1 Beta as a Measure of Portfolio Risk

Key points:

- Asset specific risk can be diversified away by forming portfolios. What remains is "portfolio risk".

- Riskiness of an asset may be judged in a portfolio context

Motivating Example

Consider investing in an equally weighted portfolio with 99 assets

$$x_i = \frac{1}{99},\ R_{99} = \text{portfolio return},\ \sigma^2_{99} = \mathrm{var}(R_{99})$$

Now consider adding a new asset, say IBM, to the portfolio

$$\begin{aligned} R_{IBM} &= \text{return on IBM},\ \sigma^2_{IBM} = \mathrm{var}(R_{IBM}), \\ \sigma_{99,IBM} &= \mathrm{cov}(R_{99}, R_{IBM}) \end{aligned}$$

Create a new equally weighted portfolio of 100 assets

$$R_{100} = (.99)R_{99} + (.01)R_{IBM}$$

Portfolio variance (risk)

$$\begin{aligned}
\mathsf{var}(R_{100}) &= (.99)^2\mathsf{var}(R_{99}) + (.01)^2\mathsf{var}(R_{IBM}) + \\
&\quad 2(.99)(.01)\mathsf{cov}(R_{99}, R_{IBM}) \\
&= (.98)\sigma^2_{99} + (.0001)\sigma^2_{IBM} + (.02)\sigma_{99,IBM} \\
&\approx (.98)\sigma^2_{99} + (.02)\sigma_{99,IBM}
\end{aligned}$$

3 Cases

1. Adding IBM does not change the risk of portfolio

$$\sigma_{100}^2 = \sigma_{99}^2$$

2. Adding IBM increases portfolio risk

$$\sigma_{100}^2 > \sigma_{99}^2$$

3. Adding IBM decreases portfolio risk

$$\sigma_{100}^2 < \sigma_{99}^2$$

Case 1

If $\sigma^2_{100} = \sigma^2_{99}$ then

$$\begin{gathered}
(.98)\sigma^2_{99} + (.02)\sigma_{99,IBM} = \sigma^2_{99} \\
\Rightarrow (.02)\sigma_{99,IBM} = (.02)\sigma^2_{99} \\
\Rightarrow \frac{\sigma_{99,IBM}}{\sigma^2_{99}} = 1 \\
\Rightarrow \frac{\text{cov}(R_{99}, R_{IBM})}{\text{var}(R_{99})} = 1 \\
\Rightarrow \beta_{IBM,99} = 1
\end{gathered}$$

Conclusion: if

$$\beta_{IBM,99} = \frac{\text{cov}(R_{99}, R_{IBM})}{\text{var}(R_{99})} = 1$$

then adding IBM to the portfolio does not change risk (variance) of the portfolio.

Case 2

If $\sigma_{100}^2 > \sigma_{99}^2$ then

$$\beta_{IBM,99} > 1$$

and adding IBM to the portfolio increases the risk (variance) of the portfolio.

Case 3

If $\sigma_{100}^2 < \sigma_{99}^2$ then

$$\beta_{IBM,99} < 1$$

and adding IBM to the portfolio decreases the risk (variance) of the portfolio.

Conclusion: beta measures portfolio risk

$$\begin{aligned} R_p &= \text{ return on any portfolio} \\ R_i &= \text{ return on any asset } i \\ \beta_{i,p} &= \frac{\text{cov}(R_i, R_p)}{\text{var}(R_p)} = \frac{\sigma_{i,p}}{\sigma_p^2} \end{aligned}$$

Conjecture: If $\beta_{i,p}$ is the appropriate measure of the risk of an asset, then the asset's expected return, μ_i, should depend on $\beta_{i,p}$. That is

$$\mu_i = f(\beta_{i,p})$$

The Capital Asset Pricing Model (CAPM) formalizes this conjecture.

2 Sharpe's Single Index Model

$$\begin{aligned} R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\ i &= 1, \ldots, N;\ t = 1, \ldots, T \end{aligned}$$

where

$$\alpha_i,\ \beta_i \text{ are constant over time}$$
$$R_{Mt} = \text{ return on market index portfolio}$$
$$\varepsilon_{it} = \text{ random error term}$$

Assumptions

$$\text{cov}(R_{Mt}, \varepsilon_{it}) = 0$$
$$\text{cov}(\varepsilon_{is}, \varepsilon_{jt}) = 0 \text{ for all } i \neq j,\ t \text{ and } s$$
$$\varepsilon_{it} \sim \text{iid } N(0, \sigma^2_{\varepsilon,i})$$
$$R_{M,t} \sim \text{iid } N(\mu_M, \sigma^2_M)$$

Intuition:

$$\varepsilon_{it} = R_{it} - \alpha_i - \beta_i R_{Mt}$$

- Return on market index, R_{Mt}, captures common "market-wide" news.
- β_i measures sensitivity to "market-wide" news
- Random error term ε_{it} captures "firm specific" news unrelated to market-wide news.
- Returns are correlated only through their exposures to common "market-wide" news captured by β_i.

