Introduction to Computational Finance and Financial Econometrics

Probability Review - Part 1

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Outline

1. Univariate Random Variables
   - Discrete Random Variables
   - Continuous Random Variables
   - The Cumulative Distribution Function (CDF)
   - Quantiles of a Distribution
   - The Standard Normal Distribution
   - Shape Characteristics of pdfs
   - Linear Functions of a Random Variable
   - Value at Risk: Introduction

Defnition: A random variable (rv) X is a variable that can take on a given set of values, called the sample space S_X, where the likelihood of the values in S_X is determined by the variable's probability distribution function (pdf).

Examples

- X = price of microsoft stock next month. $S_X = \{\mathbb{R} : 0 < X \leq M\}$
- X = simple return on a one month investment. $S_X = \{\mathbb{R} : -1 \leq X < M\}$
- $X = 1$ if stock price goes up; $X = 0$ if stock price goes down. $S_X = \{0, 1\}$

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Discrete Random Variables

Definition: A discrete rv X is one that can take on a finite number of n different values $x_1, \cdots, x_n$.

Definition: The pdf of a discrete rv X, $p(x)$, is a function such that $p(x) = \Pr(X = x)$. The pdf must satisfy:

1. $p(x) \geqslant 0$ for all $x \in S_X$; $p(x) = 0$ for all $x \notin S_X$
2. $\sum_{x \in S_X} p(x) = 1$
3. $p(x) \leqslant 1$ for all $x \in S_X$

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Continuous Random Variables

Definition: A continuous rv X is one that can take on any real value.

Definition: The pdf of a continuous rv X is a nonnegative function $f(x)$ such that for any interval A on the real line:

$$\Pr(X \in A) = \int_A f(x)dx$$

$\Pr(X \in A) =$ "Area under probability curve over the interval A". The pdf $f(x)$ must satisfy:

1. $f(x) \geqslant 0$; $\int_{-\infty}^{\infty} f(x)dx = 1$

Example

Example: Uniform distribution over $[a, b]$

Let $X \backsim U[a, b]$, where "$\backsim$" means "is distributed as". Then:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Properties: $f(x) \geq 0$, provided $b > a$, and,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{a}^{b} \frac{1}{b-a}dx = \frac{1}{b-a} \int_{a}^{b} dx$$

$$= \frac{1}{b-a}[x]_a^b = \frac{b-a}{b-a} = 1$$

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The Cumulative Distribution Function (CDF)

Definition: The CDF, F, of a rv X is $F(x) = \Pr(X \leq x)$ and:

- If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$
- $F(-\infty) = 0$ and $F(\infty) = 1$
- $\Pr(X \geq x) = 1 - F(x)$
- $\Pr(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
- $\frac{d}{dx}F(x) = f(x)$ if X is a continuous rv

Example

Example: Uniform distribution over $[0,1]$

$$X \backsim U[0,1]$$

$$f(x) = \begin{cases} \frac{1}{1-0} = 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$F(x) = \Pr(X \leq x) = \int_0^x dz$$

$$= [z]_0^x = x$$

and, for example,

$$\Pr(0 \leq X \leq 0.5) = F(0.5) - F(0)$$

$$= 0.5 - 0 = 0.5$$

Example cont.

Note,

$$\frac{d}{dx}F(x) = 1 = f(x)$$

Remark: for a continuous rv,

$$\Pr(X \leq x) = \Pr(X < x)$$

$$\Pr(X = x) = 0$$

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Quantiles of a Distribution

X is a rv with CDF $F_X(x) = \Pr(X \leq x)$.

Definition: The $\alpha * 100\%$ quantile of F_X for $\alpha \in [0, 1]$ is the value q_α such that:

$$F_X(q_\alpha) = \Pr(X \leq q_\alpha) = \alpha$$

The area under the probability curve to the left of q_α is α. If the inverse CDF F_X^{-1} exists then:

$$q_\alpha = F_X^{-1}(\alpha)$$

Example

$$1\% \text{ quantile} = q_{.01}$$

$$5\% \text{ quantile} = q_{.05}$$

$$50\% \text{ quantile} = q_{.5} = \text{ median}$$

Example: Quantile of uniform distn on [0,1]:

$$F_X(x) = x \Rightarrow q_\alpha = \alpha$$

$$q_{.01} = 0.01$$

$$q_{.5} = 0.5$$

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The Standard Normal Distribution

