Introduction to Computational Finance and Financial Econometrics

Time Series Concepts

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Outline

1 Stochastic Processes
   • Stationary Processes
   • Nonstationary Processes

2 Time series models
\{ \ldots, Y_1, Y_2, \ldots, Y_t, Y_{t+1}, \ldots \} = \{ Y_t \}_{t=-\infty}^{\infty}

sequence of random variables indexed by time

Observed time series of length $T$:

\{ Y_1 = y_1, Y_2 = y_2, \ldots, Y_T = y_T \} = \{ y_t \}_{t=1}^{T}

\begin{align*}
p(x, y) &= \Pr(X = x, Y = y) = \text{values in table} \\
e.g., \quad p(0, 0) &= \Pr(X = 0, Y = 0) = 1/8
\end{align*}
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2 Time series models
Intuition: \( \{Y_t\} \) is stationary if all aspects of its behavior are unchanged by shifts in time.

A stochastic process \( \{Y_t\}_{t=1}^{\infty} \) is *strictly stationary* if, for any given finite integer \( r \) and for any set of subscripts \( t_1, t_2, \ldots, t_r \) the joint distribution of

\[
(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_r})
\]

depends only on \( t_1 - t, t_2 - t, \ldots, t_r - t \) but not on \( t \).
Remarks

1. For example, the distribution of \((Y_1, Y_5)\) is the same as the distribution of \((Y_{12}, Y_{16})\).

2. For a strictly stationary process, \(Y_t\) has the same mean, variance (moments) for all \(t\).

3. Any function/transformation \(g(\cdot)\) of a strictly stationary process, \(\{g(Y_t)\}\) is also strictly stationary. e.g., if \(\{Y_t\}\) is strictly then \(\{Y_t^2\}\) is strictly stationary.
Covariance (weakly) Stationary Processes \{Y_t\}

- \(E[Y_t] = \mu\) for all \(t\)
- \(\text{var}(Y_t) = \sigma^2\) for all \(t\)
- \(\text{cov}(Y_t, Y_{t-j}) = \gamma_j\) depends on \(j\) and not on \(t\)

Note 1: \(\text{cov}(Y_t, Y_{t-j}) = \gamma_j\) is called the j-lag autocovariance and measures the direction of linear time dependence

Note 2: A stationary process is covariance stationary if \(\text{var}(Y_t) < \infty\) and \(\text{cov}(Y_t, Y_{t-j}) < \infty\)
Autocorrelations

\[
corr(Y_t, Y_{t-j}) = \rho_j = \frac{\text{cov}(Y_t, Y_{t-j})}{\sqrt{\text{var}(Y_t)\text{var}(Y_{t-j})}} = \frac{\gamma_j}{\sigma^2}
\]

Note 1: \(corr(Y_t, Y_{t-j}) = \rho_j\) is called the j-lag autocorrelation and measures the direction and strength of linear time dependence.

Note 2: By stationarity \(\text{var}(Y_t) = \text{var}(Y_{t-j}) = \sigma^2\).

Autocorrelation Function (ACF): Plot of \(\rho_j\) against \(j\).
Example: Gaussian White Noise Process

\[ Y_t \sim \text{iid } N(0, \sigma^2) \text{ or } Y_t \sim \text{GWN}(0, \sigma^2) \]

\[ E[Y_t] = 0, \quad \text{var}(Y_t) = \sigma^2 \]

\( Y_t \) independent of \( Y_s \) for \( t \neq s \)

\[ \Rightarrow \text{cov}(Y_t, Y_{t-s}) = 0 \text{ for } t \neq s \]

Note: “iid” = “independent and identically distributed”.

