1 Introduction to Portfolio Theory

Investment in Two Risky Assets

$$\begin{aligned} R_A &= \text{simple return on asset A} \\ R_B &= \text{simple return on asset B} \\ W_0 &= \text{initial wealth} \end{aligned}$$

Assumptions

- R_A and R_B are described by the CER model

$$R_i \sim \text{iid } N(\mu_i, \sigma_i^2),\ i = A, B$$
$$\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{cor}(R_A, R_B) = \rho_{AB}$$

- Investors like high $E[R_i] = \mu_i$
- Investors dislike high $\mathrm{var}(R_i) = \sigma_i^2$
- Investment horizon is one period (e.g., one month or one year)

Portfolios

$$\begin{aligned} x_A &= \text{share of wealth in asset A} = \frac{\text{\$ in A}}{W_0} \\ x_B &= \text{share of wealth in asset B} = \frac{\text{\$ in B}}{W_0} \end{aligned}$$

Long position

$$x_A,\ x_B > 0$$

Short position

$$x_A < 0 \text{ or } x_B < 0$$

Assumption: Allocate all wealth between assets A and B

$$x_A + x_B = 1$$

Portfolio Return

$$R_p = x_A R_A + x_B R_B$$

Portfolio Distribution

$$
\begin{aligned}
\mu_p &= E[R_p] = x_A\mu_A + x_B\mu_B \\
\sigma_p^2 &= \mathsf{var}(R_p) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B \\
& R_p \sim \text{iid } N(\mu_p, \sigma_p^2)
\end{aligned}
$$

End of Period Wealth

$$
\begin{aligned}
W_1 &= W_0(1 + R_p) = W_0(1 + x_AR_A + x_BR_B) \\
& W_1 \sim N(W_0(1 + \mu_p), \sigma_p^2 W_0^2)
\end{aligned}
$$

Portfolio Frontier

Vary investment shares x_A and x_B and compute resulting values of μ_p and σ_p^2. Plot μ_p against σ_p as functions of x_A and x_B

Example Data

$$
\begin{aligned}
\mu_A &= 0.175,\ \mu_B = 0.055 \\
\sigma_A^2 &= 0.067,\ \sigma_B^2 = 0.013 \\
\sigma_A &= 0.258,\ \sigma_B = 0.115 \\
\sigma_{AB} &= -0.004875, \\
\rho_{AB} &= \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.164
\end{aligned}
$$

Portfolio Frontier Shape

- Shape of portfolio frontier depends on correlation between assets A and B
- If $\rho_{AB} = -1$ then there exists portfolio shares x_A and x_B such that $\sigma_p^2 = 0$
- If $\rho_{AB} = 1$ then there is no benefit from diversification
- Diversification is beneficial even $0 < \rho_{AB} < 1$

Efficient Portfolios

Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios

- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios

- Which efficient portfolio an investor will hold depends on their risk preferences
  - Very risk averse investors will hold portfolios near the global minimum variance portfolio
  - Risk tolerant investors will hold portfolios that have high expected returns

2 Review of Optimization Techniques

Unconstrained Optimization

Example: finding the minimum of a univariate function

$$\begin{aligned} y &= f(x) = x^2 \\ \min_x y &= f(x) \end{aligned}$$

First order conditions for a minimum

$$\begin{aligned} 0 &= \frac{df(x)}{dx} = \frac{d}{dx}2x = 2 \cdot x \\ &\Rightarrow x = 0 \end{aligned}$$

Second order conditions for a minimum

$$0 < \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}2 \cdot x = 2$$

R function `optimize()`

- Use to optimize (maximize or minimize) functions of one variable

Example: Finding the minimum of a bivariate function

$$\begin{aligned} y &= f(x,z) = x^2 + z^2 \\ \min_{x,z} y &= f(x,z) \end{aligned}$$

First order conditions for a minimum

$$\begin{aligned} 0 &= \frac{\partial f(x,z)}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + z^2\right) = 2 \cdot x \\ 0 &= \frac{\partial f(x,z)}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + z^2\right) = 2 \cdot z \\ &\Rightarrow x = 0,\ z = 0 \end{aligned}$$

