Introduction to Computational Finance and Financial Econometrics
Return Calculations

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The Time Value of Money

Future Value

• $V$ invested for $n$ years at simple interest rate $R$ per year

• Compounding of interest occurs at end of year

\[ FV_n = V \cdot (1 + R)^n, \]

where $FV_n$ is future value after $n$ years
**Example:** Consider putting $1000 in an interest checking account that pays a simple annual percentage rate of 3%. The future value after $n = 1, 5$ and 10 years is, respectively,

\[
FV_1 = 1000 \cdot (1.03)^1 = 1030, \\
FV_5 = 1000 \cdot (1.03)^5 = 1159.27, \\
FV_{10} = 1000 \cdot (1.03)^{10} = 1343.92.
\]
FV function is a relationship between four variables: $FV_n, V, R, n$. Given three variables, you can solve for the fourth:

- **Present value:**
  \[ V = \frac{FV_n}{(1 + R)^n}. \]

- **Compound annual return:**
  \[ R = \left( \frac{FV_n}{V} \right)^{1/n} - 1. \]

- **Investment horizon:**
  \[ n = \frac{\ln(FV_n/V)}{\ln(1 + R)}. \]
Compounding occurs $m$ times per year

- $FV_n^m = V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n}$,

  $\frac{R}{m} = \text{periodic interest rate.}$

Continuous compounding

- $FV_n^\infty = \lim_{m \to \infty} V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n} = Ve^{R \cdot n}$,

  $e^1 = 2.71828$. 
**Example:** If the simple annual percentage rate is 10% then the value of $1000 at the end of one year \((n = 1)\) for different values of \(m\) is given in the table below.

<table>
<thead>
<tr>
<th>Compounding Frequency</th>
<th>Value of $1000 at end of 1 year ((R = 10%))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually ((m = 1))</td>
<td>1100.00</td>
</tr>
<tr>
<td>Quarterly ((m = 4))</td>
<td>1103.81</td>
</tr>
<tr>
<td>Weekly ((m = 52))</td>
<td>1105.06</td>
</tr>
<tr>
<td>Daily ((m = 365))</td>
<td>1105.16</td>
</tr>
<tr>
<td>Continuously ((m = \infty))</td>
<td>1105.17</td>
</tr>
</tbody>
</table>
Effective Annual Rate

Annual rate $R_A$ that equates $FV_n^m$ with $FV_n$; i.e.,

$$V \left(1 + \frac{R}{m}\right)^{m \cdot n} = V(1 + R_A)^n.$$ 

Solving for $R_A$

$$\left(1 + \frac{R}{m}\right)^m = 1 + R_A \Rightarrow R_A = \left(1 + \frac{R}{m}\right)^m - 1.$$
Continuous compounding

\[ Ve^{R \cdot n} = V (1 + R_A)^n \]
\[ \Rightarrow e^R = (1 + R_A) \]
\[ \Rightarrow R_A = e^R - 1. \]
Example. *Compute effective annual rate with semi-annual compounding*

The effective annual rate associated with an investment with a simple annual rate $R = 10\%$ and semi-annual compounding ($m = 2$) is determined by solving

$$(1 + R_A) = \left(1 + \frac{0.10}{2}\right)^2$$

$\Rightarrow R_A = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 0.1025.$
<table>
<thead>
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<th>$R_A$</th>
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<td>Annually ($m = 1$)</td>
<td>1100.00</td>
<td>10%</td>
</tr>
<tr>
<td>Quarterly ($m = 4$)</td>
<td>1103.81</td>
<td>10.38%</td>
</tr>
<tr>
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<td>1105.06</td>
<td>10.51%</td>
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Asset Return Calculations

Simple Returns

- \( P_t \) = price at the end of month \( t \) on an asset that pays no dividends

- \( P_{t-1} \) = price at the end of month \( t - 1 \)

\[
R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \% \triangle P_t = \text{net return over month } t,
\]

\[
1 + R_t = \frac{P_t}{P_{t-1}} = \text{gross return over month } t.
\]
Example. One month investment in Microsoft stock.

