Introduction to Computational Finance and Financial Econometrics
Probability Theory Review: Part 1

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Univariate Random Variables

Defnition: A random variable (rv) X is a variable that can take on a given set of values, called the sample space S_X, where the likelihood of the values in S_X is determined by the variable's probability distribution function (pdf).

Examples

- $X =$ price of microsoft stock next month. $S_X = \{\mathbb{R} : 0 < X \leq M\}$
- $X =$ simple return on a one month investment. $S_X = \{\mathbb{R} : -1 \leq X < M\}$
- $X = 1$ if stock price goes up; $X = 0$ if stock price goes down. $S_X = \{0, 1\}$

Discrete Random Variables

Definition: A discrete rv X is one that can take on a finite number of n different values $x_1, \cdots, x_n$

Definition: The pdf of a discrete rv X, $p(x)$, is a function such that $p(x) = \Pr(X = x)$. The pdf must satisfy

1. $p(x) \geqslant 0$ for all $x \in S_X$; $p(x) = 0$ for all $x \notin S_X$
2. $\sum_{x \in S_X} p(x) = 1$
3. $p(x) \leqslant 1$ for all $x \in S_X$

State of Economy	$S_X =$ Sample Space	$p(x) = \Pr(X = x)$
Depression	-0.30	0.05
Recession	0.0	0.20
Normal	0.10	0.50
Mild Boom	0.20	0.20
Major Boom	0.50	0.05

Table 1: Discrete Distribution for Annual Return

Example: Probability Distribution for Annual Return on Microsoft

Example: Bernouli Distribution

Consider two mutually exclusive events generically called “success” and “failure”.

Let $X = 1$ if success occurs and let $X = 0$ if failure occurs.

Let $\Pr(X = 1) = \pi$, where $0 < \pi < 1$, denote the probability of success. Then $\Pr(X = 0) = 1 - \pi$ is the probability of failure. A mathematical model describing this distribution is

$$p(x) = \Pr(X = x) = \pi^x(1 - \pi)^{1-x},\ x = 0, 1.$$

When $x = 0$, $p(0) = \pi^0(1 - \pi)^{1-0} = 1 - \pi$ and when $x = 1, p(1) = \pi^1(1 - \pi)^{1-1} = \pi$.

Continuous Random Variables

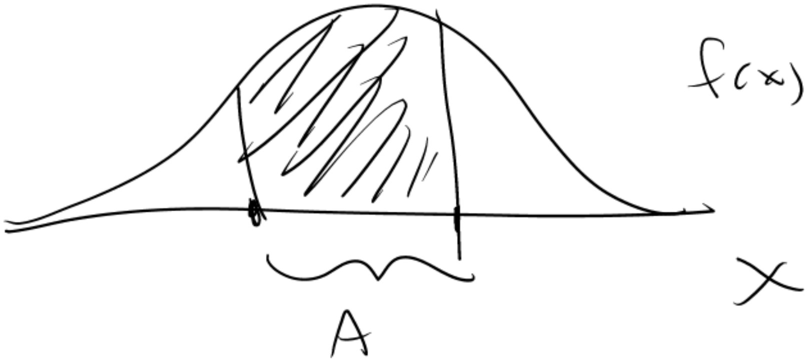
Definition: A continuous rv X is one that can take on any real value

Definition: The pdf of a continuous rv X is a nonnegative function $f(x)$ such that for any interval A on the real line

$$\Pr(X \in A) = \int_A f(x)dx$$

$\Pr(X \in A) =$ "Area under probability curve over the interval A".

The pdf $f(x)$ must satisfy



1. $f(x) \geqslant 0$; $\int_{-\infty}^{\infty} f(x)dx = 1$

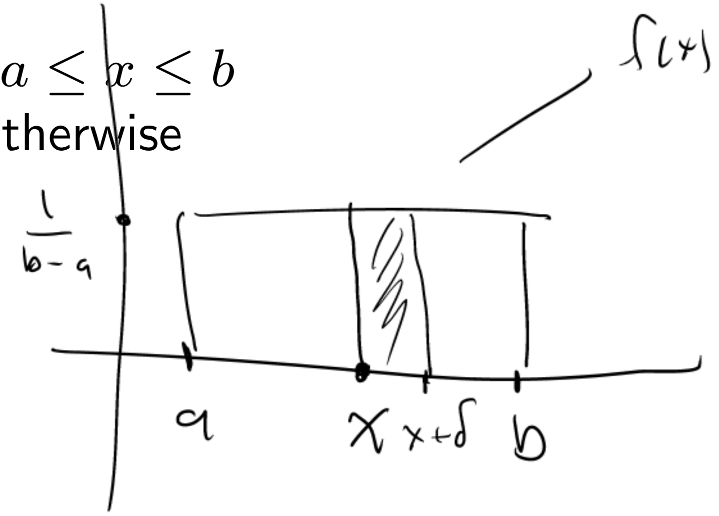
$$\Pr(X = x) = 0$$

$$\Pr(X > x) = \Pr(X \geq x)$$

Example: Uniform distribution over $[a, b]$

Let $X \backsim U[a, b]$, where "$\backsim$" means "is distributed as". Then

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Properties:

$f(x) \geq 0$, provided $b > a$, and

$$\int_{-\infty}^{\infty} f(x)dx = \int_a^b \frac{1}{b-a}dx = \frac{1}{b-a}\int_a^b dx$$
$$= \frac{1}{b-a}[x]_a^b = \frac{b-a}{b-a} = 1$$

$$\Pr(X = x) = \lim_{\delta \to 0} \Pr(x \leq X \leq x + \delta) = \int_x^{x+\delta} f(x)\,dx$$
$$= 0$$

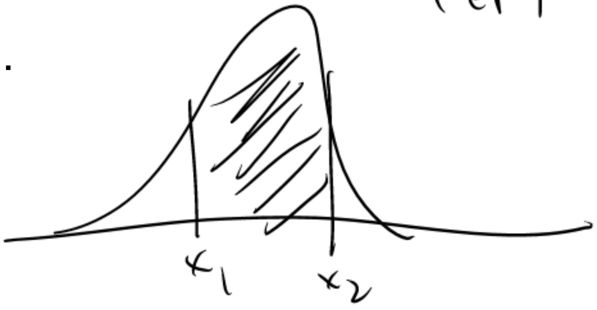
The Cumulative Distribution Function (CDF)

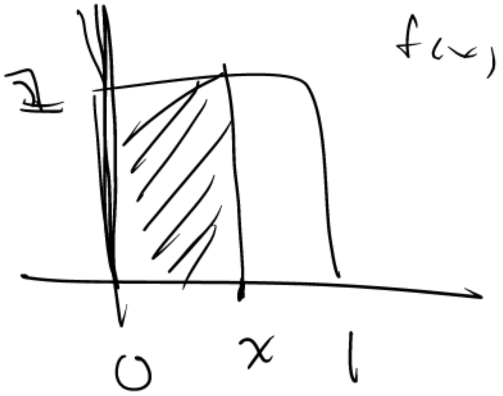
Definition The CDF, F, of a rv X is $F(x) = \text{Pr}(X \leq x)$ and

- If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$
- $F(-\infty) = 0$ and $F(\infty) = 1$
- $\text{Pr}(X \geq x) = 1 - F(x)$
- $\text{Pr}(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
- $\frac{d}{dx}F(x) = f(x)$ if X is a continuous rv.

