Econ 424/CFRM 462
Single Index Model

Eric Zivot

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Sharpe's Single Index Model

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$
$$i = 1, \ldots, N; \; t = 1, \ldots, T$$

where

$$\alpha_i, \; \beta_i \text{ are constant over time}$$
$$R_{Mt} = \text{ return on diversified market index portfolio}$$
$$\varepsilon_{it} = \text{ random error term unrelated to } R_{Mt}$$

Assumptions

- cov$(R_{Mt}, \varepsilon_{is}) = 0$ for all t, s
- cov$(\varepsilon_{is}, \varepsilon_{jt}) = 0$ for all $i \neq j$, t and s
- $\varepsilon_{it} \sim$ iid $N(0, \sigma^2_{\varepsilon,i})$
- $R_{M,t} \sim$ iid $N(\mu_M, \sigma^2_M)$

Interpretation of β_i

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$
$$\beta_i = \frac{\mathsf{cov}(R_{it}, R_{Mt})}{\mathsf{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma_M^2}$$

β_i captures the contribution of asset i to the volatility of the market index (recall risk budgeting calculations).

Derivation:

$$\begin{aligned}
\mathsf{cov}(R_{it}, R_{Mt}) &= \mathsf{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, R_{Mt}) \\
&= \mathsf{cov}(\beta_i R_{Mt}, R_{Mt}) + \mathsf{cov}(\varepsilon_{it}, R_{Mt}) \\
&= \beta_i \mathsf{var}(R_{Mt}) \quad \text{since } \mathsf{cov}(\varepsilon_{it}, R_{Mt}) = 0 \\
&\Rightarrow \beta_i = \frac{\mathsf{cov}(R_{it}, R_{Mt})}{\mathsf{var}(R_{Mt})}
\end{aligned}$$

Interpretation of ε_{it}:

$$\varepsilon_{it} = R_{it} - \alpha_i - \beta_i R_{Mt}$$

- Return on market index, R_{Mt}, captures common "market-wide" news.

- β_i measures sensitivity to "market-wide" news

- Random error term ε_{it} captures "firm specific" news unrelated to market-wide news.

- Returns are correlated only through their exposures to common "market-wide" news captured by β_i.

Remark:

- The CER model is a special case of Single Index (SI) Model where $\beta_i = 0$ for all $i = 1, \ldots, N$.

$$R_{it} = \alpha_i + \varepsilon_{it}$$

  In this case, $\alpha_i = E[R_i] = \mu_i$

- In the CER model there is only one source of news

- In the Single Index model there are two sources of news: market news and asset specific news

Single Index Model with Matrix Algebra

$$\begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \beta_1 R_{Mt} \\ \vdots \\ \beta_N R_{Mt} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{pmatrix}$$

or

$$\underset{N\times 1}{\mathbf{R}_t} = \underset{N\times 1}{\alpha} + \underset{N\times 1}{\beta} \underset{1\times 1}{R_{Mt}} + \underset{N\times 1}{\varepsilon_t}$$

where

$$\mathbf{R}_t = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix},\ \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},\ \varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{pmatrix}$$

Statistical Properties of the SI Model (Unconditional)

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

- $\mu_i = E[R_{it}] = \alpha_i + \beta_i \mu_M$

- $\sigma_i^2 = \text{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2$

- $\sigma_{ij} = \text{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j$

- $R_{it} \sim N(\mu_i, \sigma_i^2) = N(\alpha_i + \beta_i \mu_M, \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2)$

Derivations:

$$\begin{aligned}
\text{var}(R_{it}) &= \text{var}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}) \\
&= \beta_i^2 \text{var}(R_{Mt}) + \text{var}(\varepsilon_{it}) + 2\beta_i \text{cov}(R_{Mt}, \varepsilon_{it}) \\
&= \beta_i^2 \text{var}(R_{Mt}) + \text{var}(\varepsilon_{it}) \text{ (assume cov}(R_{Mt}, \varepsilon_{it}) = 0) \\
&= \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2
\end{aligned}$$

where

$$\begin{aligned}
\beta_i^2 \sigma_M^2 &= \text{variance due to market news} \\
\sigma_{\varepsilon,i}^2 &= \text{variance due to non-market news}
\end{aligned}$$