Let X be a rv such that $X \backsim N(0,1)$. Then:

$$f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad -\infty \leq x \leq \infty$$

$$\Phi(x) = \Pr(X \leq x) = \int_{-\infty}^{x} \phi(z) dz$$

Shape Characteristics

- Centered at zero
- Symmetric about zero (same shape to left and right of zero)

$$\Pr(-1 \leq x \leq 1) = \Phi(1) - \Phi(-1) = 0.67$$

$$\Pr(-2 \leq x \leq 2) = \Phi(2) - \Phi(-2) = 0.95$$

$$\Pr(-3 \leq x \leq 3) = \Phi(3) - \Phi(-3) = 0.99$$

Finding Areas Under the Normal Curve

- $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$, via change of variables formula in calculus
- $\Pr(a < X < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \Phi(b) - \Phi(a)$, cannot be computed analytically!
- Special numerical algorithms are used to calculate $\Phi(z)$

Computational tools

Excel functions:

1. `NORMSDIST` computes $\Pr(X \leq z) = \Phi(z)$ or $p(z) = \phi(z)$
2. `NORMSINV` computes the quantile $z_\alpha = \Phi^{-1}(\alpha)$

R functions

1. `pnorm` computes $\Pr(X \leq z) = \Phi(z)$
2. `qnorm` computes the quantile $z_\alpha = \Phi^{-1}(\alpha)$
3. `dnorm` computes the density $\phi(z)$

Some Tricks for Computing Area Under Normal Curve

$N(0,1)$ is symmeric about 0; total area=1

$$\Pr(X \leq z) = 1 - \Pr(X \geq z)$$

$$\Pr(X \geq z) = \Pr(X \leq -z)$$

$$\Pr(X \geq 0) = \Pr(X \leq 0) = 0.5$$

Example

In Excel use:

$$\Pr(-1 \leq X \leq 2) = \Pr(X \leq 2) - \Pr(X \leq -1)$$

$$= \texttt{NORMSDIST(2)} - \texttt{NORMSDIST(-1)}$$

$$= 0.97725 - 0.15866 = 0.81860$$

In R use:

$$\texttt{pnorm(2)} - \texttt{pnorm(-1)} = 0.81860$$

The 1%, 2.5%, 5% quantiles are:

$$\text{Excel} : z_{.01} = \Phi^{-1}(0.01) = \texttt{NORMSINV(0.01)} = -2.33$$

$$\text{R} : \texttt{qnorm(0.01)} = -2.33$$

$$\text{Excel} : z_{.025} = \Phi^{-1}(0.025) = \texttt{NORMSINV(0.025)} = -1.96$$

$$\text{R} : \texttt{qnorm(0.025)} = -1.96$$

Example cont.

$$\text{Excel} : z_{.05} = \Phi^{-1}(.05) = \texttt{NORMSINV(.05)} = -1.645$$

$$\text{R} : \texttt{qnorm(0.05)} = -1.645$$

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Shape Characteristics of pdfs

- Expected Value or Mean - Center of Mass
- Variance and Standard Deviation - Spread about mean
- Skewness - Symmetry about mean
- Kurtosis - Tail thickness

Expected Value

Expected Value - Discrete rv:

$$E[X] = \mu_X = \sum_{x \in S_X} x \cdot p(x)$$

$$= \sum_{x \in S_X} x \cdot \Pr(X = x)$$

$E[X]$ = probability weighted average of possible values of X.

Expected Value - Continuous rv:

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Note: In continuous case, $\sum_{x \in S_X}$ becames $\int_{-\infty}^{\infty}$

Expected value of discrete random variable

Using the discrete distribution for the return on Microsoft stock in Table 1, the expected return is:

$$
\begin{aligned}
E[X] &= (-0.3) \cdot (0.05) + (0.0) \cdot (0.20) + (0.1) \cdot (0.5) \\
&+ (0.2) \cdot (0.2) + (0.5) \cdot (0.05) \\
&= 0.10.
\end{aligned}
$$

Examples

Example: $X \backsim U[1,2]$:

$$E[X] = \int_1^2 x dx = \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2}[4-1] = \frac{3}{2}$$

Example: $X \backsim N(0,1)$:

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0$$

Expectation of a Function of X

Definition: Let $g(X)$ be some function of the rv X. Then,

$$E[g(X)] = \sum_{x \in S_X} g(x) \cdot p(x) \text{ Discrete case}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx \text{ Continuous case}$$

Variance and Standard Deviation

$$g(X) = (X - E[X])^2 = (X - \mu_X)^2$$

$$\text{Var}(X) = \sigma_X^2 = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$$

$$\text{SD}(X) = \sigma_X = \sqrt{\text{Var}(X)}$$

Note: $\text{Var}(X)$ is in squared units of X, and $\text{SD}(X)$ is in the same units as X. Therefore, $\text{SD}(X)$ is easier to interpret.