R function `optim()`

- Use to optimize (maximize or minimize) functions of one or more variables variable

Constrained Optimization

Example: Finding the minimum of a bivariate function subject to a linear constraint

$$\begin{aligned} y &= f(x,z) = x^2 + z^2 \\ \min_{x,z} y &= f(x,z) \\ s.t.\ x + z &= 1 \end{aligned}$$

Solution methods:

- Substitution
- Lagrange multipliers

Method of Substitution

Substitute $z = x - 1$ in $f(x, z)$ and solve univariate minimization

$$\begin{aligned} y &= f(x, x-1) = x^2 + (1-x)^2 \\ & \min_x \; f(x, x-1) \end{aligned}$$

First order conditions

$$\begin{aligned} 0 &= \frac{d}{dx}(x^2 + (1-x)) = 2x + 2(1-x)(-1) \\ &= 4x - 2 \\ &\Rightarrow x = 0.5 \end{aligned}$$

Solving for z

$$z = 1 - 0.5 = 0.5$$

Method of Lagrange Multipliers

Idea: Augment function to be minimized with extra terms to impose constraints

1. Put constraints in homogeneous form

$$x + z = 1 \Rightarrow x + z - 1 = 0$$

2. Form Lagrangian function

$$\begin{aligned} L(x, z, \lambda) &= x^2 + z^2 + \lambda(x + z - 1) \\ \lambda &= \text{Lagrange multiplier} \end{aligned}$$

3. Minimize Lagrangian function

$$\min_{x,z,\lambda} \; L(x, z, \lambda)$$

First order conditions

$$\begin{aligned} 0 &= \frac{\partial L(x, z, \lambda)}{\partial x} = 2 \cdot x + \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial z} = 2 \cdot z + \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial \lambda} = x + z - 1 \end{aligned}$$

We have three linear equations in three unknowns. Solving gives

$$\begin{aligned} 2x &= 2z = -\lambda \Rightarrow x = z \\ 2z - 1 &= 0 \Rightarrow z = 0.5, \; x = 0.5 \end{aligned}$$

Example: Finding the Global Minimum Variance Portfolio

Two methods for solution

- Analytic solution using Calculus
- Numerical solution
  - use the Solver in Excel
  - use R function `solve.QP()` in package quadprog.

Calculus Solution

Minimization problem

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$
$$s.t. \ x_A + x_B = 1$$

Use substitution method with

$$x_B = 1 - x_A$$

to give the univariate minimization

$$\min_{x_A} \ \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2 x_A (1 - x_A) \sigma_{AB}$$

First order conditions

$$
\begin{aligned}
0 &= \frac{d}{dx_A}\sigma_p^2 = 2x_A\sigma_A^2 - 2(1-x_A)\sigma_B^2 + 2\sigma_{AB}(1-2x_A) \\
&\Rightarrow \; x_A^{\mathsf{min}} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \; x_B^{\mathsf{min}} = 1 - x_A^{\mathsf{min}}
\end{aligned}
$$

Excel Solver Solution

The Solver is an Excel add-in, that can be used to numerically solve general linear and nonlinear optimization problems subject to equality or inequality constraints

- The solver is made by FrontLine Systems and is provided with Excel

- The solver add-in may not be installed in a "default installation" of Excel
  - Tools/Add-Ins and check the Solver Add-In box
  - If Solver Add-In box is not available, the Solver Add-In must be installed from original Excel installation CD

R Function quadprog()

- Use to optimize quadratic functions of several variables with linear equality or inequality constraints imposed on variables

3 Portfolios with a Risk Free Asset

Risk Free Asset

- Asset with fixed and known rate of return over investment horizon

- Usually use U.S. government T-Bill rate (horizons < 1 year) or T-Note rate (horizon > 1 yr)

