1 Matrix Algebra Review

Matrix

$$\underset{(n\times m)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$n = \text{\# of rows, } m = \text{\# of columns}$$

$$\text{Square matrix} : n = m$$

Vector

$$\underset{(n\times 1)}{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Remarks

- R is a matrix oriented programming language
- Excel can handle matrices and vectors in formulas and some functions
- Excel has special functions for working with matrices. There are called *array* functions. Must use

  `<ctrl>-<shift>-<enter>`

  to evaluate array function

1.1 Transpose of a Matrix

Interchange rows and columns of a matrix

$$\underset{(m\times n)}{\mathbf{A}}' = \text{transpose of } \underset{(n\times m)}{\mathbf{A}}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ \mathbf{A}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

R function

```
t(A)
```

Excel function

TRANSPOSE(matrix)

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1.2 Symmetric Matrix

A square matrix $\mathbf{A}$ is symmetric if

$$\mathbf{A} = \mathbf{A}'$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \ \mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Remark: Covariance and correlation matrices are symmetric

1.3 Basic Matrix Operations

1.3.1 Addition and Subtraction (element-by-element)

$$
\begin{aligned}
\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} &= \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} &= \begin{bmatrix} 4-2 & 9-0 \\ 2-0 & 1-7 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 9 \\ 2 & -6 \end{bmatrix}
\end{aligned}
$$

1.3.2 Scalar Multiplication (element-by-element)

$$
\begin{aligned}
c &= 2 = \text{scalar} \\
A &= \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} \\
2 \cdot A &= \begin{bmatrix} 2 \cdot 3 & 2 \cdot (-1) \\ 2 \cdot 0 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 0 & 10 \end{bmatrix}
\end{aligned}
$$

1.3.3 Matrix Multiplication (not element-by-element)

$$\underset{(3\times2)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad \underset{(2\times3)}{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Note: $\mathbf{A}$ and $\mathbf{B}$ are comformable matrices: $\#$ of columns in $A = \#$ of rows in B

$$\underset{(3\times2)}{\mathbf{A}} \cdot \underset{(2\times3)}{\mathbf{B}}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}$$

Remark: In general,

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

R operator

```
A%*%B
```

Excel function

`MMULT`(matrix1, matrix2)

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1.3.4 Identity Matrix

The $n-$ dimensional identity matrix has all diagonal elements equal to 1, and all off diagonal elements equal to 0.

Example

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: The identity matrix plays the roll of "1" in matrix algebra

$$\begin{aligned} \mathbf{I}_2 \cdot \mathbf{A} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ 0 + a_{21} & 0 + a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \mathbf{A} \end{aligned}$$

Similarly

$$\mathbf{A} \cdot \mathbf{I}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}$$

R function

```
diag(n)
```

creates $n-$ dimensional identity matrix

1.3.5 Matrix Inverse

Let $\underset{(n\times n)}{\mathbf{A}}$ = square matrix. $\mathbf{A}^{-1}$ = "inverse of $\mathbf{A}$" satisfies

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A} &= \mathbf{I}_n \\ \mathbf{A}\mathbf{A}^{-1} &= \mathbf{I}_n \end{aligned}$$

Remark: $\mathbf{A}^{-1}$ is similar to the inverse of a number:

$$\begin{aligned} a &= 2,\ a^{-1} = \frac{1}{2} \\ a \cdot a^{-1} &= 2 \cdot \frac{1}{2} = 1 \\ a^{-1} \cdot a &= \frac{1}{2} \cdot 2 = 1 \end{aligned}$$

R function

```
solve(A)
```

Excel function

MINVERSE(matrix)

<ctrl>-<shift>-<enter>

Representing Systems of Linear Equations Using Matrix Algebra

Consider the system of two linear equations

$$\begin{aligned} x + y &= 1 \\ 2x - y &= 1 \end{aligned}$$

The equations represent two straight lines which intersect at the point

$$x = \frac{2}{3},\ y = \frac{1}{3}$$

Matrix algebra representation:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$\mathbf{A} \cdot \mathbf{z} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can solve for z by multiplying both sides by $\mathbf{A}^{-1}$

$$\begin{aligned} \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{z} &= \mathbf{A}^{-1} \cdot \mathbf{b} \\ &\Longrightarrow \mathbf{I} \cdot \mathbf{z} = \mathbf{A}^{-1} \cdot \mathbf{b} \\ &\Longrightarrow \mathbf{z} = \mathbf{A}^{-1} \cdot \mathbf{b} \end{aligned}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Remark: As long as we can determine the elements in $\mathbf{A}^{-1}$, we can solve for the values of x and y in the vector $\mathbf{z}$. Since the system of linear equations has a solution as long as the two lines intersect, we can determine the elements in $\mathbf{A}^{-1}$ provided the two lines are not parallel.