$F_X(x) = Pr(X \leq x)$

= Area under curve to the left of x

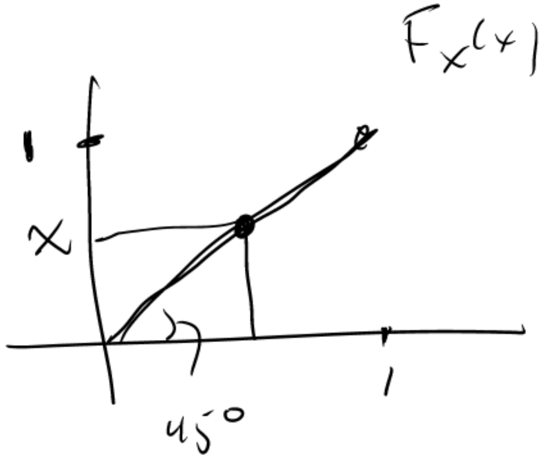




Example: Uniform distribution over $[0,1]$

$$X \backsim U\,[0,1]$$

$$f(x) = \begin{cases} \frac{1}{1-0} = 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Then

$$F(x) = \Pr(X \leq x) = \int_0^x dz$$
$$= [z]_0^x = x$$

and, for example,

$$\Pr(0 \leq X \leq 0.5) = F(0.5) - F(0)$$
$$= 0.5 - 0 = 0.5$$

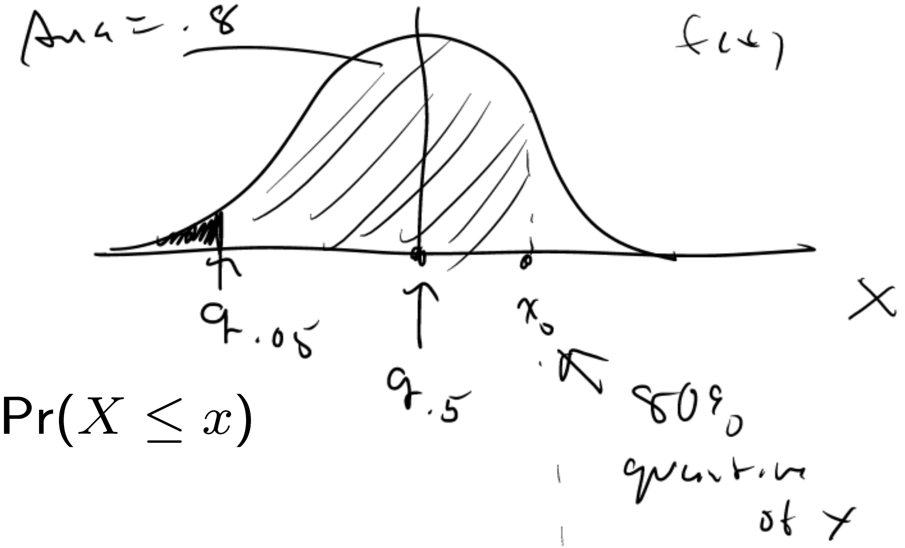
Note

$$\frac{d}{dx} F(x) = 1 = f(x)$$

Remark:

For a continuous rv

$$\mathsf{Pr}(X \leq x) = \mathsf{Pr}(X < x)$$
$$\mathsf{Pr}(X = x) = 0$$



Quantiles of a Distribution

X is a rv with continuous CDF $F_X(x) = \Pr(X \leq x)$

Definition: The $\alpha * 100\%$ quantile of F_X for $\alpha \in [0,1]$ is the value q_α such that

$$F_X(q_\alpha) = \Pr(X \leq q_\alpha) = \alpha$$

$$q_\alpha = F_X^{-1}\left(F_X(q_\alpha)\right) = F_X^{-1}(\alpha) = F_X^{-1}(\alpha)$$

The area under the probability curve to the left of q_α is α. If the inverse CDF F_X^{-1} exists then

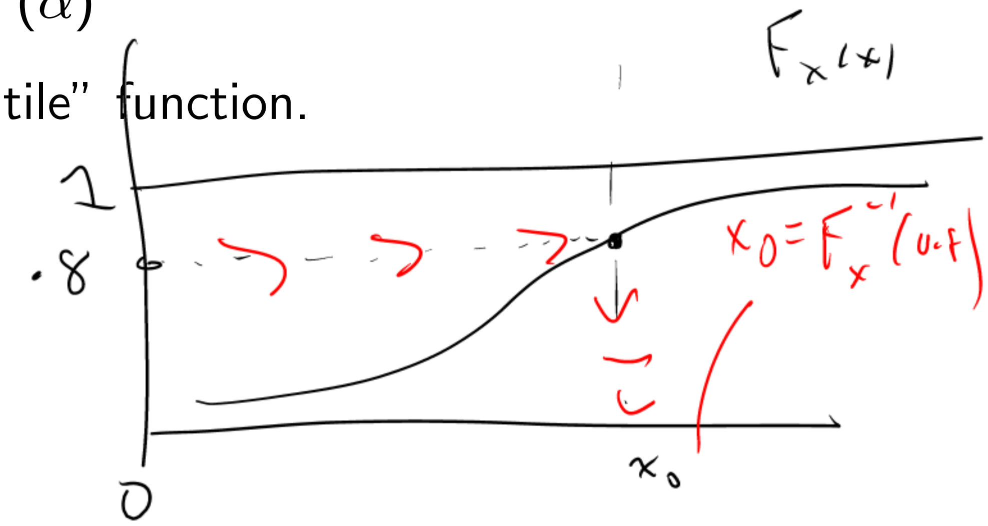
$$q_\alpha = F_X^{-1}(\alpha)$$

Note: F_X^{-1} is sometimes called the "quantile" function.

