

Bivariate Probability Distribution

Example - Two discrete rv's X and Y

		Bivariate pdf			
		Y			
		%	0	1	$\Pr(X)$
X	0	1/8	0	1/8	
	1	2/8	1/8	3/8	
	2	1/8	2/8	3/8	
	3	0	1/8	1/8	
$\Pr(Y)$		4/8	4/8	1	

$p(x, y) = \Pr(X = x, Y = y) = \text{values in table}$
e.g., $p(0, 0) = \Pr(X = 0, Y = 0) = 1/8$

Properties of joint pdf $p(x, y)$

$$S_{XY} = \{(0, 0), (0, 1), (1, 0), (1, 1), \\ (2, 0), (2, 1), (3, 0), (3, 1)\}$$

$$p(x, y) \geq 0 \text{ for } x, y \in S_{XY}$$

$$\sum_{x, y \in S_{XY}} p(x, y) = 1$$

Marginal pdfs

$$\begin{aligned} p(x) &= \Pr(X = x) = \sum_{y \in S_Y} p(x, y) \\ &= \text{sum over columns in joint table} \end{aligned}$$

$$\begin{aligned} p(y) &= \Pr(Y = y) = \sum_{x \in S_X} p(x, y) \\ &= \text{sum over rows in joint table} \end{aligned}$$

Conditional Probability

Suppose we know $Y = 0$. How does this knowledge affect the probability that $X = 0, 1, 2$, or 3 ? The answer involves conditional probability.

Example

$$\begin{aligned}\Pr(X = 0|Y = 0) &= \frac{\Pr(X = 0, Y = 0)}{\Pr(Y = 0)} \\ &= \frac{\text{joint probability}}{\text{marginal probability}} = \frac{1/8}{4/8} = 1/4\end{aligned}$$

Remark

$$\begin{aligned}\Pr(X = 0|Y = 0) &= 1/4 \neq \Pr(X = 0) = 1/8 \\ \implies &X \text{ depends on } Y\end{aligned}$$

The marginal probability, $\Pr(X = 0)$, ignores information about Y .

Definition - Conditional Probability

- The conditional pdf of X given $Y = y$ is, for all $x \in S_X$,

$$p(x|y) = \Pr(X = x|Y = y) = \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)}$$

- The conditional pdf of Y given $X = x$ is, for all values of $y \in S_Y$

$$p(y|x) = \Pr(Y = y|X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)}$$

Independence

Let X and Y be discrete rvs with pdfs $p(x)$, $p(y)$, sample spaces S_X , S_Y and joint pdf $p(x, y)$. Then X and Y are independent rv's if and only if

$$p(x, y) = p(x) \cdot p(y)$$

for all values of $x \in S_X$ and $y \in S_Y$

Result: If X and Y are independent rv's, then

$$p(x|y) = p(x) \text{ for all } x \in S_X, y \in S_Y$$
$$p(y|x) = p(y) \text{ for all } x \in S_X, y \in S_Y$$

Intuition

Knowledge of X does not influence probabilities associated with Y

Knowledge of Y does not influence probabilities associated with X

Bivariate Distributions - Continuous rv's

The joint pdf of X and Y is a non-negative function $f(x, y)$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Let $[x_1, x_2]$ and $[y_1, y_2]$ be intervals on the real line. Then

$$\begin{aligned} \Pr(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) \\ &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dx dy \\ &= \text{volume under probability surface} \\ &\quad \text{over the intersection of the intervals} \\ &\quad [x_1, x_2] \text{ and } [y_1, y_2] \end{aligned}$$

Example: Bivariate standard normal distribution

Let $X \sim N(0, 1)$, $Y \sim N(0, 1)$ and let X and Y be independent. Then

$$\begin{aligned} f(x, y) &= f(x)f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \\ &= \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}. \end{aligned}$$

To find $\Pr(-1 < X < 1, -1 < Y < 1)$ we must solve

$$\int_{-1}^1 \int_{-1}^1 \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

which, unfortunately, does not have an analytical solution. Numerical approximation methods are required to evaluate the above integral. See R package `mvtnorm`.