There are general numerical algorithms for finding the elements of $\mathbf{A}^{-1}$ and typical spreadsheet programs like Excel have these algorithms available. However, if $\mathbf{A}$ is a (2×2) matrix then there is a simple formula for $\mathbf{A}^{-1}$. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

where

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12}$$

Let's apply the above rule to find the inverse of $\mathbf{A}$ in our example:

$$\mathbf{A}^{-1} = \frac{1}{-1-2} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}.$$

Notice that

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Our solution for z is then

$$\begin{aligned} \mathbf{z} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

so that $x = \frac{2}{3}$ and $y = \frac{1}{3}$.

In general, if we have n linear equations in n unknown variables we may write the system of equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

which we may then express in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or

$$\underset{(n\times n)}{\mathbf{A}} \cdot \underset{(n\times 1)}{\mathbf{x}} = \underset{(n\times 1)}{\mathbf{b}}.$$

The solution to the system of equations is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

where $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{I}$ is the $(n \times n)$ identity matrix. If the number of equations is greater than two, then we generally use numerical algorithms to find the elements in $\mathbf{A}^{-1}$.

1.4 Representing Summation Using Matrix Notation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\underset{(n\times 1)}{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \underset{(n\times 1)}{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then

$$\begin{aligned}\mathbf{x}'\mathbf{1} &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i\end{aligned}$$

Equivalently

$$\begin{aligned}\mathbf{1}'\mathbf{x} &= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i\end{aligned}$$

Sum of Squares

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\mathbf{x}'\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1^2 + x_2^2 + \cdots + x_n^2 = \sum_{i=1}^{n} x_i^2$$

Sums of cross products

$$
\begin{aligned}
\sum_{i=1}^{n} x_i y_i &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\
\mathbf{x}'\mathbf{y} &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\
&= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \\
&= \mathbf{y}'\mathbf{x}
\end{aligned}
$$

R function

```
crossprod(x,y)
```

Excel function

```
MMULT(TRANSPOSE(x),y)
MMULT(TRANSPOSE(y),x)
<ctrl>-<shift>-<enter>
```

Portfolio Math with Matrix Algebra

Three Risky Asset Example

Let R_i $(i = A, B, C)$ denote the return on asset i and assume that R_i follows CER model:

$$
\begin{aligned}
R_i &\sim iid\ N(\mu_i, \sigma_i^2) \\
\mathrm{cov}(R_i, R_j) &= \sigma_{ij}
\end{aligned}
$$

Portfolio "$\mathbf{x}$"

$$x_i = \text{share of wealth in asset } i$$
$$x_1 + x_2 + x_3 = 1$$

Portfolio return

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$

Portfolio expected return

$$\mu_{p,x} = E[R_{p,x}] = x_A\mu_A + x_B\mu_B + x_C\mu_C$$

Portfolio variance

$$\begin{aligned}\sigma^2_{p,x} &= \text{var}(R_{p,x}) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\ &\quad + 2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}\end{aligned}$$

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma^2_{p,x})$$

Matrix Algebra Representation

$$\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}$$

Portfolio weights sum to 1

$$\begin{aligned} \mathbf{x}'\mathbf{1} &= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= x_1 + x_2 + x_3 = 1 \end{aligned}$$

Portfolio return

$$\begin{aligned} R_{p,x} &= \mathbf{x}'\mathbf{R} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix} \\ &= x_A R_A + x_B R_B + x_C R_C \\ &= \mathbf{R}'\mathbf{x} \end{aligned}$$

Portfolio expected return

$$\begin{aligned} \mu_{p,x} &= \mathbf{x}'\boldsymbol{\mu} = \begin{pmatrix} x_A & x_B & x_X \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} \\ &= x_A \mu_A + x_B \mu_B + x_C \mu_C \\ &= \boldsymbol{\mu}'\mathbf{x} \end{aligned}$$

Excel formula

```
MMULT(transpose(xvec),muvec)
<ctrl>-<shift>-<enter>
```

R formula

```
crossprod(x,mu)
t(x)%*%mu
```

Portfolio variance

$$
\begin{aligned}
\sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\
&= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\
&\quad +2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}
\end{aligned}
$$

Excel formulas

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))
MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)
<ctrl>-<shift>-<enter>
```

R formulas

```
t(x)%*%sigma%*%x
```

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma^2_{p,x})$$

Covariance Between 2 Portfolio Returns

2 portfolios

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix}$$

$$\mathbf{x}'\mathbf{1} = 1, \ \mathbf{y}'\mathbf{1} = 1$$

Portfolio returns

$$\begin{aligned} R_{p,x} &= \mathbf{x}'\mathbf{R} \\ R_{p,y} &= \mathbf{y}'\mathbf{R} \end{aligned}$$

Covariance

$$\begin{aligned} \text{cov}(R_{p,x}, R_{p,y}) &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} \\ &= \mathbf{y}'\mathbf{\Sigma}\mathbf{x} \end{aligned}$$

Excel formula

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))
MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))
<ctrl>-<shift>-<enter>
```

R formula

```
t(x)%*%sigma%*%y
```