Next

$$
\begin{aligned}
\sigma_{ij} &= \mathsf{cov}(R_{it}, R_{jt}) \\
&= \mathsf{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, \alpha_j + \beta_j R_{Mt} + \varepsilon_{jt}) \\
&= \mathsf{cov}(\beta_i R_{Mt}, \beta_j R_{Mt}) + \mathsf{cov}(\beta_i R_{Mt}, \varepsilon_{jt}) \\
&+ \mathsf{cov}(\beta_j R_{Mt}, \varepsilon_{it}) + \mathsf{cov}(\varepsilon_{it}, \varepsilon_{jt}) \\
&= \beta_i \beta_j \mathsf{cov}(R_{Mt}, R_{Mt}) \\
&= \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Implications:

- $\sigma_{ij} = 0$ if $\beta_i = 0$ or $\beta_j = 0$ (asset i or asset j do not respond to market news)

- $\sigma_{ij} > 0$ if $\beta_i, \beta_j > 0$ or $\beta_i, \beta_j < 0$ (asset i and j respond to market news in the same direction)

- $\sigma_{ij} < 0$ if $\beta_i > 0$ and $\beta_j < 0$ or if $\beta_i < 0$ and $\beta_j > 0$ (asset i and j respond to market news in opposite direction)

Statistical Properties of the SI Model (Conditional on R_{Mt})

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

Given that we observe, $R_{Mt} = r_{Mt}$

- $E[R_{it}|R_{Mt} = r_{Mt}] = \alpha_i + \beta_i r_{Mt}$

- $\text{var}(R_{it}|R_{Mt} = r_{Mt}) = \sigma^2_{\varepsilon,i}$

- $\text{cov}(R_{it}, R_{jt}|R_{Mt} = r_{Mt}) = 0$

- $R_{it}|R_{Mt} = r_{Mt} \sim N(\alpha_i + \beta_i r_{Mt}, \sigma^2_{\varepsilon,i})$

Decomposition of Total Variance

$$\begin{aligned} R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\ \sigma_i^2 = \text{var}(R_{it}) &= \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\ \text{total variance} &= \text{market variance} + \text{non-market variance} \end{aligned}$$

Divide both sides by σ_i^2

$$\begin{aligned} 1 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\ &= R_i^2 + 1 - R_i^2 \end{aligned}$$

where

$$\begin{aligned} R_i^2 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} = \text{proportion of market variance} \\ 1 - R_i^2 &= \text{proportion of non-market variance} \end{aligned}$$

Sharpe's Rule of Thumb: A typical stock has $R_i^2 = 30\%$; i.e., proportion of market variance in typical stock is 30% of total variance.

Return Covariance Matrix

3 asset example

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ i = 1, 2, 3 \\
\sigma_i^2 &= \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\
\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Covariance matrix

$$
\begin{aligned}
\mathbf{\Sigma} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1^2\sigma_M^2 + \sigma_{\varepsilon,1}^2 & \sigma_M^2\beta_1\beta_2 & \sigma_M^2\beta_1\beta_3 \\ \sigma_M^2\beta_1\beta_2 & \beta_2^2\sigma_M^2 + \sigma_{\varepsilon,2}^2 & \sigma_M^2\beta_2\beta_3 \\ \sigma_M^2\beta_1\beta_3 & \sigma_M^2\beta_2\beta_3 & \beta_3^2\sigma_M^2 + \sigma_{\varepsilon,3}^2 \end{pmatrix} \\
&= \sigma_M^2 \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_1\beta_2 & \beta_2^2 & \beta_2\beta_3 \\ \beta_1\beta_3 & \beta_2\beta_3 & \beta_3^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \sigma_{\varepsilon,2}^2 & 0 \\ 0 & 0 & \sigma_{\varepsilon,3}^2 \end{pmatrix}
\end{aligned}
$$

