# Portfolio Theory with Matrix Algebra

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## Portfolio Math with Matrix Algebra

Three Risky Asset Example

Let $R_i$ $(i = A, B, C)$ denote the return on asset $i$ and assume that $R_i$ follows CER model:

$$\begin{aligned} R_i &\sim iid\ N(\mu_i, \sigma_i^2) \\ \text{cov}(R_i, R_j) &= \sigma_{ij} \end{aligned}$$

Portfolio "$\mathbf{x}$"

$$x_i = \text{share of wealth in asset } i$$
$$x_A + x_B + x_C = 1$$

Portfolio return

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$

Portfolio expected return

$$\mu_{p,x} = E[R_{p,x}] = x_A\mu_A + x_B\mu_B + x_C\mu_C$$

Portfolio variance

$$\begin{aligned}
\sigma^2_{p,x} &= \text{var}(R_{p,x}) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\
&\quad +2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}
\end{aligned}$$

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma^2_{p,x})$$

## Matrix Algebra Representation

$$\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}$$

Portfolio weights sum to 1

$$\begin{aligned} \mathbf{x}'\mathbf{1} &= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= x_1 + x_2 + x_3 = 1 \end{aligned}$$

Portfolio return

$$
\begin{aligned}
R_{p,x} &= \mathbf{x}'\mathbf{R} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix} \\
&= x_A R_A + x_B R_B + x_C R_C \\
&= \mathbf{R}'\mathbf{x}
\end{aligned}
$$

Portfolio expected return

$$
\begin{aligned}
\mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \begin{pmatrix} x_A & x_B & x_X \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} \\
&= x_A \mu_A + x_B \mu_B + x_C \mu_C \\
&= \boldsymbol{\mu}'\mathbf{x}
\end{aligned}
$$

Excel formula

`MMULT(transpose(xvec),muvec)`

`<ctrl>-<shift>-<enter>`

Portfolio variance

$$
\begin{aligned}
\sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\
&= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\
&\quad +2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}
\end{aligned}
$$

Excel formulas

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))
MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)
<ctrl>-<shift>-<enter>
```

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2)$$

## Covariance Between 2 Portfolio Returns

2 portfolios

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix}$$
$$\mathbf{x}'\mathbf{1} = 1, \ \mathbf{y}'\mathbf{1} = 1$$

Portfolio returns

$$\begin{aligned} R_{p,x} &= \mathbf{x}'\mathbf{R} \\ R_{p,y} &= \mathbf{y}'\mathbf{R} \end{aligned}$$

Covariance

$$\begin{aligned} \mathsf{cov}(R_{p,x}, R_{p,y}) &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} \\ &= \mathbf{y}'\mathbf{\Sigma}\mathbf{x} \end{aligned}$$

Excel formula

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))
MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))
<ctrl>-<shift>-<enter>
```

## Derivatives of Simple Matrix Functions

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be an $n \times \mathbf{1}$ vectors. Then

$$\underset{n\times 1}{\frac{\partial}{\partial \mathbf{x}}} \mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{y} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{y} \end{pmatrix} = \mathbf{y}, \tag{1}$$

$$\underset{n\times 1}{\frac{\partial}{\partial \mathbf{x}}} \mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{A}\mathbf{x} \end{pmatrix} = 2\mathbf{A}\mathbf{x}. \tag{2}$$

Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

First, consider (1). Now

$$\mathbf{x}'\mathbf{y} = x_1 y_1 + x_2 y_2.$$

Then

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1} \mathbf{x}'\mathbf{y} \\ \frac{\partial}{\partial x_2} \mathbf{x}'\mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} (x_1 y_1 + x_2 y_2) \\ \frac{\partial}{\partial x_2} (x_1 y_1 + x_2 y_2) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{y}.$$

Next, consider (2). We have

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + bx_2^2.$$

Then

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}}\mathbf{x}'\mathbf{A}\mathbf{x} &= \begin{pmatrix} \frac{\partial}{\partial x_1}\left(ax_1^2 + 2bx_1x_2 + bx_2^2\right) \\ \frac{\partial}{\partial x_2}\left(ax_1^2 + 2bx_1x_2 + bx_2^2\right) \end{pmatrix} = \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2bx_2 \end{pmatrix} \\ &= 2\mathbf{A}\mathbf{x}. \end{aligned}$$