$f(x)$ inverse function $f^{-1}(\ )$

satisfies $f^{-1}(f(x)) = x$

$\ln(x)$, e^x $\quad e^{\ln(x)} = x$

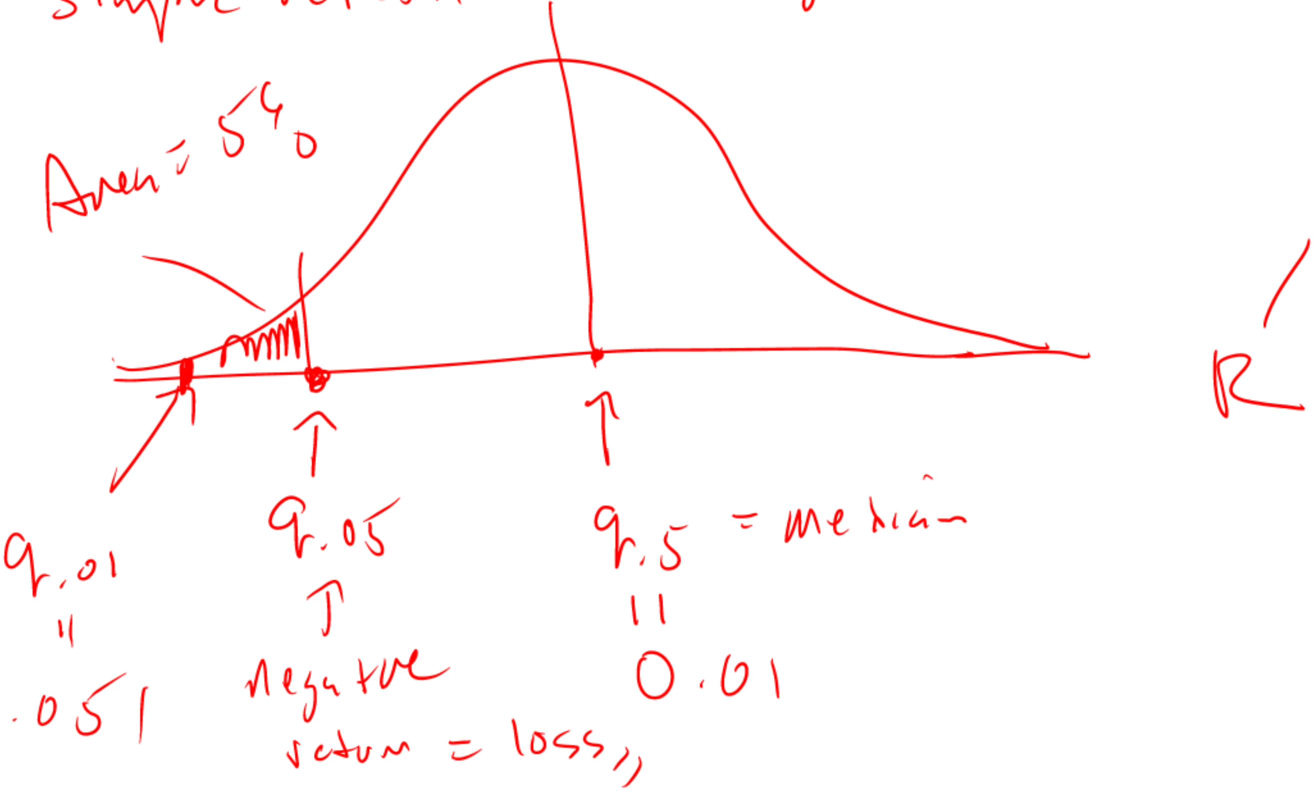


Example:

$$\begin{aligned} 1\% \text{ quantile} &= q_{.01} \\ 5\% \text{ quantile} &= q_{.05} \\ 50\% \text{ quantile} &= q_{.5} = \text{ median} \end{aligned}$$

Value at Risk

R = simple return = monthly return



Example: Quantile function of uniform distn on [0,1]

$$F_X(x) = x \Rightarrow q_\alpha = \alpha$$

$$q_{.01} = 0.01$$

$$q_{.5} = 0.5$$

The Standard Normal Distribution

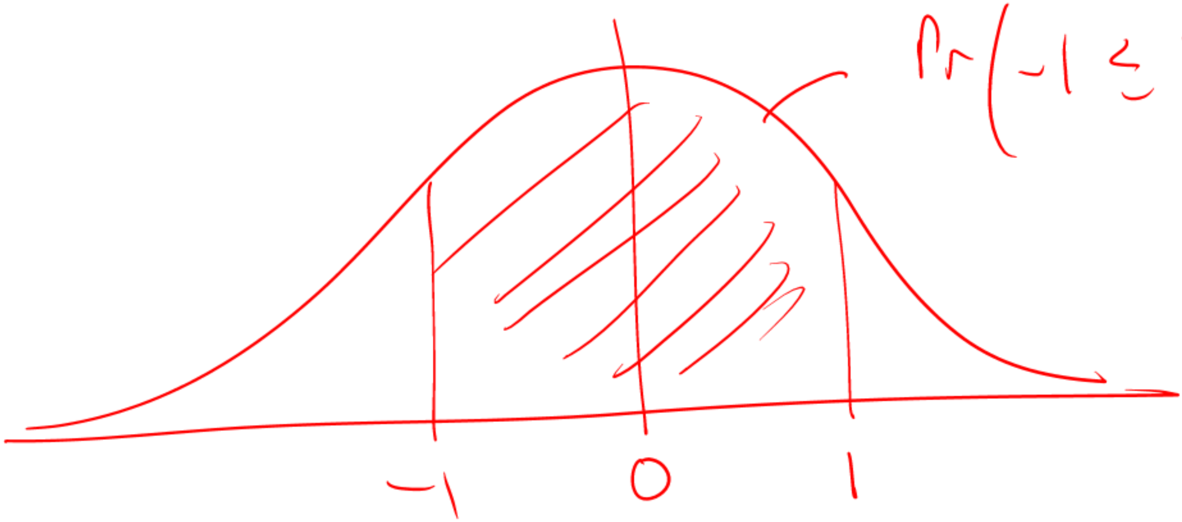
Let X be a rv such that $X \backsim N(0,1)$. Then

mean (center)

standard deviation (spread)

$$f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad -\infty \leq x \leq \infty$$

$$\Phi(x) = \Pr(X \leq x) = \int_{-\infty}^{x} \phi(z)dz$$



$\Pr(-1 \leq X \leq 1)$

$\approx .67$

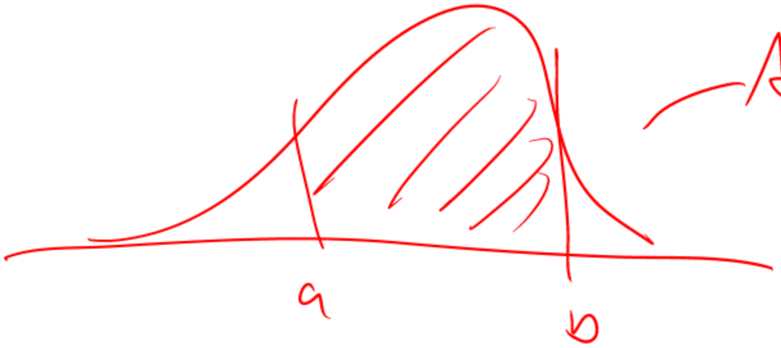
Shape Characteristics

- Centered at zero
- Symmetric about zero (same shape to left and right of zero)

$$\begin{aligned}
\Pr(-1 \leq x \leq 1) &= \Phi(1) - \Phi(-1) = 0.67 \\
\Pr(-2 \leq x \leq 2) &= \Phi(2) - \Phi(-2) \overset{\approx}{=} 0.95 \\
\Pr(-3 \leq x \leq 3) &= \Phi(3) - \Phi(-3) \overset{\approx}{=} 0.99
\end{aligned}$$

Finding Areas under the Normal Curve

- $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$, via change of variables formula in calculus
- $\Pr(a < X < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \Phi(b) - \Phi(a)$, cannot be computed analytically!
- Special numerical algorithms are used to calculate $\Phi(z)$



Area $= \Phi(b) - \Phi(a)$

Excel functions

1. `NORMSDIST` computes $\Pr(X \leq z) = \Phi(z)$ or $p(z) = \phi(z)$
2. `NORMSINV` computes the quantile $z_\alpha = \Phi^{-1}(\alpha)$