Computation of Var(X) and SD(X)

$$\sigma_X^2 = E[(X - \mu_X)^2]$$

$$= \sum_{x \in S_X} (x - \mu_X)^2 \cdot p(x) \text{ if } X \text{ is a discrete rv}$$

$$= \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f(x)dx \text{ if } X \text{ is a continuous rv}$$

$$\sigma_X = \sqrt{\sigma_X^2}$$

Remark: For "bell-shaped" data, σ_X measures the size of the typical deviation from the mean value μ_X.

Example: Variance and standard deviation for a discrete random variable

Using the discrete distribution for the return on Microsoft stock in Table 1 and the result that $\mu_X = 0.1$, we have:

$$\begin{aligned} \text{Var}(X) &= (-0.3 - 0.1)^2 \cdot (0.05) + (0.0 - 0.1)^2 \cdot (0.20) \\ &+ (0.1 - 0.1)^2 \cdot (0.5) + (0.2 - 0.1)^2 \cdot (0.2) \\ &+ (0.5 - 0.1)^2 \cdot (0.05) \\ &= 0.020 \\ \text{SD}(X) &= \sigma_X = \sqrt{0.020} = 0.141. \end{aligned}$$

Example cont.

Given that the distribution is fairly bell-shaped we can say that typical values deviate from the mean value of 0.10 by about 0.141:

$$\mu \pm \sigma = -0.10 \pm 0.141 = [-0.041,\ 2.41]$$

Example: $X \backsim N(0,1)$

$$\mu_X = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-0)^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$$

$$\sigma_X = \sqrt{1} = 1$$

$\Rightarrow$ size of typical deviation from $\mu_X = 0$ is $\sigma_X = 1$

The General Normal Distribution

$$X \backsim N(\mu_X,\ \sigma_X^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) dx,\ -\infty \leq x \leq \infty$$

$$E[X] = \mu_X = \text{ mean value}$$

$$\mathrm{Var}(X) = \sigma_X^2 = \text{ variance}$$

$$\mathrm{SD}(X) = \sigma_X = \text{ standard deviation}$$

Shape Characteristics

- Centered at μ_X
- Symmetric about μ_X

$$\Pr(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X) = 0.67$$

$$\Pr(\mu_X - 2 \cdot \sigma_X \leq X \leq \mu_X + 2 \cdot \sigma_X) = 0.95$$

$$\Pr(\mu_X - 3 \cdot \sigma_X \leq X \leq \mu_X + 3 \cdot \sigma_X) = 0.99$$

- Quantiles of the general normal distribution:

$$q_\alpha = \mu_X + \sigma_X \cdot \Phi^{-1}(\alpha) = \mu_X + \sigma_X \cdot z_\alpha$$

Remarks

- $X \backsim N(0,1)$: Standard Normal $\Longrightarrow \mu_X = 0$ and $\sigma_X^2 = 1$
- The pdf of the general Normal is completely determined by values of μ_X and σ_X^2

Finding Areas Under General Normal Curve

Excel functions:

- NORMDIST$(x, \mu_X, \sigma_X,$ cumulative$)$. If cumulative $=$ true: $\Pr(X \leq x)$ is computed; If cumulative $=$ false, $f(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2}$ is computed.
- NORMINV$(\alpha, \mu_x, \sigma_x)$ computes $q_\alpha = \mu_X + \sigma_X z_\alpha$

Finding Areas Under General Normal Curve cont.

R functions:

- simulate data: `rnorm(n, mean, sd)`
- compute CDF: `pnorm(q, mean, sd)`
- compute quantiles: `qnorm(p, mean, sd)`
- compute density: `dnorm(x, mean, sd)`

Example

Example: Why the normal distribution may not be appropriate for simple returns

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \text{simple return}$$

Assume $R_t \sim N(0.05, (0.50)^2)$

Note: $P_t \geq 0 \implies R_t \geq -1$. However, based on the assumed normal distribution:

$$\Pr(R_t < -1) = \texttt{NORMDIST(-1,0.05,0.50,TRUE)} = 0.018.$$

This implies that there is a 1.8% chance that the asset price will be negative. This is why the normal distribution may not be appropriate for simple returns.