## Computing Global Minimum Variance Portfolio

Problem: Find the portfolio $\mathbf{m} = (m_A, m_B, m_C)'$ that solves

$$\min_{m_A, m_B, m_C} \sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

1. Analytic solution using matrix algebra

2. Numerical Solution in Excel Using the Solver (see 3firmExample.xls)

**Analytic solution using matrix algebra**

The Lagrangian is

$$L(\mathbf{m},\lambda) = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} + \lambda(\mathbf{m}'\mathbf{1} - 1)$$

First order conditions (use matrix derivative results)

$$\begin{aligned}
0 &= \frac{\partial L(\mathbf{m},\lambda)}{\partial \mathbf{m}} = \frac{\partial \mathbf{m}'\mathbf{\Sigma}\mathbf{m}}{\partial \mathbf{m}} + \frac{\partial}{\partial \mathbf{m}}\lambda(\mathbf{m}'\mathbf{1} - 1) = 2 \cdot \mathbf{\Sigma}\mathbf{m} + \lambda\mathbf{1} \\
0 &= \frac{\partial L(\mathbf{m},\lambda)}{\partial \lambda} = \frac{\partial \mathbf{m}'\mathbf{\Sigma}\mathbf{m}}{\partial \lambda} + \frac{\partial}{\partial \lambda}\lambda(\mathbf{m}'\mathbf{1} - 1) = \mathbf{m}'\mathbf{1} - 1
\end{aligned}$$

Write FOCs in matrix form

$$\begin{pmatrix} 2\mathbf{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

The FOCs are the linear system

$$\mathbf{A}_m \mathbf{z}_m = \mathbf{b}$$

where

$$\mathbf{A}_m = \begin{pmatrix} 2\boldsymbol{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \ \mathbf{z}_m = \begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

The solution for $\mathbf{z}_m$ is

$$\mathbf{z}_m = \mathbf{A}_m^{-1}\mathbf{b}.$$

The first three elements of $\mathbf{z}_m$ are the portfolio weights $\mathbf{m} = (m_A, m_B, m_C)'$ for the global minimum variance portfolio with expected return $\mu_{p,m} = \mathbf{m}'\boldsymbol{\mu}$ and variance $\sigma^2_{p,m} = \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m}$.

**Efficient Portfolios of Risky Assets: Markowitz Algorithm**

**Problem 1**: find portfolio $\mathbf{x}$ that has the highest expected return for a given level of risk as measured by portfolio variance

$$\begin{aligned} \max_{x_A, x_B, x_C} \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} \text{ s.t} \\ \sigma^2_{p,x} &= \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} = \sigma^0_p = \text{ target risk} \\ \mathbf{x}'\mathbf{1} &= 1 \end{aligned}$$

**Problem 2**: find portfolio $\mathbf{x}$ that has the smallest risk, measured by portfolio variance, that achieves a target expected return.

$$\begin{aligned} \min_{x_A, x_B, x_C} \sigma^2_{p,x} &= \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \mu^0_p = \text{target return} \\ \mathbf{x}'\mathbf{1} &= 1 \end{aligned}$$

**Remark**: Problem 2 is usually solved in practice by varying the target return between a given range.

**Solving for Efficient Portfolios:**

1. Analytic solution using matrix algebra

2. Numerical solution in Excel using the solver

## Analytic solution using matrix algebra

The Lagrangian function associated with Problem 2 is

$$L(x, \lambda_1, \lambda_2) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \lambda_1(\mathbf{x}'\boldsymbol{\mu} - \mu_{p,0}) + \lambda_2(\mathbf{x}'\mathbf{1} - 1)$$

The FOCs are

$$\begin{aligned}
\frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \mathbf{x}} &= 2\mathbf{\Sigma}\mathbf{x} + \lambda_1\boldsymbol{\mu} + \lambda_2\mathbf{1} = \mathbf{0}, \\
\frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_1} &= \mathbf{x}'\boldsymbol{\mu} - \mu_{p,0} = 0, \\
\frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_2} &= \mathbf{x}'\mathbf{1} - 1 = 0.
\end{aligned}$$

These FOCs consist of five linear equations in five unknowns

$(x_A, x_B, x_C, \lambda_1, \lambda_2)$.