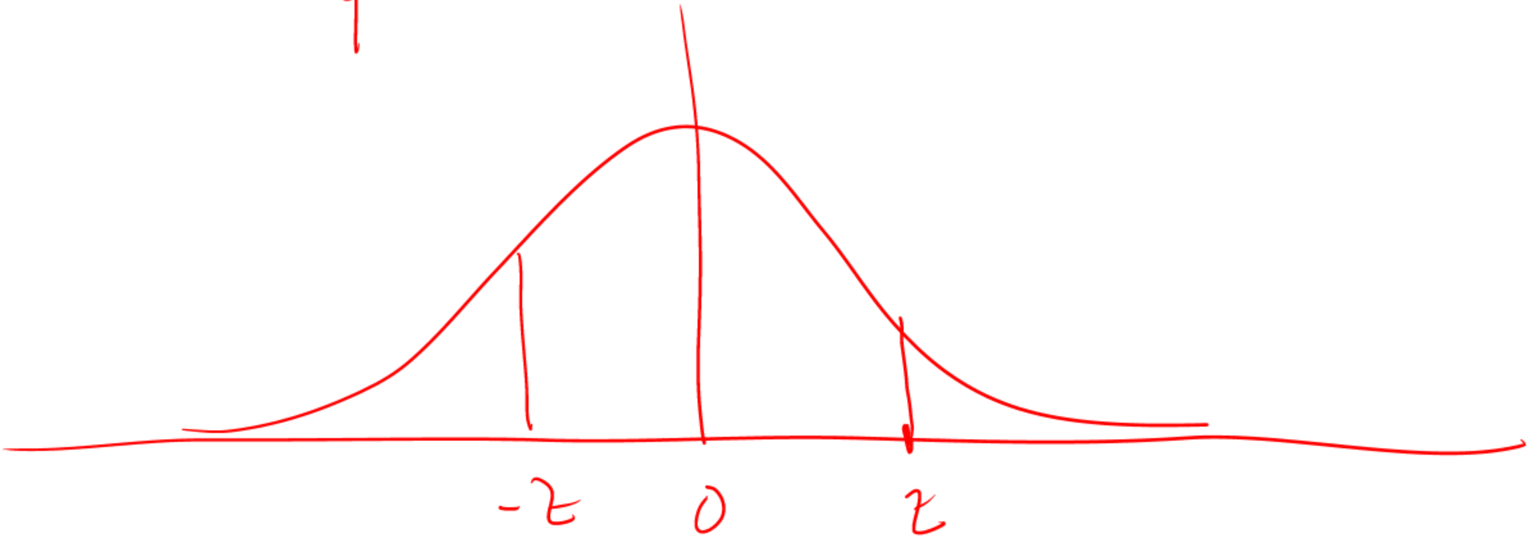
R functions

1. `pnorm` computes $\Pr(X \leq z) = \Phi(z)$
2. `qnorm` computes the quantile $z_\alpha = \Phi^{-1}(\alpha)$
3. `dnorm` computes the density $\phi(z)$

Some Tricks for Computing Area under Normal Curve

$N(0, 1)$ is symmeric about 0; total area $=1$

$$\Pr(X \le z) = 1 - \Pr(X \ge z)$$
$$\Pr(X \ge z) = \Pr(X \le -z)$$
$$\Pr(X \ge 0) = \Pr(X \le 0) = 0.5$$



Example In Excel use

$$\begin{aligned}\text{Pr}(-1 \le X \le 2) &= \text{Pr}(X \le 2) - \text{Pr}(X \le -1)\\ &= \texttt{NORMSDIST(2)} - \texttt{NORMSDIST(-1)}\\ &= 0.97725 - 0.15866 = 0.81860\end{aligned}$$

In R use

$$\texttt{pnorm(2)} - \texttt{pnorm(-1)} = 0.81860$$

The 1%, 2.5%, 5% quantiles are

$$\begin{aligned}\text{Excel:} z_{.01} &= \Phi^{-1}(0.01) = \texttt{NORMSINV(0.01)} = -2.33\\ \text{R} &: \texttt{qnorm(0.01)} = -2.33\\ \text{Excel:} z_{.025} &= \Phi^{-1}(0.025) = \texttt{NORMSINV(0.025)} = -1.96\\ \text{R} &: \texttt{qnorm(0.025)} = -1.96\\ \text{Excel:} z_{.05} &= \Phi^{-1}(.05) = \texttt{NORMSINV(.05)} = -1.645\\ \text{R} &: \texttt{qnorm(0.05)} = -1.645\end{aligned}$$

Shape Characteristics of pdfs

- Expected Value or Mean - Center of Mass
- Variance and Standard Deviation - Spread about mean
- Skewness - Symmetry about mean
- Kurtosis - Tail thickness

Expected Value - Discrete rv

$$E[X] = \mu_X = \sum_{x \in S_X} x \cdot p(x)$$
$$= \sum_{x \in S_X} x \cdot \mathrm{Pr}(X = x)$$

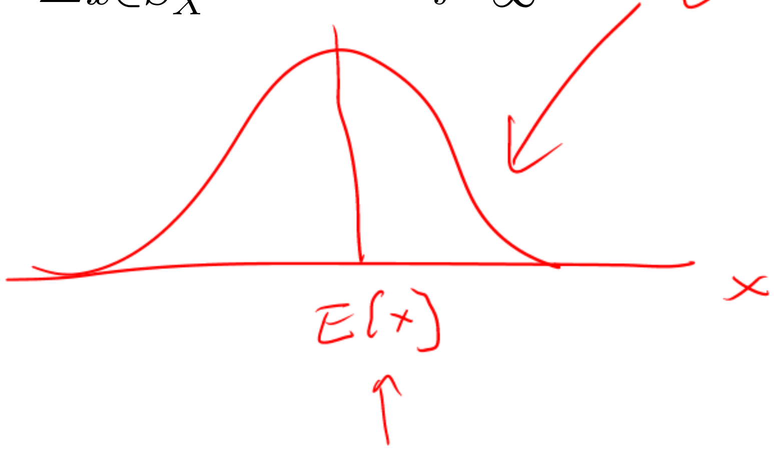
$E[X]$ = probability weighted average of possible values of X

Expected Value - Continuous rv

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Note: In continuous case, $\sum_{x \in S_X}$ becames $\int_{-\infty}^{\infty}$

$E[x]$ = "center of mass"



State of Economy	S_X = Sample Space	$p(x) = \Pr(X = x)$
Depression	-0.30	0.05
Recession	0.0	0.20
Normal	0.10	0.50
Mild Boom	0.20	0.20
Major Boom	0.50	0.05

Table 2: Discrete Distribution for Annual Return

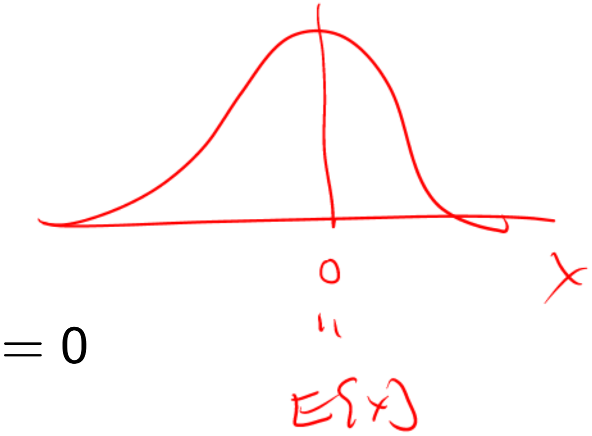
Expected value of discrete random variable

