

Time Series Concepts

Stochastic (Random) Process

$$\{\dots, Y_1, Y_2, \dots, Y_t, Y_{t+1}, \dots\} = \{Y_t\}_{t=-\infty}^{\infty}$$

sequence of random variables indexed by time

Observed time series of length T

$$\{Y_1 = y_1, Y_2 = y_2, \dots, Y_T = y_T\} = \{y_t\}_{t=1}^T$$

Stationary Processes

- Intuition: $\{Y_t\}$ is stationary if all aspects of its behavior are unchanged by shifts in time
- A stochastic process $\{Y_t\}_{t=1}^{\infty}$ is *strictly stationary* if, for any given finite integer r and for any set of subscripts t_1, t_2, \dots, t_r the joint distribution of

$$(Y_t, Y_{t_1}, Y_{t_2}, \dots, Y_{t_r})$$

depends only on $t_1 - t, t_2 - t, \dots, t_r - t$ but not on t .

Remarks

1. For example, the distribution of (Y_1, Y_5) is the same as the distribution of (Y_{12}, Y_{16}) .
2. For a strictly stationary process, Y_t has the same mean, variance (moments) for all t .
3. Any function/transformation $g(\cdot)$ of a strictly stationary process, $\{g(Y_t)\}$ is also strictly stationary. E.g., if $\{Y_t\}$ is strictly then $\{Y_t^2\}$ is strictly stationary.

Covariance (Weakly) Stationary Processes $\{Y_t\}$:

- $E[Y_t] = \mu$ for all t
- $\text{var}(Y_t) = \sigma^2$ for all t
- $\text{cov}(Y_t, Y_{t-j}) = \gamma_j$ depends on j and not on t

Note $\text{cov}(Y_t, Y_{t-j}) = \gamma_j$ is called the j -lag *autocovariance*

Autocorrelations

$$\text{corr}(Y_t, Y_{t-j}) = \rho_j = \frac{\text{cov}(Y_t, Y_{t-j})}{\sqrt{\text{var}(Y_t)\text{var}(Y_{t-j})}} = \frac{\gamma_j}{\sigma^2}$$

Note: By stationarity $\text{var}(Y_t) = \text{var}(Y_{t-j}) = \sigma^2$.

Autocorrelation Function (ACF): Plot of ρ_j against j

Example: Gaussian White Noise Process

$$Y_t \sim \text{iid } N(0, \sigma^2)$$

$$E[Y_t] = 0, \text{ var}(Y_t) = \sigma^2$$

Y_t independent of Y_s for $t \neq s$

Example: Independent White Noise Process

$$Y_t \sim \text{iid } (0, \sigma^2)$$

$$E[Y_t] = 0, \text{ var}(Y_t) = \sigma^2$$

Y_t independent of Y_s for $t \neq s$

Example: Weak White Noise Process

$$Y_t \sim WN(0, \sigma^2)$$
$$E[Y_t] = 0, \text{ var}(Y_t) = \sigma^2$$
$$\text{cov}(Y_t, Y_s) = 0 \text{ for } t \neq s$$

Nonstationary Processes

Example: Deterministically trending process

$$Y_t = \beta_0 + \beta_1 t + \varepsilon_t, \varepsilon_t \sim WN(0, \sigma^2)$$
$$E[Y_t] = \beta_0 + \beta_1 t \text{ depends on } t$$

Note: A simple detrending transformation yield a stationary process:

$$X_t = Y_t - \beta_0 - \beta_1 t = \varepsilon_t$$

Example: Random Walk

$$\begin{aligned} Y_t &= Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2), \quad Y_0 \text{ is fixed} \\ &= Y_0 + \sum_{j=1}^t \varepsilon_j \Rightarrow \text{var}(Y_t) = \sigma_\varepsilon^2 \times t \quad \text{depends on } t \end{aligned}$$

Note: A simple detrending transformation yield a stationary process:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

Moving Average (MA) Processes

MA(1) Model

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}, \quad -\infty < \theta < \infty$$
$$\varepsilon_t \sim iid N(0, \sigma_\varepsilon^2)$$