Example

Example: The normal distribution is more appropriate for cc returns

$$r_t = \ln(1 + R_t) = \text{cc return}$$

$$R_t = e^{r_t} - 1 = \text{simple return}$$

Assume $r_t \sim N(0.05, (0.50)^2)$

Unlike R_t, r_t can take on values less than -1. For example,

$$r_t = -2 \implies R_t = e^{-2} - 1 = -0.865$$

$$\Pr(r_t < -2) = \Pr(R_t < -0.865)$$

$$= \texttt{NORMDIST(-2,0.05,0.50,TRUE)} = 0.00002$$

The Log-Normal Distribution

$$X \sim N(\mu_X, \sigma_X^2), \;\; -\infty < X < \infty$$

$$Y = \exp(X) \sim \text{lognormal}(\mu_X, \sigma_X^2), \; 0 < Y < \infty$$

$$E[Y] = \mu_Y = \exp(\mu_X + \sigma_X^2/2)$$

$$\text{Var}(Y) = \sigma_Y^2 = \exp(2\mu_X + \sigma_X^2)(\exp(\sigma_X^2) - 1)$$

Example: log-normal distribution for simple returns

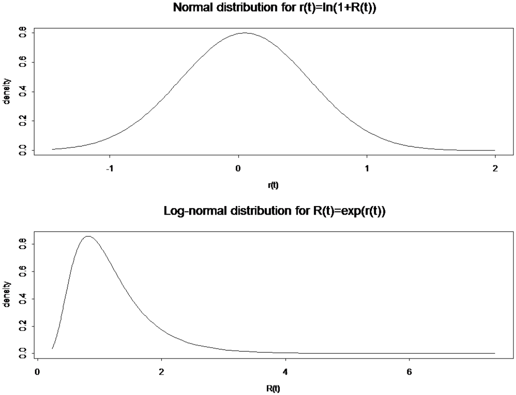
$$r_t \sim N(0.05, (0.50)^2)$$

$$1 + R_t \sim \text{lognormal}(0.05, (0.50)^2)$$

$$\mu_{1+R} = \exp(0.05 + (0.5)^2/2) = 1.191$$

$$\sigma_{1+R}^2 = \exp(2(0.05) + (0.5)^2)(\exp(0.5^2) - 1) = 0.563$$

The Log-Normal Distribution



Normal distribution for r_t and log-normal distribution for $1 + R_t = e^{r_t}$.

Skewness - Measure of symmetry

$$g(X) = ((X - \mu_X)/\sigma_X)^3$$

$$\begin{aligned}
\text{Skew}(X) &= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right] \\
&= \sum_{x \in S_X} \left(\frac{x - \mu_X}{\sigma_X}\right)^3 p(x) \text{ if } X \text{ is discrete} \\
&= \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X}\right)^3 f(x)dx \text{ if } X \text{ is continuous}
\end{aligned}$$

Intuition

- If X has a symmetric distribution about μ_X then $\text{Skew}(X) = 0$
- $\text{Skew}(X) > 0 \Longrightarrow$ pdf has long right tail, and median $<$ mean
- $\text{Skew}(X) < 0 \Longrightarrow$ pdf has long left tail, and median $>$ mean

Example

Example: Using the discrete distribution for the return on Microsoft stock in Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have:

$$\begin{aligned} \text{skew}(X) &= [(-0.3 - 0.1)^3 \cdot (0.05) + (0.0 - 0.1)^3 \cdot (0.20) \\ &+ (0.1 - 0.1)^3 \cdot (0.5) + (0.2 - 0.1)^3 \cdot (0.2) \\ &+ (0.5 - 0.1)^3 \cdot (0.05)]/(0.141)^3 \\ &= 0.0 \end{aligned}$$

Example

Example: $X \backsim N(\mu_X, \sigma_X^2)$. Then,

$$\text{Skew}(X) = \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X} \right)^3 \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2} \right) dx = 0$$

Example: $Y \backsim \text{lognormal}(\mu_X, \sigma_X^2)$. Then,

$$\text{Skew}(Y) = \left(\exp(\sigma_X^2) + 2 \right) \sqrt{\exp(\sigma_X^2) - 1} > 0$$