Remark: CER model is a special case of Single Index (SI) Model where $\beta_i = 0$ for all $i = 1, \ldots, N$. In this case, $\alpha_i = E[R_i] = \mu_i$

2.1 Statistical Properties of the SI Model

- $\mu_i = E[R_{it}] = \alpha_i + \beta_i \mu_M$

- $\sigma_i^2 = \mathsf{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2$

- $\sigma_{ij} = \mathsf{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j$

Derivations:

$$\begin{aligned}
\text{var}(R_{it}) &= \text{var}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}) \\
&= \beta_i^2 \text{var}(R_{Mt}) + \text{var}(\varepsilon_{it}) \\
&= \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2
\end{aligned}$$

where

$$\begin{aligned}
\beta_i^2 \sigma_M^2 &= \text{variance due to market news} \\
\sigma_{\varepsilon,i}^2 &= \text{variance due to non-market news}
\end{aligned}$$

Next

$$
\begin{aligned}
\sigma_{ij} &= \mathsf{cov}(R_{it}, R_{jt}) \\
&= \mathsf{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, \alpha_j + \beta_j R_{Mt} + \varepsilon_{jt}) \\
&= \mathsf{cov}(\beta_i R_{Mt}, \beta_j R_{Mt}) + \mathsf{cov}(\beta_i R_{Mt}, \varepsilon_{jt}) \\
&\quad + \mathsf{cov}(\beta_j R_{Mt}, \varepsilon_{it}) + \mathsf{cov}(\varepsilon_{it}, \varepsilon_{jt}) \\
&= \beta_i \beta_j \mathsf{cov}(R_{Mt}, R_{Mt}) \\
&= \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Implications:

- $\sigma_{ij} = 0$ if $\beta_i = 0$ or $\beta_j = 0$ (asset i or asset j do not respond to market news)

- $\sigma_{ij} > 0$ if $\beta_i, \beta_j > 0$ or $\beta_i, \beta_j < 0$ (asset i and j respond to market news in the same direction)

- $\sigma_{ij} < 0$ if $\beta_i > 0$ and $\beta_j < 0$ or if $\beta_i < 0$ and $\beta_j > 0$ (asset i and j respond to market news in opposite direction)

2.1.1 Interpretation of β_i

$$\beta_i = \frac{\mathsf{cov}(R_{it}, R_{Mt})}{\mathsf{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma_M^2}$$

β_i captures the contribution of asset i to the variance/risk of the market index.

Derivation:

$$\begin{aligned}
\mathsf{cov}(R_{it}, R_{Mt}) &= \mathsf{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, R_{Mt}) \\
&= \mathsf{cov}(\beta_i R_{Mt}, R_{Mt}) \\
&= \beta_i \mathsf{var}(R_{Mt}) \\
&\Rightarrow \beta_i = \frac{\mathsf{cov}(R_{it}, R_{Mt})}{\mathsf{var}(R_{Mt})}
\end{aligned}$$

2.1.2 Decomposition of Total Risk

$$\sigma_i^2 = \text{var}(R_{it}) = \beta_i^2\sigma_M^2 + \sigma_{\varepsilon,i}^2$$
$$\text{total risk} = \text{market risk} + \text{non-market risk}$$

Divide both sides by σ_i^2

$$\begin{aligned} 1 &= \frac{\beta_i^2\sigma_M^2}{\sigma_i^2} + \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\ &= R_i^2 + 1 - R_i^2 \end{aligned}$$

where

$$\begin{aligned} R_i^2 &= \frac{\beta_i^2\sigma_M^2}{\sigma_i^2} = \text{proportion of market risk} \\ 1 - R_i^2 &= \text{proportion of non-market risk} \end{aligned}$$

Sharpe's Rule of Thumb: A typical stock has $R^2 = 30\%$; i.e., proportion of market risk in typical stock is 30% of total risk.