Covariance and Correlation - Measuring linear dependence between two rv's

Covariance: Measures direction but not strength of linear relationship between 2 rv's

$$\begin{aligned}\sigma_{XY} &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \sum_{x,y \in S_{XY}} (x - \mu_X)(y - \mu_Y) \cdot p(x, y) \quad (\text{discrete}) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x, y) dx dy \quad (\text{cts})\end{aligned}$$

Example: For the data in Table 2, we have

$$\begin{aligned}\sigma_{XY} = \text{Cov}(X, Y) &= (0 - 3/2)(0 - 1/2) \cdot 1/8 \\ &\quad + (0 - 3/2)(1 - 1/2) \cdot 0 + \dots \\ &\quad + (3 - 3/2)(1 - 1/2) \cdot 1/8 = 1/4\end{aligned}$$

Properties of Covariance

$$\text{Cov}(X, Y) = \text{Cov}(Y, X)$$

$$\text{Cov}(aX, bY) = a \cdot b \cdot \text{Cov}(X, Y) = a \cdot b \cdot \sigma_{XY}$$

$$\text{Cov}(X, X) = \text{Var}(X)$$

$$X, Y \text{ independent} \implies \text{Cov}(X, Y) = 0$$

$$\text{Cov}(X, Y) = 0 \not\Rightarrow X \text{ and } Y \text{ are independent}$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Correlation: Measures direction and strength of linear relationship between 2 rv's

$$\begin{aligned}\rho_{XY} &= \text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{SD}(X) \cdot \text{SD}(Y)} \\ &= \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \text{scaled covariance}\end{aligned}$$

Example: For the Data in Table 2

$$\rho_{XY} = \text{Cor}(X, Y) = \frac{1/4}{\sqrt{(3/4) \cdot (1/2)}} = 0.577$$

Properties of Correlation

$$-1 \leq \rho_{XY} \leq 1$$

$$\rho_{XY} = 1 \text{ if } Y = aX + b \text{ and } a > 0$$

$$\rho_{XY} = -1 \text{ if } Y = aX + b \text{ and } a < 0$$

$$\rho_{XY} = 0 \text{ if and only if } \sigma_{XY} = 0$$

$$\rho_{XY} = 0 \not\Rightarrow X \text{ and } Y \text{ are independent in general}$$

$$\rho_{XY} = 0 \implies \text{independence if } X \text{ and } Y \text{ are normal}$$

Bivariate normal distribution

Let X and Y be distributed bivariate normal. The joint pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X} \right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right] \right\}$$

where $E[X] = \mu_X$, $E[Y] = \mu_Y$, $SD(X) = \sigma_X$, $SD(Y) = \sigma_Y$, and $\rho = \text{cor}(X, Y)$.

Linear Combination of 2 rv's

Let X and Y be rv's. Define a new rv Z that is a linear combination of X and Y :

$$Z = aX + bY$$

where a and b are constants. Then

$$\begin{aligned}\mu_Z &= E[Z] = E[aX + bY] \\ &= aE[X] + bE[Y] \\ &= a \cdot \mu_X + b \cdot \mu_Y\end{aligned}$$

and

$$\begin{aligned}\sigma_Z^2 &= \text{Var}(Z) = \text{Var}(a \cdot X + b \cdot Y) \\ &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2a \cdot b \cdot \text{Cov}(X, Y) \\ &= a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2a \cdot b \cdot \sigma_{XY}\end{aligned}$$

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then $Z \sim N(\mu_Z, \sigma_Z^2)$

Example: Portfolio returns

R_A = return on asset A with $E[R_A] = \mu_A$ and $\text{Var}(R_A) = \sigma_A^2$

R_B = return on asset B with $E[R_B] = \mu_B$ and $\text{Var}(R_B) = \sigma_B^2$

$\text{Cov}(R_A, R_B) = \sigma_{AB}$ and $\text{Cor}(R_A, R_B) = \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \cdot \sigma_B}$

Portfolio

x_A = share of wealth invested in asset A , x_B = share of wealth invested in asset B

$x_A + x_B = 1$ (exhaust all wealth in 2 assets)

$R_P = x_A \cdot R_A + x_B \cdot R_B$ = portfolio return

Portfolio Problem: How much wealth should be invested in assets A and B ?

Portfolio expected return (gain from investing)

$$\begin{aligned} E[R_P] &= \mu_P = E[x_A \cdot R_A + x_B \cdot R_B] \\ &= x_A E[R_A] + x_B E[R_B] \\ &= x_A \mu_A + x_B \mu_B \end{aligned}$$

Portfolio variance (risk from investing)

$$\begin{aligned} \text{Var}(R_P) &= \sigma_P^2 = \text{Var}(x_A R_A + x_B R_B) \\ &= x_A^2 \text{Var}(R_A) + x_B^2 \text{Var}(R_B) + \\ &\quad 2 \cdot x_A \cdot x_B \cdot \text{Cov}(R_A, R_B) \\ &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \\ \text{SD}(R_P) &= \sqrt{\text{Var}(R_P)} = \sigma_P \\ &= \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \right)^{1/2} \end{aligned}$$

Linear Combination of N rv's.