Kurtosis - Measure of tail thickness

$$g(X) = ((X - \mu_X)/\sigma_X)^4$$

$$\begin{aligned}
\text{Kurt}(X) &= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^4\right] \\
&= \sum_{x \in S_X} \left(\frac{x - \mu_X}{\sigma_X}\right)^4 p(x) \text{ if } X \text{ is discrete} \\
&= \int_{-\infty}^{\infty} \left(\frac{x - \mu_X}{\sigma_X}\right)^4 f(x)dx \text{ if } X \text{ is continuous}
\end{aligned}$$

Intuition:

- Values of x far from μ_X get blown up resulting in large values of kurtosis
- Two extreme cases: fat tails (large kurtosis); thin tails (small kurtosis)

Example

Example: Kurtosis for a discrete random variable

Using the discrete distribution for the return on Microsoft stock in Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have:

$$\begin{aligned}
\text{Kurt}(X) &= [(-0.3 - 0.1)^4 \cdot (0.05) + (0.0 - 0.1)^4 \cdot (0.20) \\
&+ (0.1 - 0.1)^4 \cdot (0.5) + (0.2 - 0.1)^4 \cdot (0.2) \\
&+ (0.5 - 0.1)^4 \cdot (0.05)]/(0.141)^4 \\
&= 6.5
\end{aligned}$$

Example

Example: $X \backsim N(\mu_X, \sigma_X^2)$

$$\text{Kurt}(X) = \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^4 \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2} dx = 3$$

Definition: Excess kurtosis $= \text{Kurt}(X) - 3 =$ kurtosis value in excess of kurtosis of normal distribution.

- Excess kurtosis $(X) > 0 \Rightarrow X$ has fatter tails than normal distribution
- Excess kurtosis $(X) < 0 \Rightarrow X$ has thinner tails than normal distribution

Standard Deviation as a Measure of Risk

$$R_A = \text{monthly return on asset A}$$

$$R_B = \text{monthly return on assetB}$$

$$R_A \backsim N(\mu_A, \sigma_A^2), R_B \backsim N(\mu_B, \sigma_B^2)$$

where,

$$\mu_A = E[R_A] = \text{ expected monthly return on asset A}$$

$$\sigma_A = \text{SD}(R_A)$$

$$= \text{ std. deviation of monthly return on asset A}$$

Suppose,

$$\mu_A > \mu_B$$

$$\sigma_A > \sigma_B$$

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Linear Functions of a Random Variable

Let X be a discrete or continuous rv with $\mu_X = E[X]$, and $\sigma_X^2 = \text{Var}(X)$. Define a new rv Y to be a linear function of X :

$$Y = g(X) = a \cdot X + b$$

where a and b are known constants. Then,

$$\begin{aligned}
\mu_Y &= E[Y] = E[a \cdot X + b] \\
&= a \cdot E[X] + b = a \cdot \mu_X + b \\
\sigma_Y^2 &= \text{Var}(Y) = \text{Var}(a \cdot X + b) \\
&= a^2 \cdot \text{Var}(X) \\
&= a^2 \cdot \sigma_X^2 \\
\sigma_Y &= a \cdot \sigma_X
\end{aligned}$$

Linear Function of a Normal rv

Let $X \backsim N(\mu_X, \sigma_X^2)$ and define $Y = a \cdot X + b$. Then,

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

with,

$$\mu_Y = a \cdot \mu_X + b$$

$$\sigma_Y^2 = a^2 \cdot \sigma_X^2$$

Remarks:

- Proof of result relies on change-of-variables formula for determining pdf of a function of a rv
- Result may or may not hold for random variables whose distributions are not normal

Example

Example: Standardizing a Normal rv

Let $X \sim N(\mu_X, \sigma_X^2)$. The standardized rv Z is created using:

$$Z = \frac{X - \mu_X}{\sigma_X} = \frac{1}{\sigma_X} \cdot X - \frac{\mu_X}{\sigma_X}$$

$$= a \cdot X + b$$

$$a = \frac{1}{\sigma_X},\ b = -\frac{\mu_X}{\sigma_X}$$

Example cont.