- T-Bill or T-Note rate is only nominally risk free

Properties of Risk-Free Asset

$$\begin{aligned} R_f &= \text{return on risk-free asset} \\ E[R_f] &= r_f = \text{constant} \\ \text{var}(R_f) &= 0 \\ \text{cov}(R_f, R_i) &= 0,\ R_i = \text{return on any asset} \end{aligned}$$

Portfolios of Risky Asset and Risk Free Asset

$$\begin{aligned} x_f &= \text{share of wealth in T-Bills} \\ x_B &= \text{share of wealth in asset B} \\ x_f + x_B &= 1 \\ x_f &= 1 - x_B \end{aligned}$$

Portfolio return

$$
\begin{aligned}
R_p &= x_f r_f + x_B R_B \\
&= (1 - x_B) r_f + x_B R_B \\
&= r_f + x_B (R_B - r_f)
\end{aligned}
$$

Portfolio excess return

$$R_p - r_f = x_B (R_B - r_f)$$

Portfolio Distribution

$$
\begin{aligned}
\mu_p &= E[R_p] = r_f + x_B(\mu_B - r_f) \\
\sigma_p^2 &= \mathrm{var}(R_p) = x_B^2 \sigma_B^2 \\
\sigma_p &= x_B \sigma_B \\
&\quad R_p \sim N(\mu_p, \sigma_p^2)
\end{aligned}
$$

Risk Premium

$$\mu_p - r_f = x_B(\mu_B - r_f)$$

$$= \text{expected return over T-Bill}$$

Leveraged Investment

$$x_f < 0,\ x_B > 1$$

Borrow at T-Bill rate to buy more of asset B

Result: Leverage increases portfolio risk

$$\sigma_p = x_B \sigma_B$$
$$x_B \uparrow \ \Rightarrow \sigma_p \uparrow$$

Determining Portfolio Frontier

Goal: Plot μ_p vs. σ_p

$$\begin{aligned}
\sigma_p &= x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B} \\
\mu_p &= r_f + x_B(\mu_B - r_f) \\
&= r_f + \frac{\sigma_p}{\sigma_B}(\mu_B - r_f) \\
&= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p
\end{aligned}$$

where

$$\left(\frac{\mu_B - r_f}{\sigma_B}\right) = \text{Asset B Sharpe Ratio}$$
$$= \text{excess expected return per unit risk}$$

Efficient Portfolios with 2 Risky Assets and a Risk Free Asset

Investment in 2 Risky Assets and T-Bill

$$
\begin{aligned}
R_A &= \text{cc return on asset A} \\
R_B &= \text{cc return on asset B} \\
R_f &= r_f = \text{ return on T-Bill}
\end{aligned}
$$

Assumptions

- R_A and R_B are described by the CER model

$$R_i \sim iid\ N(\mu_i, \sigma_i^2),\ i = A, B$$
$$\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{corr}(R_A, R_B) = \rho_{AB}$$

Mutual Fund Separation Theorem

Efficient portfolios are combinations of two portfolios (mutual funds)

- T-Bill portfolio
- Tangency portfolio - portfolio of assets A and B that has the maximum Shape slope

Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.

Finding the tangency portfolio

$$\max_{x_A,\, x_B} \frac{\mu_p - r_f}{\sigma_p} \text{ subject to}$$

$$\begin{aligned} \mu_p &= x_A\mu_A + x_B\mu_B \\ \sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB} \\ 1 &= x_A + x_B \end{aligned}$$

Using the substitution method it can be shown that

$$x_A^{\text{tan}} = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}$$

$$x_B^{\text{tan}} = 1 - x_A^{\text{tan}}$$

Portfolio characteristics

$$
\begin{aligned}
\mu_p^{\mathsf{tan}} &= x_A^{\mathsf{tan}}\mu_A + x_B^{\mathsf{tan}}\mu_B \\
\left(\sigma_p^{\mathsf{tan}}\right)^2 &= \left(x_A^{\mathsf{tan}}\right)^2\sigma_A^2 + \left(x_B^{\mathsf{tan}}\right)^2\sigma_B^2 + 2x_A^{\mathsf{tan}}x_B^{\mathsf{tan}}\sigma_{AB}
\end{aligned}
$$