We can represent the FOCs in matrix notation as

$$\begin{pmatrix} 2\mathbf{\Sigma} & \boldsymbol{\mu} & \mathbf{1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

or

$$\mathbf{A}_x \mathbf{z}_x = \mathbf{b}_0$$

where

$$\mathbf{A}_x = \begin{pmatrix} 2\mathbf{\Sigma} & \boldsymbol{\mu} & \mathbf{1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix}, \ \mathbf{z}_x = \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b}_0 = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

The solution for $\mathbf{z}_x$ is then

$$\mathbf{z}_x = \mathbf{A}_x^{-1}\mathbf{b}_0.$$

The first three elements of $\mathbf{z}_x$ are the portfolio weights $\mathbf{x} = (x_A, x_B, x_C)'$ for the efficient portfolio with expected return $\mu_{p,x} = \mu_{p,0}$.

**Computing the Portfolio Frontier**

Result: The portfolio frontier can be represented as convex combinations of any two frontier portfolios. Let $\mathbf{x}$ be a frontier portfolio that solves

$$\begin{aligned}\min_{\mathbf{x}} \sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \mu_p^0 \\ \mathbf{x}'\mathbf{1} &= 1\end{aligned}$$

Let $\mathbf{y} \neq \mathbf{x}$ be another frontier portfolio that solves

$$\begin{aligned}\min_{\mathbf{y}} \sigma_{p,y}^2 &= \mathbf{y}'\mathbf{\Sigma}\mathbf{y} \text{ s.t.} \\ \mu_{p,y} &= \mathbf{y}'\boldsymbol{\mu} = \mu_p^1 \neq \mu_p^0 \\ \mathbf{y}'\mathbf{1} &= 1\end{aligned}$$

Let $\alpha$ be any constant. Then the portfolio

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y}$$

is a frontier portfolio. Furthermore

$$\begin{aligned}
\mu_{p,z} &= \mathbf{z}'\boldsymbol{\mu} = \alpha \cdot \mu_{p,x} + (1 - \alpha)\mu_{p,y} \\
\sigma^2_{p,z} &= \mathbf{z}'\boldsymbol{\Sigma}\mathbf{z} \\
&= \alpha^2\sigma^2_{p,x} + (1 - \alpha)^2\sigma^2_{p,y} + 2\alpha(1 - \alpha)\sigma_{x,y} \\
\sigma_{x,y} &= \mathrm{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}
\end{aligned}$$

**Example**: 3 asset case

$$
\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y} \\
&= \alpha \cdot \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} + (1-\alpha) \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \\
&= \begin{pmatrix} \alpha x_A + (1-\alpha) y_A \\ \alpha x_B + (1-\alpha) y_B \\ \alpha x_C + (1-\alpha) y_C \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}
\end{aligned}
$$

## Strategy for Plotting Portfolio Frontier

1. Set global minimum variance portfolio = first frontier portfolio

$$\min_{\mathbf{m}} \sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

   and compute $\mu_{p,m} = \mathbf{m}'\boldsymbol{\mu}$

2. Find asset $i$ that has highest expected return. Set target return to $\mu^0 = \max(\boldsymbol{\mu})$ and solve

$$\begin{aligned} \min_{\mathbf{x}} \sigma^2_{p,x} &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \mu^0_p = \max(\boldsymbol{\mu}) \\ \mathbf{x}'\mathbf{1} &= 1 \end{aligned}$$