Simplification using matrix algebra

$$\underset{3\times 1}{\mathbf{R}_t} = \underset{3\times 1}{\alpha} + \underset{3\times 1}{\beta}\underset{1\times 1}{R_{Mt}} + \underset{3\times 1}{\varepsilon_t}$$

$$\mathbf{R}_t = \begin{pmatrix} R_{1t} \\ R_{2t} \\ R_{3t} \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},\ \varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

Then

$$\begin{aligned} \mathbf{\Sigma} &= var(\mathbf{R}) = \beta \mathsf{var}(R_{Mt})\beta' + var(\varepsilon_t) \\ &= \sigma_M^2 \cdot \underset{(3\times 1)}{\beta}\underset{(1\times 3)}{\beta}{}' + \underset{(3\times 3)}{\mathbf{D}} \end{aligned}$$

where

$$\begin{aligned} \sigma_M^2 \cdot \beta\beta' &= \text{ covariance due to market} \\ \mathbf{D} &= \mathsf{diag}(\sigma^2_{\varepsilon,1}, \sigma^2_{\varepsilon,2}, \sigma^2_{\varepsilon,3}) = \text{asset specific variances} \end{aligned}$$

SI Model and Portfolios

2 asset example

$$
\begin{aligned}
R_{1t} &= \alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t} \\
R_{2t} &= \alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t} \\
x_1 &= \text{share invested in asset 1} \\
x_2 &= \text{share invested in asset 2} \\
x_1 + x_2 &= 1
\end{aligned}
$$

Portfolio return

$$\begin{aligned}
R_{p,t} &= x_1 R_{1t} + x_2 R_{2t} \\
&= x_1(\alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t}) \\
&+ x_2(\alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t}) \\
&= (x_1\alpha_1 + x_2\alpha_2) + (x_1\beta_1 + x_2\beta_2)\, R_{Mt} \\
&+ (x_1\varepsilon_{1t} + x_2\varepsilon_{2t}) \\
&= \alpha_p + \beta_p R_{Mt} + \varepsilon_{p,t}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_p &= x_1\alpha_1 + x_2\alpha_2 \\
\beta_p &= x_1\beta_1 + x_2\beta_2 \\
\varepsilon_{p,t} &= x_1\varepsilon_{1t} + x_2\varepsilon_{2t}
\end{aligned}$$

SI Model with Large Portfolios

$$
\begin{aligned}
i &= 1, \ldots, N \text{ assets (e.g. } N = 500) \\
x_i &= \frac{1}{N} = \text{equal investment shares} \\
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}
\end{aligned}
$$

Portfolio return

$$
\begin{aligned}
R_{p,t} &= \sum_{i=1}^{N} x_i R_{it} \\
&= \sum_{i=1}^{N} x_i \left(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}\right) \\
&= \sum_{i=1}^{N} x_i \alpha_i + \left(\sum_{i=1}^{N} x_i \beta_i\right) R_{Mt} + \sum_{i=1}^{N} x_i \varepsilon_{it} \\
&= \frac{1}{N} \sum_{i=1}^{N} \alpha_i + \left(\frac{1}{N} \sum_{i=1}^{N} \beta_i\right) R_{Mt} + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} \\
&= \bar{\alpha} + \bar{\beta} R_{Mt} + \bar{\varepsilon}_t
\end{aligned}
$$

where

$$\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N} \alpha_i$$
$$\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N} \beta_i$$
$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_{it}$$

Result: For large N,

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_{it} \approx E[\varepsilon_{it}] = 0$$

because $\varepsilon_{it} \sim$ iid $N(0, \sigma^2_{\varepsilon,i})$.

Implications

In a large well diversified portfolio, the following results hold:

- $R_{p,t} \approx \bar{\alpha} + \bar{\beta} R_{Mt}$: all non-market risk is diversified away

- $\text{var}(R_{p,t}) = \bar{\beta}^2 \text{var}(R_{Mt})$: Magnitude of portfolio variance is proportional to market variance. Magnitude of portfolio variance is determined by portfolio beta $\bar{\beta}$

- $R_p^2 \approx 1$: Approximately 100% of portfolio variance is due to market variance