Buy stock at end of month $t - 1$ at $P_{t-1} = $85 and sell stock at end of next month for $P_t = $90. Assuming that Microsoft does not pay a dividend between months $t - 1$ and $t$, the one-month simple net and gross returns are

$$R_t = \frac{$90 - $85}{$85} = \frac{$90}{$85} - 1 = 1.0588 - 1 = 0.0588,$$

$$1 + R_t = 1.0588.$$

The one month investment in Microsoft yielded a 5.88% per month return.
Multi-period Returns

Simple two-month return

\[ R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1. \]

Relationship to one month returns

\[ R_t(2) = \frac{P_t}{P_{t-2}} - 1 = \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 = (1 + R_t) \cdot (1 + R_{t-1}) - 1. \]
Here

\[ 1 + R_t = \text{one-month gross return over month } t, \]
\[ 1 + R_{t-1} = \text{one-month gross return over month } t - 1, \]
\[ \implies 1 + R_t(2) = (1 + R_t) \cdot (1 + R_{t-1}). \]

two-month gross return = the product of two one-month gross returns

Note: two-month returns are not additive:

\[ R_t(2) = R_t + R_{t-1} + R_t \cdot R_{t-1} \]
\[ \approx R_t + R_{t-1} \text{ if } R_t \text{ and } R_{t-1} \text{ are small} \]
Example: Two-month return on Microsoft

Suppose that the price of Microsoft in month $t - 2$ is $80$ and no dividend is paid between months $t - 2$ and $t$. The two-month net return is

$$R_t(2) = \frac{90 - 80}{80} = \frac{90}{80} - 1 = 1.1250 - 1 = 0.1250,$$

or 12.50% per two months. The two one-month returns are

$$R_{t-1} = \frac{85 - 80}{80} = 1.0625 - 1 = 0.0625$$

$$R_t = \frac{90 - 85}{85} = 1.0588 - 1 = 0.0588,$$

and the geometric average of the two one-month gross returns is

$$1 + R_t(2) = 1.0625 \times 1.0588 = 1.1250.$$
Simple $k$-month Return

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1$$

$$1 + R_t(k) = (1 + R_t) \cdot (1 + R_{t-1}) \cdots (1 + R_{t-k+1})$$

$$= \prod_{j=0}^{k-1} (1 + R_{t-j})$$

Note

$$R_t(k) \neq \sum_{j=0}^{k-1} R_{t-j}$$
Portfolio Returns

• Invest $V$ in two assets: A and B for 1 period

• $x_A = \text{share of }$ $V$ invested in A; $V \times x_A = \$ \text{ amount}$

• $x_B = \text{share of }$ $V$ invested in B; $V \times x_B = \$ \text{ amount}$

• Assume $x_A + x_B = 1$

• Portfolio is defined by investment shares $x_A$ and $x_B$
At the end of the period, the investments in A and B grow to

$$V(1 + R_{p,t}) = V \left[ x_A(1 + R_A,t) + x_B(1 + R_B,t) \right]$$

$$= V \left[ x_A + x_B + x_A R_A,t + x_B R_B,t \right]$$

$$= V \left[ 1 + x_A R_A,t + x_B R_B,t \right]$$

$$\Rightarrow R_{p,t} = x_A R_A,t + x_B R_B,t$$

The simple portfolio return is a share weighted average of the simple returns on the individual assets.
Example: *Portfolio of Microsoft and Starbucks stock*

Purchase ten shares of each stock at the end of month $t-1$ at prices

\[ P_{msft,t-1} = 85, \quad P_{sbux,t-1} = 30, \]

The initial value of the portfolio is

\[ V_{t-1} = 10 \times 85 + 10 \times 30 = 1,150. \]

The portfolio shares are

\[ x_{msft} = 850/1150 = 0.7391, \quad x_{sbux} = 300/1150 = 0.2609. \]

The end of month $t$ prices are $P_{msft,t} = 90$ and $P_{sbux,t} = 28$. 
Assuming Microsoft and Starbucks do not pay a dividend between periods \( t - 1 \) and \( t \), the one-period returns are

\[
R_{msft,t} = \frac{90 - 85}{85} = 0.0588
\]

\[
R_{sbux,t} = \frac{28 - 30}{30} = -0.0667
\]

The return on the portfolio is

\[
R_{p,t} = (0.7391)(0.0588) + (0.2609)(-0.0667) = 0.02609
\]

and the value at the end of month \( t \) is

\[
V_t = 1,150 \times 1.02609 = 1,180
\]
In general, for a portfolio of $n$ assets with investment shares $x_i$ such that $x_1 + \cdots + x_n = 1$

$$1 + R_{p,t} = \sum_{i=1}^{n} x_i (1 + R_{i,t})$$

$$R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$$

$$= x_1 R_{1t} + \cdots + x_n R_{nt}$$
Adjusting for Dividends

\[ D_t = \text{dividend payment between months } t - 1 \text{ and } t \]

\[ R_{t}^{\text{total}} = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}} \]

\[ = \text{capital gain return + dividend yield (gross)} \]

\[ 1 + R_{t}^{\text{total}} = \frac{P_t + D_t}{P_{t-1}} \]
Example. *Total return on Microsoft stock.*