Using the discrete distribution for the return on Microsoft stock in the above table, the expected return is

$$\begin{aligned} E[X] &= (-0.3)\cdot(0.05) + (0.0)\cdot(0.20) + (0.1)\cdot(0.5) \\ &+ (0.2)\cdot(0.2) + (0.5)\cdot(0.05) \\ &= 0.10. \end{aligned}$$

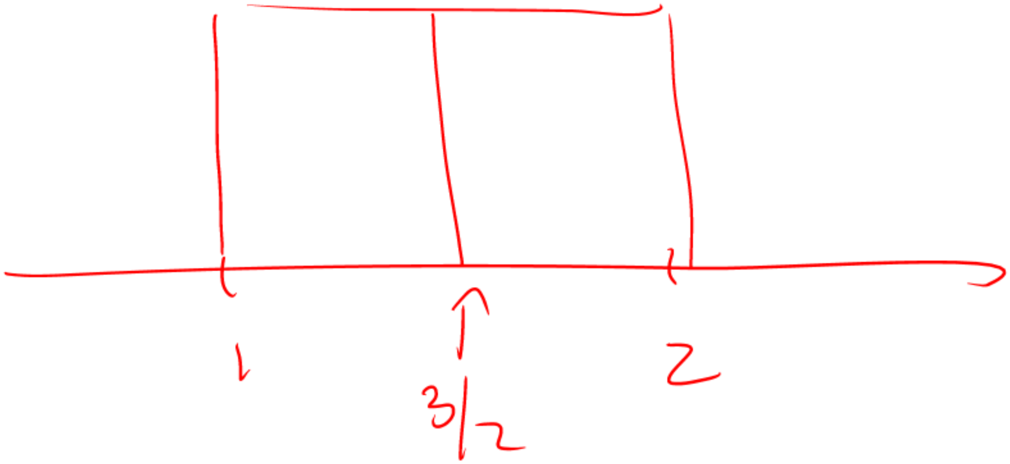
Example: $X \backsim U[1,2]$

$$E[X] = \int_1^2 x dx = \left[\frac{x^2}{2}\right]_1^2$$
$$= \frac{1}{2}[4-1] = \frac{3}{2}$$



Example: $X \backsim N(0,1)$

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0$$



Expectation of a Function of X

Definition: Let $g(X)$ be some function of the rv X. Then

$$E[g(X)] = \sum_{x \in S_X} g(x) \cdot p(x) \text{ Discrete case}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx \text{ Continuous case}$$

$X \sim$ cts with pdf $f(x)$

$g(x) = x^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

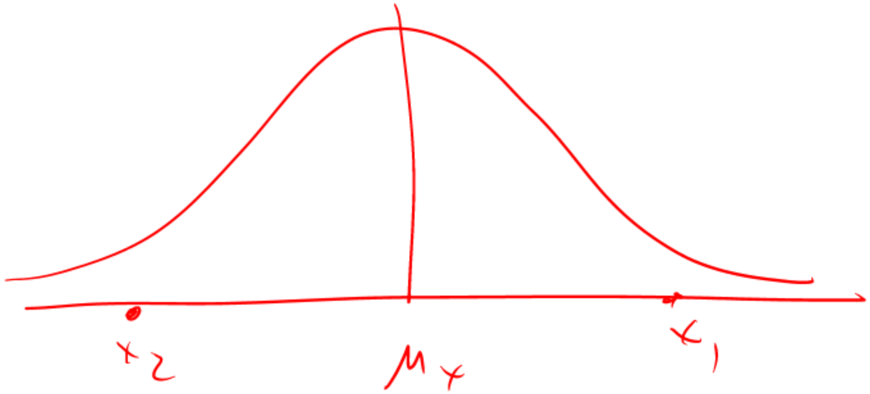
Variance and Standard Deviation

$$g(X) = (X - E[X])^2 = (X - \mu_X)^2$$

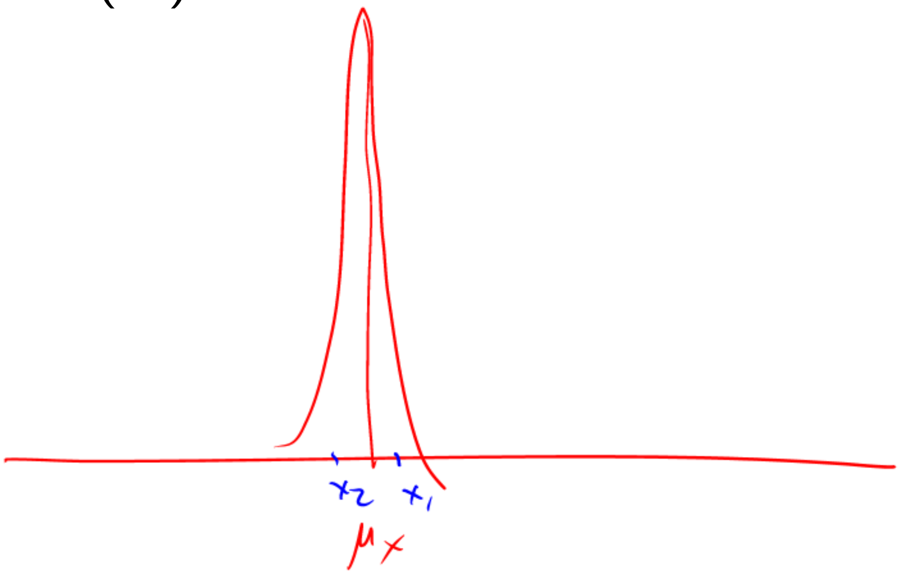
$$\mathrm{Var}(X) = \sigma_X^2 = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$$

$$\mathrm{SD}(X) = \sigma_X = \sqrt{\mathrm{Var}(X)}$$

Note: Var(X) is in squared units of X, and SD(X) is in the same units as X. Therefore, SD(X) is easier to interpret.



Big Var(X) and big SD(X)



small Var(X)
small SD(X)

Computation of Var(X) and SD(X)

$$\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] \\
&= \sum_{x \in S_X} (x-\mu_X)^2 \cdot p(x) \text{ if } X \text{ is a discrete rv} \\
&= \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)dx \text{ if } X \text{ is a continuous rv} \\
\sigma_X &= \sqrt{\sigma_X^2}
\end{aligned}$$

Remark: For "bell-shaped" data, σ_X measures the size of the typical deviation from the mean value μ_X.

Example: Variance and standard deviation for a discrete random variable

Using the discrete distribution for the return on Microsoft stock in Table 1 and the result that $\mu_X = 0.1$, we have

$$\begin{aligned}\mathrm{Var}(X) &= (-0.3-0.1)^2 \cdot (0.05) + (0.0-0.1)^2 \cdot (0.20) \\ &+ (0.1-0.1)^2 \cdot (0.5) + (0.2-0.1)^2 \cdot (0.2) \\ &+ (0.5-0.1)^2 \cdot (0.05) \\ &= 0.020 \\ \mathrm{SD}(X) &= \sigma_X = \sqrt{0.020} = 0.141.\end{aligned}$$

Given that the distribution is fairly bell-shaped we can say that typical values deviate from the mean value of 0.10 by about 0.141

$$\mu \pm \sigma = 0.10 \pm 0.141 = [-0.041,\ 0.241]$$

Example: $X \backsim N(0,1)$.

$$\begin{aligned}
\mu_X &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0 \\
\sigma_X^2 &= \int_{-\infty}^{\infty} (x-0)^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1 \\
\sigma_X &= \sqrt{1} = 1
\end{aligned}$$