Properties

$$E[Y_t] = \mu + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}]$$
$$= \mu + 0 + 0 = \mu$$

$$\begin{aligned}
\text{var}(Y_t) &= \sigma^2 = E[(Y_t - \mu)^2] \\
&= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^2] + 2\theta E[\varepsilon_t\varepsilon_{t-1}] + \theta^2 E[\varepsilon_{t-1}^2] \\
&= \sigma_\varepsilon^2 + 0 + \theta^2\sigma_\varepsilon^2 = \sigma_\varepsilon^2(1 + \theta^2) \\
\text{cov}(Y_t, Y_{t-1}) &= \gamma_1 = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \\
&= E[\varepsilon_t\varepsilon_{t-1}] + \theta E[\varepsilon_t\varepsilon_{t-2}] \\
&\quad + \theta E[\varepsilon_{t-1}^2] + \theta^2 E[\varepsilon_{t-1}\varepsilon_{t-2}] \\
&= 0 + 0 + \theta\sigma_\varepsilon^2 + 0 = \theta\sigma_\varepsilon^2
\end{aligned}$$

Furthermore,

$$\begin{aligned}\text{cov}(Y_t, Y_{t-2}) &= \gamma_2 = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-2} + \theta\varepsilon_{t-3})] \\ &= E[\varepsilon_t\varepsilon_{t-2}] + \theta E[\varepsilon_t\varepsilon_{t-3}] \\ &\quad + \theta E[\varepsilon_{t-1}\varepsilon_{t-2}] + \theta^2 E[\varepsilon_{t-1}\varepsilon_{t-3}] \\ &= 0 + 0 + 0 + 0 = 0\end{aligned}$$

Similar calculation show that

$$\text{cov}(Y_t, Y_{t-j}) = \gamma_j = 0 \text{ for } j > 1$$

Autocorrelations

$$\rho_1 = \frac{\gamma_1}{\sigma^2} = \frac{\theta\sigma_\varepsilon^2}{\sigma_\varepsilon^2(1 + \theta^2)} = \frac{\theta}{(1 + \theta^2)}$$
$$\rho_j = \frac{\gamma_j}{\sigma^2} = 0 \text{ for } j > 1$$

Note:

$$\rho_1 = 0 \text{ if } \theta = 0$$

$$\rho_1 > 0 \text{ if } \theta > 0$$

$$\rho_1 < 0 \text{ if } \theta < 0$$

Result: MA(1) is covariance stationary for any value of θ

Example: MA(1) model for overlapping returns

Let r_t denote the 1-month cc return and assume that

$$r_t \sim \text{iid } N(\mu_r, \sigma_r^2)$$

Consider creating a monthly time series of 2-month cc returns using

$$r_t(2) = r_t + r_{t-1}$$

These 2-month returns observed monthly overlap by 1 month

$$\begin{aligned} r_t(2) &= r_t + r_{t-1} \\ r_{t-1}(2) &= r_{t-1} + r_{t-2} \\ r_{t-2}(2) &= r_{t-2} + r_{t-3} \\ &\vdots \end{aligned}$$

Claim: The stochastic process $\{r_t(2)\}$ follows a MA(1) process

Autoregressive (AR) Processes

AR(1) Model (mean-adjusted form)

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t, \quad -1 < \phi < 1$$
$$\varepsilon_t \sim \text{iid } N(0, \sigma_\varepsilon^2)$$

Result: AR(1) model is covariance stationary provided $-1 < \phi < 1$

Properties

$$\begin{aligned}E[Y_t] &= \mu \\ \text{var}(Y_t) &= \sigma^2 = \sigma_\varepsilon^2 / (1 - \phi^2) \\ \text{cov}(Y_t, Y_{t-1}) &= \gamma_1 = \sigma^2 \phi \\ \text{corr}(Y_t, Y_{t-1}) &= \rho_1 = \gamma_1 / \sigma^2 = \phi \\ \text{cov}(Y_t, Y_{t-j}) &= \gamma_j = \sigma^2 \phi^j \\ \text{corr}(Y_t, Y_{t-j}) &= \rho_j = \gamma_j / \sigma^2 = \phi^j\end{aligned}$$

Note: Since $|\phi| < 1$

$$\lim_{j \rightarrow \infty} \rho_j = \phi^j = 0$$

AR(1) Model (regression model form)

$$\begin{aligned} Y_t - \mu &= \phi(Y_{t-1} - \mu) + \varepsilon_t \Rightarrow \\ Y_t &= \mu - \phi\mu + \phi Y_{t-1} + \varepsilon_t \\ &= c + \phi Y_{t-1} + \varepsilon_t \end{aligned}$$

where

$$c = (1 - \phi)\mu \Rightarrow \mu = \frac{c}{1 - \phi}$$

Remarks:

- Regression model form is convenient for estimation by linear regression

The AR(1) model and Economic and Financial Time Series

The AR(1) model is a good description for the following time series

- Interest rates
- Growth rate of macroeconomic variables
 - Real GDP, industrial production
 - Money, velocity
 - Real wages, unemployment