Buy stock in month $t-1$ at $P_{t-1} = $85 and sell the stock the next month for $P_t = $90. Assume Microsoft pays a $1 dividend between months $t-1$ and $t$. The capital gain, dividend yield and total return are then

$$R_{t}^{total} = \frac{\$90 + \$1 - \$85}{\$85} = \frac{\$90 - \$85}{\$85} + \frac{\$1}{\$85}$$

$$= 0.0588 + 0.0118$$

$$= 0.0707$$

The one-month investment in Microsoft yields a 7.07% per month total return. The capital gain component is 5.88%, and the dividend yield component is 1.18%.
Adjusting for Inflation

The computation of real returns on an asset is a two step process:

- Deflate the nominal price $P_t$ of the asset by an index of the general price level $CPI_t$

- Compute returns in the usual way using the deflated prices
\( P_t^{\text{Real}} = \frac{P_t}{CPI_t} \)

\( R_t^{\text{Real}} = \frac{P_t^{\text{Real}} - P_{t-1}^{\text{Real}}}{P_{t-1}^{\text{Real}}} = \frac{P_t}{CPI_t} - \frac{P_{t-1}}{CPI_{t-1}} \)

\[ = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} - 1 \]

Alternatively, define inflation as

\[ \pi_t = \%\Delta CPI_t = \frac{CPI_t - CPI_{t-1}}{CPI_{t-1}} \]

Then

\[ R_t^{\text{Real}} = \frac{1 + R_t}{1 + \pi_t} - 1 \]
**Example.** *Compute real return on Microsoft stock.*

Suppose the CPI in months $t - 1$ and $t$ is 1 and 1.01, respectively, representing a 1% monthly growth rate in the overall price level. The real prices of Microsoft stock are

$$P_{t-1}^{\text{Real}} = \frac{85}{1} = 85, \quad P_t^{\text{Real}} = \frac{90}{1.01} = 89.1089$$

The real monthly return is

$$R_t^{\text{Real}} = \frac{89.10891 - 85}{85} = 0.0483$$
The nominal return and inflation over the month are

\[ R_t = \frac{\$90 - \$85}{\$85} = 0.0588, \quad \pi_t = \frac{1.01 - 1}{1} = 0.01 \]

Then the real return is

\[ R_t^{\text{Real}} = \frac{1.0588}{1.01} - 1 = 0.0483 \]

Notice that simple real return is almost, but not quite, equal to the simple nominal return minus the inflation rate

\[ R_t^{\text{Real}} \approx R_t - \pi_t = 0.0588 - 0.01 = 0.0488 \]
Annualizing Returns

Returns are often converted to an annual return to establish a standard for comparison

**Example:** Assume same monthly return $R_m$ for 12 months:

Compound annual gross return (CAGR) $= 1 + R_A = 1 + R_t(12) = (1 + R_m)^{12}$

Compound annual net return $= R_A = (1 + R_m)^{12} - 1$

Note: We don’t use $R_A = 12R_m$ because this ignores compounding.
Example. *Annualized return on Microsoft*

Suppose the one-month return, $R_t$, on Microsoft stock is 5.88%. If we assume that we can get this return for 12 months then the compounded annualized return is

$$R_A = (1.0588)^{12} - 1 = 1.9850 - 1 = 0.9850$$

or 98.50% per year. Pretty good!
Average Returns

For investments over a given horizon, it is often of interest to compute a measure of average return over the horizon.