$\Rightarrow$ size of typical deviation from $\mu_X = 0$ is $\sigma_X = 1$

The General Normal Distribution

$$
\begin{aligned}
X &\backsim N(\mu_X,\ \sigma_X^2) \\
f(x) &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right),\ -\infty \leq x \leq \infty \\
E[X] &= \mu_X = \text{ mean value} \\
\text{Var}(X) &= \sigma_X^2 = \text{ variance} \\
\text{SD}(X) &= \sigma_X = \text{ standard deviation}
\end{aligned}
$$

Shape Characteristics

- Centered at μ_X
- Symmetric about μ_X

$$\Pr(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X) = 0.67$$
$$\Pr(\mu_X - 2 \cdot \sigma_X \leq X \leq \mu_X + 2 \cdot \sigma_X) = 0.95$$
$$\Pr(\mu_X - 3 \cdot \sigma_X \leq X \leq \mu_X + 3 \cdot \sigma_X) = 0.99$$

- Quantiles of the general normal distribution:

$$q_\alpha = \mu_X + \sigma_X \cdot \Phi^{-1}(\alpha) = \mu_X + \sigma_X \cdot z_\alpha$$

Remarks:

- $X \backsim N(0,1)$: Standard Normal $\Longrightarrow \mu_X = 0$ and $\sigma_X^2 = 1$
- The pdf of the general Normal is completely determined by values of μ_X and σ_X^2

Finding Areas under General Normal Curve

Excel Functions

- `NORMDIST`$(x, \mu_X, \sigma_X,$`cumulative`$)$. If `cumulative` = `true`: $\Pr(X \leq x)$ is computed; If `cumulative` = `false`, $f(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2}$ is computed

- `NORMINV`$(\alpha, \mu_x, \sigma_x)$ computes $q_\alpha = \mu_X + \sigma_X z_\alpha$

R Runctions

- simulate data: `rnorm(n, mean, sd)`
- compute CDF: `pnorm(q, mean, sd)`
- compute quantiles: `qnorm(p, mean, sd)`
- compute density: `dnorm(x, mean, sd)`

Standard Deviation as a Measure of Risk

$$R_A = \text{monthly return on asset A}$$
$$R_B = \text{monthly return on assetB}$$
$$R_A \backsim N(\mu_A, \sigma_A^2), R_B \backsim N(\mu_B, \sigma_B^2)$$

where

$$\mu_A = E[R_A] = \text{ expected monthly return on asset A}$$
$$\sigma_A = \text{SD}(R_A)$$
$$= \text{ std. deviation of monthly return on asset A}$$

Typically, if

$$\mu_A > \mu_B$$

then

$$\sigma_A > \sigma_B$$

Example: Why the normal distribution may not be appropriate for simple returns

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \text{simple return}$$

$$\text{Assume } R_t \sim N(0.05, (0.50)^2)$$

Note: $P_t \geq 0 \implies R_t \geq -1$. However, based on the assumed normal distribution

$$\Pr(R_t < -1) = \texttt{NORMDIST(-1,0.05,0.50,TRUE)} = 0.018$$
$$= \texttt{pnorm(-1, 0.05, 0.50)} = 0.018$$

This implies that there is a 1.8% chance that the asset price will be negative. This is why the normal distribution may not be appropriate for simple returns.

Example: The normal distribution is more appropriate for cc returns

$$\begin{aligned} r_t &= \ln(1 + R_t) = \text{cc return} \\ R_t &= e^{r_t} - 1 = \text{ simple return} \\ \text{Assume } r_t &\sim N(0.05, (0.50)^2) \end{aligned}$$

Unlike R_t, r_t can take on values less than -1. For example,

$$\begin{aligned} r_t = -2 &\implies R_t = e^{-2} - 1 = -0.865 \\ \Pr(r_t < -2) &= \Pr(R_t < -0.865) \\ &= \texttt{NORMDIST(-2,0.05,0.50,TRUE)} = 0.00002 \end{aligned}$$

The Log-Normal Distribution

$$\begin{aligned} X &\sim N(\mu_X, \sigma_X^2), \quad -\infty < X < \infty \\ Y = \exp(X) &\sim \text{lognormal}(\mu_X, \sigma_X^2), \; 0 < Y < \infty \\ E[Y] = \mu_Y &= \exp(\mu_X + \sigma_X^2/2) \\ \text{Var}(Y) = \sigma_Y^2 &= \exp(2\mu_X + \sigma_X^2)(\exp(\sigma_X^2) - 1) \end{aligned}$$

Example: log-normal distribution for simple returns

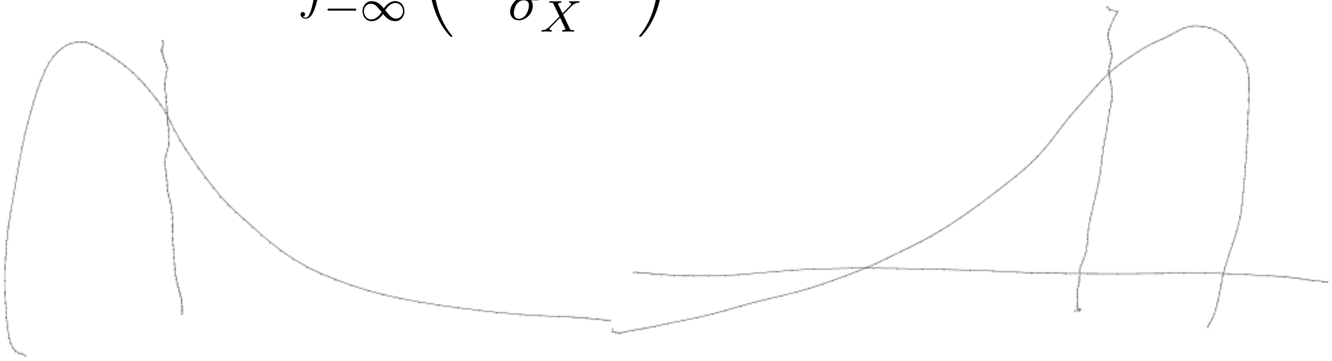
$$\begin{aligned} r_t &\sim N(0.05, (0.50)^2), \; r_t = \ln(1 + R_t) \\ \exp(r_t) = 1 + R_t &\sim \text{lognormal}(0.05, (0.50)^2) \\ \mu_{1+R} &= \exp(0.05 + (0.5)^2/2) = 1.191 \\ \sigma_{1+R}^2 &= \exp(2(0.05) + (0.5)^2)(\exp(0.5^2) - 1) = 0.563 \end{aligned}$$

R Runctions

- simulate data: `rlnorm(n, mean, sd)`
- compute CDF: `plnorm(q, mean, sd)`
- compute quantiles: `qlnorm(p, mean, sd)`
- compute density: `dlnorm(y, mean, sd)`

Skewness - Measure of symmetry

$$
\begin{aligned}
g(X) &= ((X - \mu_X)/\sigma_X)^3 \\
\text{Skew}(X) &= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right] \\
&= \sum_{x \in S_X} \left(\frac{x - \mu_X}{\sigma_X}\right)^3 p(x) \text{ if } X \text{ is discrete} \\
&= \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X}\right)^3 f(x)dx \text{ if } X \text{ is continuous}
\end{aligned}
$$