Properties of Z,

$$E[Z] = \frac{1}{\sigma_X}E[X] - \frac{\mu_X}{\sigma_X}$$

$$= \frac{1}{\sigma_X} \cdot \mu_X - \frac{\mu_X}{\sigma_X} = 0$$

$$\mathrm{Var}(Z) = (\frac{1}{\sigma_X})^2 \cdot \mathrm{Var}(X)$$

$$= (\frac{1}{\sigma_X})^2 \cdot \sigma_X^2 = 1$$

$$Z \sim N(0, 1)$$

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Value at Risk: Introduction

Consider a $\$10,000$ investment in Microsoft for 1 month. Assume,

$$R = \text{simple monthly return on Microsoft}$$

$$R \sim N(0.05, (0.10)^2),\ \mu_R = 0.05,\ \sigma_R = 0.10$$

Goal: Calculate how much we can lose with a specified probability α.

Questions

1. What is the probability distribution of end of month wealth, $W_1 = \$10,000 \cdot (1 + R)$?
2. What is $\Pr(W_1 < \$9,000)$?
3. What value of R produces $W_1 = \$9,000$?
4. What is the monthly value-at-risk (VaR) on the \$10,000 investment with 5% probability? That is, how much can we lose if $R \leq q_{.05}$?

Answers

1. $W_1 = \$10,000 \cdot (1 + R)$ is a linear function of R, and R is a normally distributed rv. Therefore, W_1 is normally distributed with:

$$E[W_1] = \$10,000 \cdot (1 + E[R])$$

$$= \$10,000 \cdot (1 + 0.05) = \$10,500,$$

$$\text{Var}(W_1) = (\$10,000)^2 \text{Var}(R)$$

$$= (\$10,000)^2 (0.1)^2 = 1,000,000$$

$$W_1 \sim N(\$10,500, (\$1,000)^2)$$

2. Using $W_1 \sim N(\$10,500, (\$1,000)^2)$

$$\Pr\left(W_1 < \$9,000\right)$$

$$= \texttt{NORMDIST(9000,10500,1000,TRUE)} = 0.067$$

Answers cont.

3. To find R that produces $W_1 = \$9,000$ solve:

$$R = \frac{\$9,000 - \$10,000}{\$10,000} = -0.10.$$

Notice that -0.10 is the 6.7% quantile of the distribution of R:

$$q_{.067} = \Pr(R < -0.10) = 0.067$$

4. Use $R \sim N(0.05, (0.10)^2)$ and solve for the the 5% quantile:

$$\Pr(R < q_{.05}^R) = 0.05 \Rightarrow$$

$$q_{.05}^R = \texttt{NORMINV(0.05, 0.05,0.10)} = -0.114.$$

If $R = -11.4\%$ the loss in investment value is at least,

$$\$10,000 \cdot (-0.114) = -\$1,144$$
$$= 5\% \text{ VaR}$$

Answers cont.

In general, the $\alpha \times 100\%$ Value-at-Risk (VaR_α) for an initial investment of $\$W_0$ is computed as:

$$\text{VaR}_\alpha = \$W_0 \times q_\alpha$$

$$q_\alpha = \alpha \times 100 \text{ quantile of simple return distn}$$

Remark: Because VaR represents a loss, it is often reported as a positive number. For example, -\$1,144 represents a loss of \$1,144. So the VaR is reported as \$1,144.

VaR for Continuously Compounded Returns

$$r = \ln(1 + R), \text{ cc monthly return}$$

$$R = e^r - 1, \text{ simple monthly return}$$

Assume,

$$r \sim N(\mu_r, \sigma_r^2)$$

$$W_0 = \text{initial investment}$$

Example

Example: $100 \cdot \alpha\%$ VaR Computation

- Compute α quantile of Normal Distribution for r:

$$q_\alpha^r = \mu_r + \sigma_r z_\alpha$$

- Convert α quantile for r into α quantile for R:

$$q_\alpha^R = e^{q_\alpha^r} - 1$$

- Compute $100 \cdot \alpha\%$ VaR using q_α^R:

$$\text{VaR}_\alpha = \$W_0 \cdot q_\alpha^R$$

Example

Example: Conpute 5% VaR assuming:

$$r_t \sim N(0.05, (0.10)^2), W_0 = \$10,000$$

The 5% cc return quantile is:

$$q^r_{.05} = \mu_r + \sigma_r z_{.05}$$

$$= 0.05 + (0.10)(-1.645) = -0.114$$

The 5% simple return quantile is:

$$q^R_{.05} = e^{q^r_{.05}} - 1 = e^{-.114} - 1 = -0.108$$

The 5% VaR based on a \$10,000 initial investment is:

$$\text{VaR}_{.05} = \$10,000 \cdot (-0.108) = -\$1,077$$

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