Consider a sequence of monthly investments over the year with monthly returns

\[ R_1, R_2, \ldots, R_{12} \]

The annual return is

\[ R_A = R(12) = (1 + R_1)(1 + R_2) \cdots (1 + R_{12}) - 1 \]

Q: What is the average monthly return?
Two possibilities

1. Arithmetic average (can be misleading)

$$\bar{R} = \frac{1}{12}(R_1 + \cdots + R_{12})$$

2. Geometric average (better measure of average return)

$$(1 + \bar{R})^{12} = (1 + R_A) = (1 + R_1)(1 + R_2) \cdots (1 + R_{12})$$

$$\Rightarrow \bar{R} = (1 + R_A)^{1/12} - 1$$

$$= [(1 + R_1)(1 + R_2) \cdots (1 + R_{12})]^{1/12} - 1$$
Example: Consider a two period investment with returns

\[ R_1 = 0.5, \quad R_2 = -0.5 \]

$1$ invested over two periods grows to

\[ FV = \$1 \times (1 + R_1)(1 + R_2) = (1.5)(0.5) = 0.75 \]

for a 2-period return of

\[ R(2) = 0.75 - 1 = -0.25 \]

Hence, the 2-period investment loses 25%
The arithmetic average return is

\[ \bar{R} = \frac{1}{2}(0.5 + -0.5) = 0 \]

This is misleading because the actual investment lost money over the 2 period horizon. The compound 2-period return based on the arithmetic average is

\[ (1 + \bar{R})^2 - 1 = 1^2 - 1 = 0 \]

The geometric average is

\[ [(1.5)(0.5)]^{1/2} - 1 = (0.75)^{1/2} - 1 = -0.1340 \]

This is a better measure because it indicates that the investment eventually lost money. The compound 2-period return is

\[ (1 + \bar{R})^2 - 1 = (0.867)^2 - 1 = -0.25 \]
**Contnuously Compounded (cc) Returns**

\[ r_t = \ln(1 + R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) \]

\( \ln(\cdot) = \text{natural log function} \)

Note:

\[ \ln(1 + R_t) = r_t : \text{given } R_t \text{ we can solve for } r_t \]

\[ R_t = e^{r_t} - 1 : \text{given } r_t \text{ we can solve for } R_t \]

\( r_t \) is always smaller than \( R_t \)
Digression on natural log and exponential functions

- \( \ln(0) = -\infty, \ln(1) = 0 \)

- \( e^{-\infty} = 0, \ e^0 = 1, \ e^1 = 2.7183 \)

- \( \frac{d}{dx} \ln(x) = \frac{1}{x}, \ \frac{d}{dx} e^x = e^x \)

- \( \ln(e^x) = x, \ e^{\ln(x)} = x \)

- \( \ln(x \cdot y) = \ln(x) + \ln(y); \ \ln(\frac{x}{y}) = \ln(x) - \ln(y) \)
\begin{itemize}
  \item $\ln(x^y) = y \ln(x)$
  
  \item $e^x e^y = e^{x+y}$, $e^x e^{-y} = e^{x-y}$
  
  \item $(e^x)^y = e^{xy}$
\end{itemize}
Intuition

\[ e^{r_t} = e^{\ln(1+R_t)} = e^{\ln(P_t/P_{t-1})} \]

\[ = \frac{P_t}{P_{t-1}} \]

\[ \implies P_{t-1} \cdot e^{r_t} = P_t \]

\[ \implies r_t = \text{cc growth rate in prices between months } t - 1 \text{ and } t \]
**Result.** If \( R_t \) is small then

\[
r_t = \ln(1 + R_t) \approx R_t
\]

**Proof.** For a function \( f(x) \), a first order Taylor series expansion about \( x = x_0 \) is

\[
f(x) = f(x_0) + \frac{d}{dx} f(x_0)(x - x_0) + \text{remainder}
\]

Let \( f(x) = \ln(1 + x) \) and \( x_0 = 0 \). Note that

\[
\frac{d}{dx} \ln(1 + x) = \frac{1}{1 + x}, \quad \frac{d}{dx} \ln(1 + x_0) = 1
\]

Then

\[
\ln(1 + x) \approx \ln(1) + 1 \cdot x = 0 + x = x
\]
Computational Trick

\[ r_t = \ln \left( \frac{P_t}{P_{t-1}} \right) = \ln(P_t) - \ln(P_{t-1}) = p_t - p_{t-1} = \text{difference in log prices} \]

where

\[ p_t = \ln(P_t) \]
Example. *Compute cc return*

Let $P_{t-1} = 85$, $P_t = 90$ and $R_t = 0.0588$. Then the cc monthly return can be computed in two ways:

$$
\begin{align*}
    r_t &= \ln(1.0588) = 0.0571 \\
    r_t &= \ln(90) - \ln(85) = 4.4998 - 4.4427 = 0.0571.
\end{align*}
$$

Notice that $r_t$ is slightly smaller than $R_t$. 
Multi-period Returns

\[ r_t(2) = \ln(1 + R_t(2)) \]

\[ = \ln \left( \frac{P_t}{P_{t-2}} \right) \]

\[ = p_t - p_{t-2} \]