Intuition

- If X has a symmetric distribution about μ_X then Skew$(X) = 0$
- Skew$(X) > 0 \Longrightarrow$ pdf has long right tail, and median $<$ mean
- Skew$(X) < 0 \Longrightarrow$ pdf has long left tail, and median $>$ mean

Example: Using the discrete distribution for the return on Microsoft stock in Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have

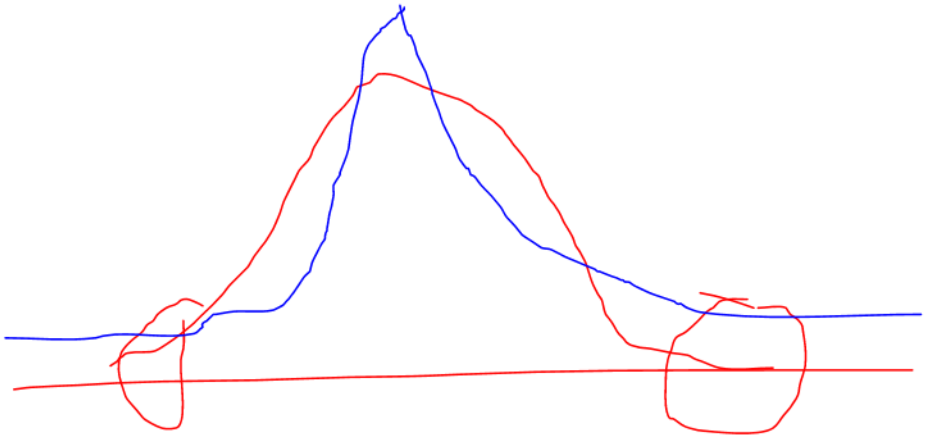
$$\begin{aligned}\text{skew}(X) &= [(-0.3-0.1)^3 \cdot (0.05) + (0.0-0.1)^3 \cdot (0.20) \\ &+ (0.1-0.1)^3 \cdot (0.5) + (0.2-0.1)^3 \cdot (0.2) \\ &+ (0.5-0.1)^3 \cdot (0.05)]/(0.141)^3 \\ &= 0.0\end{aligned}$$

Example: $X \backsim N(\mu_X, \sigma_X^2)$. Then

$$\mathrm{Skew}(X) = \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X} \right)^3 \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2} \right) dx = 0$$

Example: $Y \backsim \mathrm{lognormal}(\mu_X, \sigma_X^2)$. Then

$$\mathrm{Skew}(Y) = \left(\exp(\sigma_X^2) + 2 \right) \sqrt{\exp(\sigma_X^2) - 1} > 0$$



Kurtosis - Measure of tail thickness

$$g(X) = ((X - \mu_X)/\sigma_X)^4$$

$$\text{Kurt}(X) = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^4\right]$$

$$= \sum_{x \in S_X} \left(\frac{x - \mu_X}{\sigma_X}\right)^4 p(x) \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X}\right)^4 f(x)dx \text{ if } X \text{ is continuous}$$

Intuition

- Values of x far from μ_X get blown up resulting in large values of kurtosis

- Two extreme cases: fat tails (large kurtosis); thin tails (small kurtosis)

Example: Kurtosis for a discrete random variable

Using the discrete distribution for the return on Microsoft stock in Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have

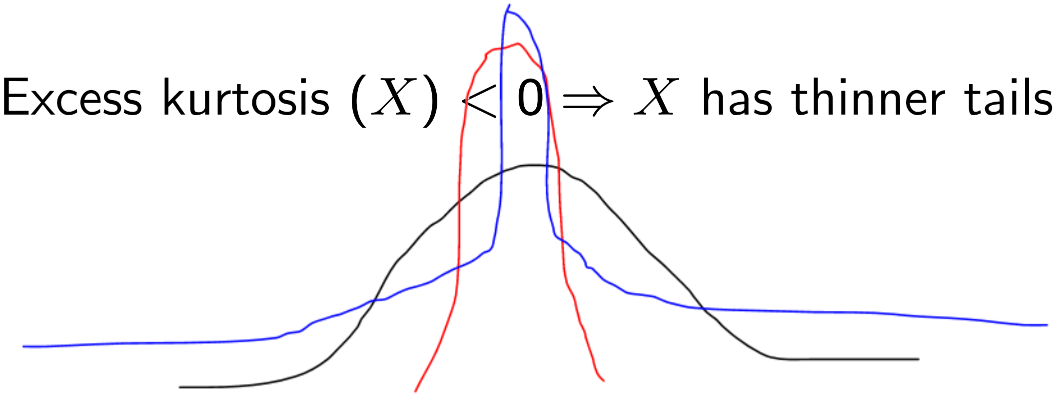
$$\begin{aligned} \text{Kurt}(X) &= [(-0.3 - 0.1)^4 \cdot (0.05) + (0.0 - 0.1)^4 \cdot (0.20) \\ &+ (0.1 - 0.1)^4 \cdot (0.5) + (0.2 - 0.1)^4 \cdot (0.2) \\ &+ (0.5 - 0.1)^4 \cdot (0.05)]/(0.141)^4 \\ &= 6.5 \end{aligned}$$

Example: $X \backsim N(\mu_X, \sigma_X^2)$

$$\text{Kurt}(X) = \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^4 \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2} dx = 3$$

Definition: Excess kurtosis = $\text{Kurt}(X) - 3$ = kurtosis value in excess of kurtosis of normal distribution.

- Excess kurtosis $(X) > 0 \Rightarrow X$ has fatter tails than normal distribution
- Excess kurtosis $(X) < 0 \Rightarrow X$ has thinner tails than normal distribution



The Student's-t Distribution

A distribution similar to the standard normal distribution but with fatter tails, and hence larger kurtosis, is the Student's t distribution. If X has a Student's t distribution with degrees of freedom parameter v, denoted $X \sim t_v$, then its pdf has the form

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{x^2}{v}\right)^{-\left(\frac{v+1}{2}\right)}, \quad -\infty < x < \infty,\; v > 0.$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ denotes the gamma function.

It can be shown that

$$\begin{aligned} E[X] &= 0,\ v > 1 \\ \text{var}(X) &= \frac{v}{v-2},\ v > 2, \\ \text{skew}(X) &= 0,\ v > 3, \\ \text{kurt}(X) &= \frac{6}{v-4} + 3,\ v > 4. \end{aligned}$$

The parameter v controls the scale and tail thickness of distribution. If v is close to four, then the kurtosis is large and the tails are thick. If $v < 4$, then $\text{kurt}(X) = \infty$. As $v \to \infty$ the Student's t pdf approaches that of a standard normal random variable and $\text{kurt}(X) = 3$.

R Runctions

- simulate data: `rt(n, df)`
- compute CDF: `pt(q, df)`
- compute quantiles: `qt(p, df)`
- compute density: `dt(x, df)`

Here df is the degrees of freedom parameter v.

