Econ 424/CFRM 462
Statistical Analysis of Efficient Portfolios

Eric Zivot

August 14, 2014
The CER Model and Efficient Portfolios

Let $R_{it}$ denote the return on asset $i$ in month $t$ and assume that $R_{it}$ follows CER model:

$$R_{it} \sim iid \ N(\mu_i, \sigma_i^2),$$

$i = 1, \ldots, N$ (assets)

$t = 1, \ldots, T$ (months)

$$cov(R_{it}, R_{jt}) = \sigma_{ij}$$

We estimate the CER model parameters using sample statistics giving

$$\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}$$

Remember, the estimates $\hat{\mu}_i, \hat{\sigma}_i^2$ are $\hat{\sigma}_{ij}$ are random variables and are subject to error

Key result: Sharpe ratios and efficient portfolios are functions of $\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}$; they are random variables and are subject to error
Statistical Properties of Efficient portfolios

- Inputs to portfolio theory are estimates from CER model $\hat{\mu}$ and $\hat{\Sigma}$

- Sharpe ratios and efficient portfolios are functions of $\hat{\mu}$ and $\hat{\Sigma}$.

- The estimated Sharpe ratio is

$$\widehat{SR}_i = \frac{\hat{\mu}_i - r_f}{\hat{\sigma}_i}$$

- No easy formula for $SE(\widehat{SR}_i)$
• The estimated global minimum variance portfolio is

$$\hat{m} = \frac{\hat{\Sigma}^{-1}1}{1'\hat{\Sigma}^{-1}1}$$

$\hat{m}$ is estimated with error because we estimate $\Sigma$ using $\hat{\Sigma}$.

• No easy analytic formulas for the standard errors of the elements of $\hat{m} = (\hat{m}_1, \ldots, \hat{m}_n)'$; i.e. no easy formula for $SE(\hat{m}_i)$

• In addition, the expected return and standard deviation of $R_{p,\hat{m}} = \hat{m}'R$ have additional sources of error due to the error in $\hat{m}$. That is,

$$\hat{\mu}_{p,\hat{m}} = \hat{m}'\hat{\mu}$$
$$\hat{\sigma}_{p,\hat{m}} = (\hat{m}'\hat{\Sigma}\hat{m})^{1/2}$$

No easy analytic formulas for $SE(\hat{\mu}_{p,\hat{m}})$ and $SE(\hat{\sigma}_{p,\hat{m}})$
Optimizers are Error Maximizers

• From our analysis of the CER model, $\mu_i$ is estimated less precisely than $\sigma_i$. That is, there is more estimation error in $\hat{\mu}_i$ than $\hat{\sigma}_i$.

• Large estimation error in $\hat{\mu}_i$ greatly impacts efficient portfolios
  
  – Large positive errors ($\hat{\mu}_i$ much greater than $\mu_i$) leads to efficient portfolios being concentrated in asset $i$
  
  – Large negative errors ($\hat{\mu}_i$ much less than $\mu_i$) leads to efficient portfolios that avoid asset $i$ or shorts asset $i$

• Constraints on portfolio weights can offset the impact of estimation error in $\hat{\mu}_i$

  Portfolio of assets have smaller estimation error in $\hat{\mu}$ than individual assets
Bootstrapping Efficient Portfolios

The bootstrap can be used to evaluate the sampling uncertainty of Sharpe ratios and efficient portfolios.

Portfolio statistics to bootstrap:

- Portfolio weights
- Portfolio expected returns and standard deviations
Are Efficient Portfolios Constant Over Time?

Result: We have seen evidence that the parameters of the CER model for various assets are not constant over time:

- Rolling estimates of $\mu$, $\sigma$, and $\sigma_{ij}$ show variation over time

Implication: Since estimates of $\mu$, $\sigma$, and $\sigma_{ij}$ are inputs to efficient portfolio calculations, then time variation in $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\sigma}_{ij}$ imply time variation in efficient portfolios
Rolling Efficient Portfolios

Idea: Using rolling estimates of $\mu$ and $\Sigma$ compute rolling efficient portfolios

- global minimum variance portfolio
- efficient portfolio for target return
- tangency portfolio
- efficient frontier

Look at time variation in resulting portfolio weights
Rolling Global Minimum Variance Portfolio

Idea: compute estimates of portfolio weights $\mathbf{m}$ over rolling windows of length $n < T$:

$$
\min_{\mathbf{m}(n)} \mathbf{m}_t(n)'\hat{\Sigma}_t(n)\mathbf{m}_t(n) \quad \text{s.t.} \quad \mathbf{m}_t(n)'\mathbf{1} = 1
$$

$$
t = n, \ldots, T
$$

$\hat{\Sigma}_t(n) = \text{rolling estimate of } \Sigma \text{ in month } t$

If

$$
\hat{\Sigma}_n(n) \approx \hat{\Sigma}_{n+1}(n) \approx \cdots \approx \hat{\Sigma}_T(n)
$$

then

$$
\mathbf{m}_n(n) \approx \mathbf{m}_{n+1}(n) \approx \cdots \approx \mathbf{m}_T(n)
$$
Rolling Efficient Portfolios

Idea: compute estimates of portfolio weights $\mathbf{x}$ over rolling windows of length $n < T$ for $t = n, \ldots, T$:

$$\min_{\mathbf{x}(n)} \mathbf{x}_t(n)' \hat{\Sigma}_t(n) \mathbf{x}_t(n)$$

s.t. $\mathbf{x}_t(n)' \mathbf{1} = 1$, $\mathbf{x}_t(n)' \hat{\mu}_t(n) = \mu_p^{\text{target}}$

$\hat{\mu}_t(n) =$ rolling estimate of $\mu$ in month $t$

$\hat{\Sigma}_t(n) =$ rolling estimate of $\Sigma$ in month $t$

If

$$\hat{\mu}_n(n) \approx \hat{\mu}_{n+1}(n) \approx \cdots \approx \hat{\mu}_T(n)$$

$$\hat{\Sigma}_n(n) \approx \hat{\Sigma}_{n+1}(n) \approx \cdots \approx \hat{\Sigma}_T(n)$$

then

$$\mathbf{x}_n(n) \approx \mathbf{x}_{n+1}(n) \approx \cdots \approx \mathbf{x}_T(n)$$