.0009     4.7468  0.0000
      vexmx -0.0282  0.0126    -2.2290  0.0297

Residual standard error: 0.00688 on 58 degrees of freedom
Multiple R-Squared: 0.0789
F-statistic: 4.969 on 1 and 58 degrees of freedom, the p-value is 0.0297
```

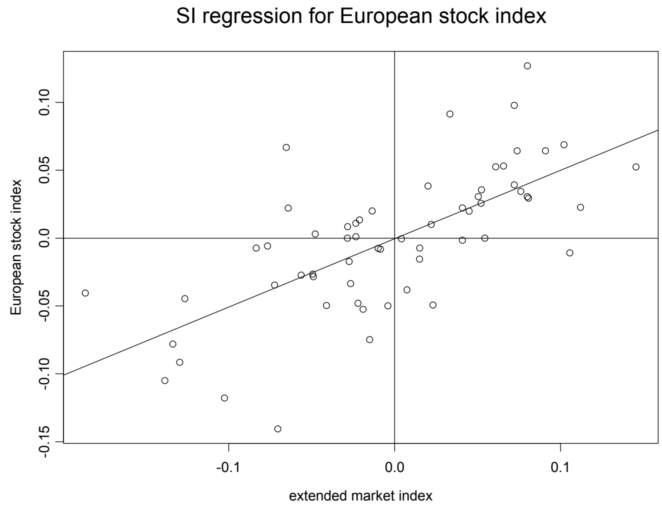

```
> veurx.fit = lm(veurx~vexmx)
> summary(veurx.fit,cor=F)

Call: lm(formula = veurx ~ vexmx)
Residuals:
     Min       1Q     Median      3Q Max
 -0.1047 -0.02203 -0.0009752 0.02241 0.1

Coefficients:
              Value Std. Error t value Pr(>|t|)
```

```
(Intercept) -0.0004  0.0049    -0.0875  0.9305
      vexmx  0.5039  0.0699     7.2054  0.0000