Linear Functions of a Random Variable

Let X be a discrete or continuous rv with $\mu_X = E[X]$, and $\sigma_X^2 = \text{Var}(X)$. Define a new rv Y to be a linear function of X :

$$Y = g(X) = a \cdot X + b$$

a and b are known constants

Then

$$\begin{aligned}
\mu_Y &= E[Y] = E[a \cdot X + b] \\
&= a \cdot E[X] + b = a \cdot \mu_X + b \\
\sigma_Y^2 &= \text{Var}(Y) = \text{Var}(a \cdot X + b) \\
&= a^2 \cdot \text{Var}(X) \\
&= a^2 \cdot \sigma_X^2 \\
\sigma_Y &= a \cdot \sigma_X
\end{aligned}$$

Linear Function of a Normal rv

Let $X \backsim N(\mu_X, \sigma_X^2)$ and define $Y = a \cdot X + b$. Then

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

with

$$\begin{aligned} \mu_Y &= a \cdot \mu_X + b \\ \sigma_Y^2 &= a^2 \cdot \sigma_X^2 \end{aligned}$$

Remarks

- Proof of result relies on change-of-variables formula for determining pdf of a function of a rv
- Result may or may not hold for random variables whose distributions are not normal

Example - Standardizing a Normal rv

Let $X \sim N(\mu_X, \sigma_X^2)$. The standardized rv Z is created using

$$
\begin{aligned}
Z &= \frac{X - \mu_X}{\sigma_X} = \frac{1}{\sigma_X} \cdot X - \frac{\mu_X}{\sigma_X} \\
&= a \cdot X + b \\
a &= \frac{1}{\sigma_X},\ b = -\frac{\mu_X}{\sigma_X}
\end{aligned}
$$

Properties of Z

$$\begin{aligned} E[Z] &= \frac{1}{\sigma_X} E[X] - \frac{\mu_X}{\sigma_X} \\ &= \frac{1}{\sigma_X} \cdot \mu_X - \frac{\mu_X}{\sigma_X} = 0 \\ \mathsf{Var}(Z) &= \left(\frac{1}{\sigma_X}\right)^2 \cdot \mathsf{Var}(X) \\ &= \left(\frac{1}{\sigma_X}\right)^2 \cdot \sigma_X^2 = 1 \\ Z &\sim N(0,1) \end{aligned}$$

Value at Risk: Introduction

Consider a $W_0 = \$10,000$ investment in Microsoft for 1 month. Assume

$$\begin{aligned} R &= \text{simple monthly return on Microsoft} \\ R &\sim N(0.05, (0.10)^2),\ \mu_R = 0.05,\ \sigma_R = 0.10 \end{aligned}$$

Goal: Calculate how much we can lose with a specified probability α

Questions:

1. What is the probability distribution of end of month wealth, $W_1 = \$10,000 \cdot (1+R)$?

2. What is $\Pr(W_1 < \$9,000)$?

3. What value of R produces $W_1 = \$9,000$?

4. What is the monthly value-at-risk (VaR) on the \$10,000 investment with 5% probability? That is, how much can we lose if $R \leq q_{.05}$?

Answers:

1. $W_1 = \$10,000 \cdot (1 + R)$ is a linear function of R, and R is a normally distributed rv. Therefore, W_1 is normally distributed with

$$\begin{gathered} E[W_1] = \$10,000 \cdot (1 + E[R]) \\ = \$10,000 \cdot (1 + 0.05) = \$10,500, \\ \text{Var}(W_1) = (\$10,000)^2 \text{Var}(R) \\ = (\$10,000)^2 (0.1)^2 = 1,000,000 \\ W_1 \sim N(\$10,500, (\$1,000)^2) \end{gathered}$$

2. Using $W_1 \sim N(\$10,500, (\$1,000)^2)$

$$\begin{aligned} &\Pr\left(W_1 < \$9,000\right) \\ &= \texttt{NORMDIST(9000,10500,1000)} = 0.067 \end{aligned}$$

3. To find R that produces $W_1 = \$9,000$ solve

$$R = \frac{\$9,000 - \$10,000}{\$10,000} = -0.10.$$

Notice that -0.10 is the 6.7% quantile of the distribution of R :

$$q_{.067} = \Pr(R < -0.10) = 0.067$$

4. Use $R \sim N(0.05, (0.10)^2)$ and solve for the the 5% quantile:

$$\begin{aligned} \Pr(R < q_{.05}^R) = 0.05 \Rightarrow \\ q_{.05}^R = \texttt{NORMINV(0.05, 0.05,0.10)} = -0.114. \end{aligned}$$

If $R = -11.4\%$ the loss in investment value is at least

$$\begin{aligned} \$10,000 \cdot (-0.114) &= -\$1,144 \\ &= 5\% \text{ VaR} \end{aligned}$$

In general, the $\alpha \times 100\%$ Value-at-Risk (VaR_α) for an initial investment of $\$W_0$ is computed as

$$\begin{aligned} \mathsf{VaR}_\alpha &= \$W_0 \times q_\alpha^R \\ q_\alpha^R &= \alpha \times 100\% \text{ quantile of simple return distn} \end{aligned}$$

Remarks:

1. If $R \sim N(\mu_R, \sigma_R^2)$ then $q_\alpha^R = \mu_R + \sigma_R q_\alpha^Z$, $q_\alpha^Z = \alpha \times 100\%$ quantile of $Z \sim N(0,1)$ and

   $$\mathsf{VaR}_\alpha = \$W_0 \times \left(\mu_R + \sigma_R q_\alpha^Z\right)$$

   For example, let $W_0 = \$10,000$, $\mu_R = 0.05$,and $\sigma_R = 0.10$. Then for $\alpha = 0.05, q_{0.05}^Z = -1.645$ and

   $$\mathsf{VaR}_\alpha = \$10,000 \times (0.05 + 0.10 \times (-1.645)) = -1,144$$

2. Because VaR represents a loss, it is often reported as a positive number. For example, $-\$1,144$ represents a loss of $\$1,144$. So the VaR is reported as $\$1,144$.

VaR for Continuously Compounded Returns

$$r = \ln(1+R), \text{ cc monthly return}$$
$$R = e^{r} - 1, \text{ simple monthly return}$$

Assume

$$r \sim N(\mu_r, \sigma_r^2)$$
$$W_0 = \text{initial investment}$$

Note: The distribution of R is log-normal so the $\alpha-$quantile of the distribution of R is not $\mu_R + \sigma_R q_\alpha^Z$. That is,

$$q_\alpha^R \neq \mu_R + \sigma_R q_\alpha^Z$$

Q: What is q_α^R?

$100 \cdot \alpha\%$ VaR Computation

- Compute α quantile of Normal Distribution for r:

$$q_\alpha^r = \mu_r + \sigma_r z_\alpha$$

- Convert α quantile for r into α quantile for R (quantiles are preserved under increasing transformations):

$$q_\alpha^R = e^{q_\alpha^r} - 1$$

- Compute $100 \cdot \alpha\%$ VaR using q_α^R:

$$\mathrm{VaR}_\alpha = \$W_0 \cdot q_\alpha^R$$

Example: Compute 5% VaR assuming

$$r_t \sim N(0.05, (0.10)^2), W_0 = \$10,000$$

The 5% cc return quantile is

$$\begin{aligned} q^r_{.05} &= \mu_r + \sigma_r z_{.05} \\ &= 0.05 + (0.10)(-1.645) = -0.114 \end{aligned}$$

The 5% simple return quantile is

$$q^R_{.05} = e^{q^r_{.05}} - 1 = e^{-.114} - 1 = -0.108$$

The 5% VaR based on a $10,000 initial investment is

$$\text{VaR}_{.05} = \$10,000 \cdot (-0.108) = -\$1,077$$