Note that

\[ e^{r_t(2)} = e^{\ln(P_t/P_{t-2})} \]

\[ \Rightarrow P_{t-2} e^{r_t(2)} = P_t \]

\[ \implies r_t(2) = \text{cc growth rate in prices between months } t - 2 \text{ and } t \]
**Result:** cc returns are additive

\[ r_t(2) = \ln \left( \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \right) \]

\[ = \ln \left( \frac{P_t}{P_{t-1}} \right) + \ln \left( \frac{P_{t-1}}{P_{t-2}} \right) \]

\[ = r_t + r_{t-1} \]

where \( r_t = \) cc return between months \( t - 1 \) and \( t \), \( r_{t-1} = \) cc return between months \( t - 2 \) and \( t - 1 \)
Example. Compute cc two-month return

Suppose \( P_{t-2} = 80 \), \( P_{t-1} = 85 \) and \( P_t = 90 \). The cc two-month return can be computed in two equivalent ways: (1) take difference in log prices

\[
    r_{t}(2) = \ln(90) - \ln(80) = 4.4998 - 4.3820 = 0.1178.
\]

(2) sum the two cc one-month returns

\[
    r_t = \ln(90) - \ln(85) = 0.0571
\]
\[
    r_{t-1} = \ln(85) - \ln(80) = 0.0607
\]
\[
    r_t(2) = 0.0571 + 0.0607 = 0.1178.
\]

Notice that \( r_t(2) = 0.1178 < R_t(2) = 0.1250 \).
General Result

\[ r_t(k) = \ln(1 + R_t(k)) = \ln\left(\frac{P_t}{P_{t-k}}\right) \]

\[ = \sum_{j=0}^{k-1} r_{t-j} \]

\[ = r_t + r_{t-1} + \cdots + r_{t-k+1} \]
Portfolio Returns

\[ R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t} \]

\[ r_{p,t} = \ln(1 + R_{p,t}) = \ln(1 + \sum_{i=1}^{n} x_i R_{i,t}) \neq \sum_{i=1}^{n} x_i r_{i,t} \]

\[ \Rightarrow \text{portfolio returns are not additive} \]

Note: If \( R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t} \) is not too large, then \( r_{p,t} \approx R_{p,t} \) otherwise, \( R_{p,t} > r_{p,t} \).
Example. Compute cc return on portfolio

Consider a portfolio of Microsoft and Starbucks stock with

\[ x_{msft} = 0.25, \ x_{sbux} = 0.75, \]

\[ R_{msft,t} = 0.0588, \ R_{sbux,t} = -0.0503 \]

\[ R_{p,t} = x_{msft} R_{msft,t} + x_{sbux,t} R_{sbux,t} = -0.02302 \]

The cc portfolio return is

\[ r_{p,t} = \ln(1 - 0.02302) = \ln(0.977) = -0.02329 \]

Note

\[ r_{msft,t} = \ln(1 + 0.0588) = 0.05714 \]

\[ r_{sbux,t} = \ln(1 - 0.0503) = -0.05161 \]

\[ x_{msft} r_{msft} + x_{sbux} r_{sbux} = -0.02442 \neq r_{p,t} \]
Adjusting for Inflation

The cc one period real return is

\[ r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}}) \]

\[ 1 + R_t^{\text{Real}} = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} \]

It follows that

\[ r_t^{\text{Real}} = \ln \left( \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} \right) = \ln \left( \frac{P_t}{P_{t-1}} \right) + \ln \left( \frac{CPI_{t-1}}{CPI_t} \right) \]

\[ = \ln(P_t) - \ln(P_{t-1}) + \ln(CPI_{t-1}) - \ln(CPI_t) = r_t - \pi_t^{cc} \]

where

\[ r_t = \ln(P_t) - \ln(P_{t-1}) = \text{nominal cc return} \]

\[ \pi_t^{cc} = \ln(CPI_t) - \ln(CPI_{t-1}) = \text{cc inflation} \]
Example. Compute cc real return

Suppose:

\[ R_t = 0.0588 \]
\[ \pi_t = 0.01 \]
\[ R_t^{\text{Real}} = 0.0483 \]

The real cc return is

\[ r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}}) = \ln(1.0483) = 0.047. \]

Equivalently,

\[ r_t^{\text{Real}} = r_t - \pi_t^{cc} = \ln(1.0588) - \ln(1.01) = 0.047 \]