Residual standard error: 0.03805 on 58 degrees of freedom
Multiple R-Squared: 0.4723
F-statistic: 51.92 on 1 and 58 degrees of freedom, the p-value is 1.324e
-009
```



a. For the short term bond index and the European equity index, what are the estimated values of α and β and what are the estimated standard errors for these estimates?

	$\hat{\alpha}$	$SE(\hat{\alpha})$	$\hat{\beta}$	$SE(\hat{\beta})$
Short-term bond	0.0042	0.009	-0.0282	0.0126
European equity	-0.0004	0.0049	0.5039	0.0699

b. Based on the estimated values of β, what can you say about the risk characteristics of the short-term bond index and the European equity index relative to the extended market index?

The estimated beta for the short-term bond is negative and close to zero. This means that adding the short-term bond index to the extended market portfolio will lower the variability of the extended market index. Thus the short-term bond is much less risky than the extended market index. Since the European equity beta is about 0.5, adding it to the market index will slightly lower the variability. Thus the European equity index is less risky than the extended market index.

c. Is β for the European equity index estimated more precisely than β for the short-term bond index? Why or why not?

Yes. $SE(\hat{\beta})$ *for European equity is about half the size of* $SE(\hat{\beta})$ *for the short-term bond index. Also, since European equity is a portfolio of stocks it should have a higher percentage of market* *risk than the bond index and the power of diversification should concentrate the risk on market risk. This is verified by comparing the R-square values for the two regressions.*

d. For each regression, what is the proportion of market or systematic risk and what is the proportion of firm specific or unsystematic risk? For each regression, what does the Residual Standard Error represent?

	R-square (% market risk)	*1-(R-square) (% non-market risk)*
Short-term bond	0.08	0.92
European Equity	0.47	0.53

For each regression, the residual standard error is an estimate of the typical size of ε in the SI model regression. That is, it is an estimate of σ_ε.

e. Why should the European equity index have a greater proportion of systematic risk and a larger standard error than the short-term bond index?

The European equity index is a large diversified portfolio of stocks, whereas the short-term bond index is a small, not diversified, index of short-term bonds. The diversification effect concentrates the market risk of the European equity index and reduces the amount of non-market risk. This tends to increase the R-square (% of market risk). Short-term bonds generally do not have much market risk since short-term interest rates are fairly stable. Hence the R-square of the bond index is expected to be low (close to zero). This is what we see in the data.

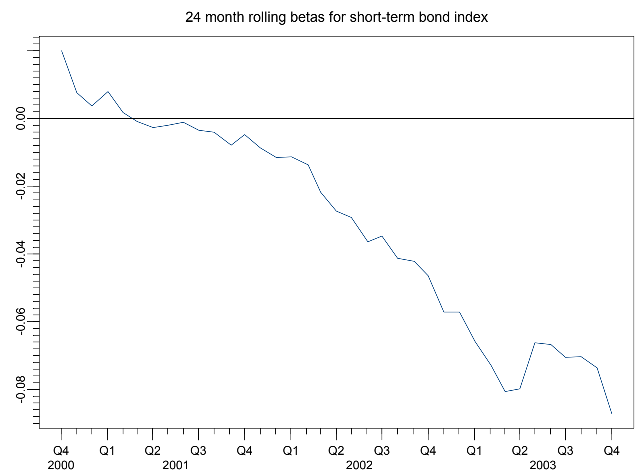
f. For the short-term bond index and the European equity index, test the null hypothesis that $\beta = 0$ against the alternative hypothesis that $\beta \neq 0$ using a 5% significance level. What do you conclude?

	$t_{\beta=0} = \frac{\hat{\beta} - 0}{SE(\hat{\beta})}$	*Decision: reject if |t-stat| > 2*
Short-term bonds	$\frac{-0.0282 - 0}{0.0126} = -2.229$	*Reject* $H_0 : \beta = 0$ *at 5% level*
European equity	$\frac{0.5039 - 0}{0.0699} = 7.205$	*Reject* $H_0 : \beta = 0$ *at 5% level*

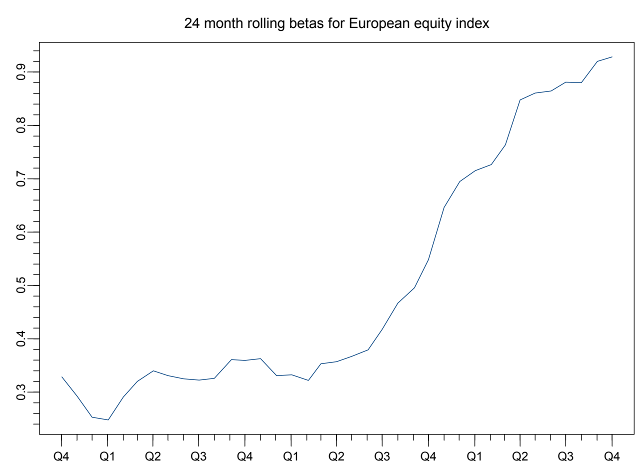
g. For Short term bond index and the European equity index, test the null hypothesis that $\beta = 1$ against the alternative hypothesis that $\beta \neq 1$ using a 5% significance level. What do you conclude?

	$t_{\beta=1} = \dfrac{\hat{\beta}-1}{SE(\hat{\beta})}$	*Decision: reject if |t-stat| > 2*
Short-term bonds	$\dfrac{-0.0282-1}{0.0126} = -81.60$	*Reject* $H_0 : \beta = 1$ *at 5% level*
European equity	$\dfrac{0.5039-1}{0.0699} = -7.097$	*Reject* $H_0 : \beta = 1$ *at 5% level*

h. The following graphs show the 24-month rolling estimates of β for the SI models for the short-term bond index and the European equity index. Using these graphs, what can you say about the stability of β over time?



The rolling betas for the short-term bond index start out slightly positive and decline steadily to about -0.08. There appears to be clear evidence that the beta has declined over the sample (is not constant).



The rolling betas for the European equity index starts out low, around 0.3, and steadily increases to about 0.8 at the end of the sample. The European equity returns seem to have followed the overall market by declining during the sample, which results in a beta value closer to one at the end of the sample. The beta does not appear constant over the sample period.

IV. CAPM (20 points)

a. What are the main assumptions underlying the CAPM?

- *Many identical investors who are price takers*
- *All investors plan to invest over the same time horizon*
- *No taxes or transactions costs*
- *Can borrow or lend at the risk free rate*
- *Investors only care about expected return and variance*
- *Market consists of all publicly traded goods*

b. What is the security market line (SML) pricing relationship? Draw a graph showing this relationship, and indicate the intercept and slope.

$$E[R_{it}] = r_f + \beta_i(E[R_{Mt}] - r_f)$$

c. Suppose you have a history of returns on N assets for T months and a market index, and you estimate the excess returns single index model regression

$$R_{it} - r_{ft} = \alpha_i^* + \beta_i(R_{Mt} - r_{ft}) + \varepsilon_{it}$$

where r_f denotes the risk-free rate and R_{Mt} denotes the return on a value-weighted market index. How would you test the null hypothesis that the CAPM holds for every asset?

Use a t-test to test the null hypothesis that $\alpha_i^* = 0$ *for all assets.*

d. The prediction test of the CAPM plots estimates of average excess returns $\hat{\mu}_i - r_f$ against beta estimates $\hat{\beta}_i$. Based on these estimates one may estimate the simple linear regression equation

$$\hat{\mu}_i - r_f = \gamma_0 + \gamma_1\hat{\beta}_i + error_i,\ i = 1,\ldots,N$$

If the CAPM were true, what should the estimated values for γ_0 and γ_1 be?

If the CAPM were true, then $\gamma_0 = 0$ and $\gamma_1 = \hat{\mu}_M - r_f$