Efficient Portfolios: tangency portfolio plus T-Bills

$$\begin{aligned} x_{\mathsf{tan}} &= \text{share of wealth in tangency portfolio} \\ x_f &= \text{share of wealth in T-bills} \\ x_{\mathsf{tan}} + x_f &= 1 \\ \mu_p^e &= r_f + x_{\mathsf{tan}}(\mu_p^{\mathsf{tan}} - r_f) \\ \sigma_p^e &= x_{\mathsf{tan}}\sigma_p^{\mathsf{tan}} \end{aligned}$$

Result: The weights x_{tan} and x_f are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills
- Risk tolerant investors hold mostly tangency portfolio

Example

For the two asset example, the tangency portfolio is

$$\begin{aligned}
x_A^{\text{tan}} &= .46,\ x_B^{\text{tan}} = 0.54 \\
\mu_p^{\text{tan}} &= (.46)(.175) + (.54)(.055) = 0.11 \\
\left(\sigma_p^{\text{tan}}\right)^2 &= (.46)^2(.067) + (.54)^2(.013) \\
&\quad +2(.46)(.54)(-.005) \\
&= 0.015 \\
\sigma_p^{\text{tan}} &= \sqrt{.015} = 0.124
\end{aligned}$$

Efficient portfolios have the following characteristics

$$\begin{aligned}
\mu_p^e &= r_f + x_{\text{tan}}(\mu_p^{\text{tan}} - r_f) \\
&= 0.03 + x_{\text{tan}}(0.11 - 0.03) \\
\sigma_p^e &= x_{\text{tan}}\sigma_p^{\text{tan}} \\
&= x_{\text{tan}}(0.124)
\end{aligned}$$

Problem: Find the efficient portfolio that has the same risk (SD) as asset B? That is, determine x_{tan} and x_f such that

$$\sigma_p^e = \sigma_B = 0.114 = \text{target risk}$$

Note: The efficient portfolio will have a higher expected return than asset B

Solution:

$$
\begin{aligned}
.114 &= \sigma_p^e = x_{\text{tan}}\sigma_p^{\text{tan}} \\
&= x_{\text{tan}}(.124) \\
&\Rightarrow x_{\text{tan}} = \frac{.114}{.124} = .92 \\
x_f &= 1 - x_{\text{tan}} = .08
\end{aligned}
$$

Efficient portfolio with same risk as asset B has

$$
\begin{aligned}
(.92)(.46) &= .42 \text{ in asset A} \\
(.92)(.54) &= .50 \text{ in asset B} \\
& \quad\ .08 \text{ in T-Bills}
\end{aligned}
$$

If $r_f = 0.03$, then expected Return on efficient portfolio is

$$
\mu_p^e = .03 + (.92)(.11) = .13
$$

Problem: Assume that $r_f = 0.03$. Find the efficient portfolio that has the same expected return as asset B. That is, determine $x_{\textsf{tan}}$ and x_f such that

$$\mu_p^e = \mu_B = 0.055 = \text{target expected return}$$

Note: The efficient portfolio will have a lower SD than asset B

Solution:

$$
\begin{aligned}
0.055 &= \mu_p^e = 0.03 + x_{\mathsf{tan}}(.11 - .03) \\
x_{\mathsf{tan}} &= \frac{0.055 - 0.03}{.11 - .03} = .31 \\
x_f &= 1 - x_{\mathsf{tan}} = .69
\end{aligned}
$$

Efficient portfolio with same expected return as asset B has

$$
\begin{aligned}
(.31)(.46) &= .14 \text{ in asset A} \\
(.31)(.54) &= .17 \text{ in asset B} \\
&\ .69 \text{ in T-Bills}
\end{aligned}
$$

The SD of the efficient portfolio is

$$\sigma_p^e = .31(.124) = .038$$