3. Create grid of $\alpha$ values, initially between 0 and 1, and compute

$$\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{m} \\
\mu_{p,z} &= \alpha \cdot \mu_{p,m} + (1-\alpha)\mu_{p,x} \\
\sigma^2_{p,z} &= \alpha^2 \sigma^2_{p,m} + (1-\alpha)^2 \sigma^2_{p,x} + 2\alpha(1-\alpha)\sigma_{m,x} \\
\sigma_{m,x} &= \mathbf{m}'\mathbf{\Sigma}\mathbf{x}
\end{aligned}$$

4. Plot $\mu_{p,z}$ against $\sigma_{p,z}$. Expand the grid of $\alpha$ values if necessary to improve the plot

**Finding the Tangency Portfolio**

The tangency portfolio $\mathbf{t}$ is the portfolio of risky assets that maximizes Sharpe's slope:

$$\max_{\mathbf{t}} \text{ Sharpe's slope } = \frac{\mu_{p,t} - r_f}{\sigma_{p,t}}$$

subject to

$$\mathbf{t}'\mathbf{1} = \mathbf{1}$$

In matrix notation,

$$\text{Sharpe's slope } = \frac{\mathbf{t}'\boldsymbol{\mu} - r_f}{(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{1/2}}$$

**Solving for Efficient Portfolios:**

1. Analytic solution using matrix algebra
2. Numerical solution in Excel using the solver

## Analytic solution using matrix algebra

The Lagrangian for this problem is

$$L(\mathbf{t}, \lambda) = \left(\mathbf{t}'\boldsymbol{\mu} - r_f\right)(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-\frac{1}{2}} + \lambda(\mathbf{t}'\mathbf{1} - 1)$$

Using the chain rule, the first order conditions are

$$\begin{aligned} \frac{\partial L(\mathbf{t}, \lambda)}{\partial \mathbf{t}} &= \boldsymbol{\mu}(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-\frac{1}{2}} - \left(\mathbf{t}'\boldsymbol{\mu} - r_f\right)(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-3/2}\boldsymbol{\Sigma}\mathbf{t} + \lambda\mathbf{1} = 0 \\ \frac{\partial L(\mathbf{t}, \lambda)}{\partial \lambda} &= \mathbf{t}'\mathbf{1} - 1 = 0 \end{aligned}$$

After much tedious algebra, it can be shown that the solution for $\mathbf{t}$ is

$$\mathbf{t} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}$$

Remarks:

- If the risk free rate, $r_f$, is ~~greater~~ less than the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio has a positive Sharpe slope

- If the risk free rate, $r_f$, is equal to the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio is not defined

- If the risk free rate, $r_f$, is ~~less~~ greater than the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio has a negative Sharpe slope.

## Mutual Fund Separation Theorem Again

Efficient Portfolios of T-bills and Risky assets are combinations of two portfolios (mutual funds)

- T-bills
- Tangency portfolio

Efficient Portfolios

$$
\begin{aligned}
x_t &= \text{ share of wealth in tangency portfolio } \mathbf{t} \\
x_f &= \text{share of wealth in T-bills} \\
x_t + x_f &= 1 \\
\mu_p^e &= r_f + x_t(\mu_{p,t} - r_f),\ \mu_{p,t} = \mathbf{t}'\boldsymbol{\mu} \\
\sigma_p^e &= x_t \sigma_{p,t},\ \sigma_{p,t} = \left(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}\right)^{1/2}
\end{aligned}
$$

**Remark**: The weights $x_t$ and $x_f$ are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills ($x_t \approx 0$)
- Risk tolerant investors hold mostly tangency portfolio ($x_t \approx 1$)
- If Sharpe's slope for the tangency portfolio is negative then the efficient portfolio involve shorting the tangency portfolio

## Portfolio Value-at-Risk

Let $\mathbf{x} = (x_1, \ldots, x_n)'$ denote a vector of asset share for a portfolio. Portfolio risk is measured by $\text{VaR}(R_{p,x}) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x}$. Alternatively, portfolio risk can be measured using Value-at-Risk:

$$\begin{aligned} \text{VaR}_\alpha &= W_0 \cdot (e^{q_\alpha} - 1) \\ W_0 &= \text{initial investment} \\ q_\alpha &= 100 \cdot \alpha\% \text{ cc return quantile} \\ \alpha &= \text{ loss probability} \end{aligned}$$