Let X_1, X_2, \dots, X_N be rvs and let a_1, a_2, \dots, a_N be constants. Define

$$Z = a_1X_1 + a_2X_2 + \dots + a_NX_N = \sum_{i=1}^N a_iX_i$$

Then

$$\begin{aligned}\mu_Z &= E[Z] = a_1E[X_1] + a_2E[X_2] + \dots + a_NE[X_N] \\ &= \sum_{i=1}^N a_iE[X_i] = \sum_{i=1}^N a_i\mu_i\end{aligned}$$

For the variance,

$$\begin{aligned}\sigma_Z^2 = \text{Var}(Z) &= a_1^2 \text{Var}(X_1) + \cdots + a_N^2 \text{Var}(X_N) \\ &+ 2a_1a_2 \text{Cov}(X_1, X_2) + 2a_1a_3 \text{Cov}(X_1, X_3) + \cdots \\ &+ 2a_2a_3 \text{Cov}(X_2, X_3) + 2a_2a_4 \text{Cov}(X_2, X_4) + \cdots \\ &+ 2a_{N-1}a_N \text{Cov}(X_{N-1}, X_N)\end{aligned}$$

Note: N variance terms and $N(N - 1) = N^2 - N$ covariance terms. If $N = 100$, there are $100 \times 99 = 9900$ covariance terms!

Result: If X_1, X_2, \dots, X_N are each normally distributed random variables then

$$Z = \sum_{i=1}^N a_i X_i \sim N(\mu_Z, \sigma_Z^2)$$

Example: Portfolio variance with three assets

R_A, R_B, R_C are simple returns on assets A, B and C

x_A, x_B, x_C are portfolio shares such that $x_A + x_B + x_C = 1$

$$R_p = x_A R_A + x_B R_B + x_C R_C$$

Portfolio variance

$$\begin{aligned} \sigma_P^2 = & x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 \\ & + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} \end{aligned}$$

Note: Portfolio variance calculation may be simplified using matrix layout

$$\begin{array}{cccc} & x_A & x_B & x_C \\ x_A & \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ x_B & \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ x_C & \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{array}$$

Example: Multi-period continuously compounded returns

$$r_t = \ln(1 + R_t) = \text{monthly cc return}$$

$$r_t \sim N(\mu, \sigma^2) \quad \text{for all } t$$

$$\text{Cov}(r_t, r_s) = 0 \quad \text{for all } t \neq s$$

Annual return

$$\begin{aligned} r_t(12) &= \sum_{j=0}^{11} r_{t-j} \\ &= r_t + r_{t-1} + \cdots + r_{t-11} \end{aligned}$$

Then

$$\begin{aligned} E[R_t(12)] &= \sum_{j=0}^{11} E[R_{t-j}] \\ &= \sum_{j=0}^{11} \mu \quad (E[R_t] = \mu \text{ for all } t) \\ &= 12\mu \quad (\mu = \text{mean of monthly return}) \end{aligned}$$

$$\begin{aligned}\text{Var}(r_t(12)) &= \text{Var}\left(\sum_{j=0}^{11} r_{t-j}\right) \\ &= \sum_{j=0}^{11} \text{Var}(r_{t-j}) = \sum_{j=0}^{11} \sigma^2 \\ &= 12 \cdot \sigma^2 \quad (\sigma^2 = \text{monthly variance}) \\ \text{SD}(r_t(12)) &= \sqrt{12} \cdot \sigma\end{aligned}$$

Then

$$r_t(12) \sim N(12\mu, 12\sigma^2)$$

For example, suppose

$$r_t \sim N(0.01, (0.10)^2)$$

Then

$$\begin{aligned} E[r_t(12)] &= 12 \times (0.01) = 0.12 \\ \text{Var}(r_t(12)) &= 12 \times (0.10)^2 = 0.12 \\ \text{SD}(r_t(12)) &= \sqrt{0.12} = 0.346 \\ r_t(12) &\sim N(0.12, (0.346)^2) \end{aligned}$$