2.1.3 Return Covariance Matrix

3 asset example

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ i = 1, 2, 3 \\
\sigma_i^2 &= \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\
\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Covariance matrix

$$
\begin{aligned}
\boldsymbol{\Sigma} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1^2\sigma_M^2 + \sigma_{\varepsilon,1}^2 & \sigma_M^2\beta_1\beta_2 & \sigma_M^2\beta_1\beta_3 \\ \sigma_M^2\beta_1\beta_2 & \beta_2^2\sigma_M^2 + \sigma_{\varepsilon,2}^2 & \sigma_M^2\beta_2\beta_3 \\ \sigma_M^2\beta_1\beta_3 & \sigma_M^2\beta_2\beta_3 & \beta_3^2\sigma_M^2 + \sigma_{\varepsilon,3}^2 \end{pmatrix} \\
&= \sigma_M^2 \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_1\beta_2 & \beta_2^2 & \beta_2\beta_3 \\ \beta_1\beta_3 & \beta_2\beta_3 & \beta_3^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \sigma_{\varepsilon,2}^2 & 0 \\ 0 & 0 & \sigma_{\varepsilon,3}^2 \end{pmatrix}
\end{aligned}
$$

Simplification using matrix algebra

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} \sigma^2_{\varepsilon,1} & 0 & 0 \\ 0 & \sigma^2_{\varepsilon,2} & 0 \\ 0 & 0 & \sigma^2_{\varepsilon,3} \end{pmatrix}$$

Then

$$\underset{(3\times 3)}{\boldsymbol{\Sigma}} = \sigma^2_M \cdot \underset{(3\times 1)}{\boldsymbol{\beta}} \underset{(1\times 3)}{\boldsymbol{\beta}}' + \underset{(3\times 3)}{\mathbf{D}}$$

where

$$\begin{aligned} \sigma^2_M \cdot \boldsymbol{\beta}\boldsymbol{\beta}' &= \text{covariance due to market} \\ \mathbf{D} &= \text{asset specific variances} \end{aligned}$$

2.2 SI Model and Portfolios

2 asset example

$$
\begin{aligned}
R_{1t} &= \alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t} \\
R_{2t} &= \alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t} \\
x_1 &= \text{share invested in asset 1} \\
x_2 &= \text{share invested in asset 2} \\
x_1 + x_2 &= 1
\end{aligned}
$$

Portfolio return

$$
\begin{aligned}
R_{p,t} &= x_1 R_{1t} + x_2 R_{2t} \\
&= x_1(\alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t}) \\
&\quad + x_2(\alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t}) \\
&= (x_1\alpha_1 + x_2\alpha_2) + (x_1\beta_1 + x_2\beta_2)\, R_{Mt} \\
&\quad + (x_1\varepsilon_{1t} + x_2\varepsilon_{2t}) \\
&= \alpha_p + \beta_p R_{Mt} + \varepsilon_{p,t}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_p &= x_1\alpha_1 + x_2\alpha_2 \\
\beta_p &= x_1\beta_1 + x_2\beta_2 \\
\varepsilon_{p,t} &= x_1\varepsilon_{1t} + x_2\varepsilon_{2t}
\end{aligned}
$$

2.2.1 SI Model with Large Portfolios

$$
\begin{aligned}
i &= 1, \ldots, N \text{ assets (e.g. } N = 500) \\
x_i &= \frac{1}{N} = \text{equal investment shares} \\
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}
\end{aligned}
$$

Portfolio return

$$
\begin{aligned}
R_{p,t} &= \sum_{i=1}^{N} x_i R_{it} \\
&= \sum_{i=1}^{N} x_i \left(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}\right) \\
&= \sum_{i=1}^{N} x_i \alpha_i + \left(\sum_{i=1}^{N} x_i \beta_i\right) R_{Mt} + \sum_{i=1}^{N} x_i \varepsilon_{it} \\
&= \frac{1}{N} \sum_{i=1}^{N} \alpha_i + \left(\frac{1}{N} \sum_{i=1}^{N} \beta_i\right) R_{Mt} + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} \\
&= \bar{\alpha} + \bar{\beta} R_{Mt} + \bar{\varepsilon}_t
\end{aligned}
$$

where

$$\begin{aligned} \bar{\alpha} &= \frac{1}{N}\sum_{i=1}^{N}\alpha_i \\ \bar{\beta} &= \frac{1}{N}\sum_{i=1}^{N}\beta_i \\ \bar{\varepsilon}_t &= \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it} \end{aligned}$$

Result: For large N,

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it} \approx 0$$

because $\varepsilon_{it} \sim$ iid $N(0, \sigma^2_{\varepsilon,i})$.

Implications

In a large well diversified portfolio, the following results hold:

- $R_{p,t} \approx \bar{\alpha} + \bar{\beta} R_{Mt}$: all non-market risk is diversified away

- $\text{var}(R_{p,t}) = \bar{\beta}^2 \text{var}(R_{Mt})$: Magnitude of portfolio risk is proportional to market risk. Magnitude of portfolio risk is determined by portfolio beta $\bar{\beta}$

- $R_p^2 \approx 1$: Approximately 100% of portfolio risk is due to market risk