**Result**: Assuming the CER model holds,

$$\begin{aligned} q_\alpha &= \mu_{p,x} + \sigma_{p,x} q_\alpha^Z \\ \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} \\ \sigma_{p,x} &= \left(\mathbf{x}'\mathbf{\Sigma}\mathbf{x}\right)^{1/2} \\ q_\alpha^Z &= 100 \cdot \alpha\% \text{ quantile from } N(0,1) \end{aligned}$$

**Example**: Using VaR to evaluate an efficient portfolio

Invest in 3 risky assets (Microsoft, Starbucks, Nordstrom) and T-bills. Assume $r_f = 0.005$

1. Determine efficient portfolio that has same expected return as Starbucks

2. Compare $\text{VaR}_{.05}$ for Starbucks and efficient portfolio based on \$100,000 investment

Solution for 1.

$$\begin{aligned}
\mu_{\text{SBUX}} &= 0.0285 \\
\mu_p^e &= r_f + x_t(\mu_{p,t} - r_f) \\
r_f &= 0.005 \\
\mu_{p,t} &= \mathbf{t}'\boldsymbol{\mu} = .05186, \sigma_{p,t} = 0.111
\end{aligned}$$

Solve

$$\begin{aligned}
0.0285 &= 0.005 + x_t(0.05186 - 0.005) \\
x_t &= \frac{0.0285 - .005}{0.05186 - .005} = 0.501 \\
x_f &= 1 - 0.501 = 0.499
\end{aligned}$$

Note:

$$\begin{aligned}
\mu_p^e &= 0.005 + 0.501 \cdot (0.05186 - 0.005) = 0.0285 \\
\sigma_p^e &= x_t \sigma_{p,t} = (0.501)(0.111) = 0.057
\end{aligned}$$

Solution for 2.

$$
\begin{aligned}
q_{.05}^{\text{SBUX}} &= \mu_{\text{SBUX}} + \sigma_{\text{SBUX}} \cdot (-1.645) \\
&= 0.0285 + (0.141) \cdot (-1.645) \\
&= -0.203 \\
q_{.05}^{e} &= \mu_p^e + \sigma_p^e \cdot (-1.645) \\
&= .0285 + (.057) \cdot (-1.645) \\
& \quad -0.063
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{VaR}_{.05}^{SBUX} &= \$100,000 \cdot \left( e^{q_{.05}^{\text{SBUX}}} - 1 \right) \\
&= \$100,000 \cdot \left( e^{-.203} - 1 \right) = -\$18,402 \\
\text{VaR}_{.05}^{e} &= \$100,000 \cdot \left( e^{q_{.05}^{e}} - 1 \right) \\
&= \$100,000 \cdot \left( e^{-.063} - 1 \right) = -\$6,124
\end{aligned}
$$

**Efficient Portfolios without Short-Sales**

Short Sale

- Borrow asset from broker and sell now
- To close short position, buy back asset and return to broker
- Profit if asset price drops after short sale
- If asset $i$ is sold short then

$$x_i < 0$$

where $x_i =$ share of wealth in asset $i$

## No Short Sale Restrictions

- Exchanges (e.g. NYSE, NASDAQ) may prevent short sales in some assets
- Some institutions (e.g. pension funds) are prevented from short-selling assets
- Certain accounts do not allow short sales (e.g. retirement accounts)
- Short selling often requires substantial credit qualifications

**Markowitz Algorithm with No Short Sales Restrictions**

$$\begin{aligned}
\min_{\mathbf{x}} \sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\
\mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \mu_p^0 \\
\mathbf{x}'\mathbf{1} &= 1 \\
x_i &\geq 0
\end{aligned}$$

Remarks:

- Problem must be solved numerically (e.g. using the Solver) :(
- Portfolio Frontier can no longer be constructed from any two efficient portfolios :(
- No short sale portfolio frontier must lie “inside” the portfolio frontier that allows short sales