Here, \( \{Y_t\} \) represents random draws from the same \( N(0, \sigma^2) \) distribution.
Example: Independent White Noise Process

\[ Y_t \sim \text{iid } (0, \sigma^2) \text{ or } Y_t \sim IWN(0, \sigma^2) \]

\[ E[Y_t] = 0, \quad \text{var}(Y_t) = \sigma^2 \]

\[ Y_t \text{ independent of } Y_s \text{ for } t \neq s \]

Here, \( \{Y_t\} \) represents random draws from the same distribution. However, we don’t specify exactly what the distribution is - only that it has mean zero and variance \( \sigma^2 \). For example, \( Y_t \) could be iid Student’s t with variance equal to \( \sigma^2 \). This is like GWN but with fatter tails (i.e., more extreme observations).
Example: Weak White Noise Process

\[ Y_t \sim WN(0, \sigma^2) \]

\[ E[Y_t] = 0, \; \text{var}(Y_t) = \sigma^2 \]

\[ \text{cov}(Y_t, Y_s) = 0 \text{ for } t \neq s \]

Here, \( \{Y_t\} \) represents an uncorrelated stochastic process with mean zero and variance \( \sigma^2 \). Recall, the uncorrelated assumption does not imply independence. Hence, \( Y_t \) and \( Y_s \) can exhibit non-linear dependence (e.g. \( Y_t^2 \) can be correlated with \( Y_s^2 \)).
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Nonstationary Processes

Definition: A nonstationary stochastic process is a stochastic process that is not covariance stationary.

Note: A non-stationary process violates one or more of the properties of covariance stationarity.

Example: Deterministically trending process

\[ Y_t = \beta_0 + \beta_1 t + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2) \]

\[ E[Y_t] = \beta_0 + \beta_1 t \text{ depends on } t \]

Note: A simple detrending transformation yield a stationary process:

\[ X_t = Y_t - \beta_0 - \beta_1 t = \varepsilon_t \]
**Example:** Random Walk

\[ Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2), \quad Y_0 \text{ is fixed} \]

\[ = Y_0 + \sum_{j=1}^{t} \varepsilon_j \Rightarrow \text{var}(Y_t) = \sigma_\varepsilon^2 \times t \text{ depends on } t \]

Note: A simple detrending transformation yield a stationary process:

\[ \Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t \]
Outline

1 Stochastic Processes

2 Time series models
   - Moving Average (MA) Processes
   - Autoregressive (AR) Processes
Definition: A time series model is a probability model to describe the behavior of a stochastic process \( \{Y_t\} \).

Note: Typically, a time series model is a simple probability model that describes the time dependence in the stochastic process \( \{Y_t\} \).
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1 Stochastic Processes

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Moving Average (MA) Processes

Idea: Create a stochastic process that only exhibits one period linear time dependence.

MA(1) Model:

\[ Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}, \quad -\infty < \theta < \infty \]

\[ \varepsilon_t \sim iid \ N(0, \sigma^2) \] (i.e., \( \varepsilon_t \sim GWN(0, \sigma^2) \))

\( \theta \) determines the magnitude of time dependence

Properties:

\[ E[Y_t] = \mu + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}] \]

\[ = \mu + 0 + 0 = \mu \]
Moving Average (MA) Processes cont.

\[ \text{var}(Y_t) = \sigma^2 = E[(Y_t - \mu)^2] \]

\[ = E[(\varepsilon_t + \theta \varepsilon_{t-1})^2] \]

\[ = E[\varepsilon_t^2] + 2\theta E[\varepsilon_t \varepsilon_{t-1}] + \theta^2 E[\varepsilon_{t-1}^2] \]

\[ = \sigma_{\varepsilon}^2 + 0 + \theta^2 \sigma_{\varepsilon}^2 = \sigma_{\varepsilon}^2 (1 + \theta^2) \]

\[ \text{cov}(Y_t, Y_{t-1}) = \gamma_1 = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-1} + \theta \varepsilon_{t-2})] \]

\[ = E[\varepsilon_t \varepsilon_{t-1}] + \theta E[\varepsilon_t \varepsilon_{t-2}] \]

\[ + \theta E[\varepsilon_{t-1}^2] + \theta^2 E[\varepsilon_{t-1} \varepsilon_{t-2}] \]

\[ = 0 + 0 + \theta \sigma_{\varepsilon}^2 + 0 = \theta \sigma_{\varepsilon}^2 \]
Furthermore,

$$\text{cov}(Y_t, Y_{t-2}) = \gamma_2 = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-2} + \theta \varepsilon_{t-3})]$$

$$= E[\varepsilon_t \varepsilon_{t-2}] + \theta E[\varepsilon_t \varepsilon_{t-3}]$$

$$+ \theta E[\varepsilon_{t-1} \varepsilon_{t-2}] + \theta^2 E[\varepsilon_{t-1} \varepsilon_{t-3}]$$

$$= 0 + 0 + 0 + 0 = 0$$

Similar calculation show that:

$$\text{cov}(Y_t, Y_{t-j}) = \gamma_j = 0 \text{ for } j > 1$$
Autocorrelations:

\[ \rho_1 = \frac{\gamma_1}{\sigma^2} = \frac{\theta \sigma^2}{\sigma^2 (1 + \theta^2)} = \frac{\theta}{1 + \theta^2} \]

\[ \rho_j = \frac{\gamma_j}{\sigma^2} = 0 \text{ for } j > 1 \]

Note:

\[ \rho_1 = 0 \text{ if } \theta = 0 \]

\[ \rho_1 > 0 \text{ if } \theta > 0 \]

\[ \rho_1 < 0 \text{ if } \theta < 0 \]

Result: MA(1) is covariance stationary for any value of \( \theta \).
Example: MA(1) model for overlapping returns

Let \( r_t \) denote the 1-month cc return and assume that:
\[
r_t \sim \text{iid } N(\mu_r, \sigma_r^2)
\]
Consider creating a monthly time series of 2-month cc returns using:
\[
r_t(2) = r_t + r_{t-1}
\]
These 2-month returns observed monthly overlap by 1 month:
\[
r_t(2) = r_t + r_{t-1}
\]
\[
r_{t-1}(2) = r_{t-1} + r_{t-2}
\]
\[
r_{t-2}(2) = r_{t-2} + r_{t-3}
\]
\[
\vdots
\]
Claim: The stochastic process \( \{r_t(2)\} \) follows a MA(1) process.
Outline

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Autoregressive (AR) Processes

Idea: Create a stochastic process that exhibits multi-period geometrically decaying linear time dependence.

AR(1) Model (mean-adjusted form):

\[ Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t, \quad -1 < \phi < 1 \]

\[ \varepsilon_t \sim \text{iid } N(0, \sigma_\varepsilon^2) \]

Result: AR(1) model is covariance stationary provided \(-1 < \phi < 1\).
Autoregressive (AR) Processes cont.

Properties:

\[ E[Y_t] = \mu \]

\[ \text{var}(Y_t) = \sigma^2 = \sigma^2_\varepsilon/(1 - \phi^2) \]

\[ \text{cov}(Y_t, Y_{t-1}) = \gamma_1 = \sigma^2 \phi \]

\[ \text{corr}(Y_t, Y_{t-1}) = \rho_1 = \gamma_1/\sigma^2 = \phi \]

\[ \text{cov}(Y_t, Y_{t-j}) = \gamma_j = \sigma^2 \phi^j \]

\[ \text{corr}(Y_t, Y_{t-j}) = \rho_j = \gamma_j/\sigma^2 = \phi^j \]

Note: Since \(|\phi| < 1\),

\[ \lim_{j \to \infty} \rho_j = \phi^j = 0 \]
AR(1) Model (regression model form)

\[ Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t \Rightarrow \]

\[ Y_t = \mu - \phi\mu + \phi Y_{t-1} + \varepsilon_t \]

\[ = c + \phi Y_{t-1} + \varepsilon_t \]

where,

\[ c = (1 - \phi)\mu \Rightarrow \mu = \frac{c}{1 - \phi} \]

Remarks:

- Regression model form is convenient for estimation by linear regression
The AR(1) model is a good description for the following time series:

- Interest rates on U.S. Treasury securities, dividend yields, unemployment
- Growth rate of macroeconomic variables
  - Real GDP, industrial production, productivity
  - Money, velocity, consumer prices
  - Real and nominal wages