

# Chapter 1

## Introduction to Portfolio Theory

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This chapter introduces modern portfolio theory in a simplified setting where there are only two risky assets and a single risk-free asset.

### 1.1 Portfolios of Two Risky Assets

Consider the following investment problem. We can invest in two non-dividend paying stocks Amazon (A) and Boeing (B) over the next month. Let  $R_A$  denote monthly simple return on Amazon and  $R_B$  denote the monthly simple return on stock Boeing. These returns are to be treated as random variables because the returns will not be realized until the end of the month. We assume that the returns  $R_A$  and  $R_B$  are jointly normally distributed, and that we have the following information about the means, variances and covariances of the probability distribution of the two returns:

$$\mu_A = E[R_A], \sigma_A^2 = \text{var}(R_A), \mu_B = E[R_B], \sigma_B^2 = \text{var}(R_B), \quad (1.1)$$

$$\sigma_{AB} = \text{cov}(R_A, R_B), \rho_{AB} = \text{cor}(R_A, R_B) = \frac{\sigma_{AB}}{\sigma_A \sigma_B}. \quad (1.2)$$

We assume that these values are taken as given. Typically, they are estimated from historical return data for the two stocks. However, they can also be subjective guesses by an analyst.

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The expected returns,  $\mu_A$  and  $\mu_B$ , are our best guesses for the monthly returns on each of the stocks. However, because the investment returns are random variables we must recognize that the realized returns may be different from our expectations. The variances,  $\sigma_A^2$  and  $\sigma_B^2$ , provide measures of the uncertainty associated with these monthly returns. We can also think of the variances as measuring the risk associated with the investments. Assets with high return variability (or volatility) are often thought to be risky, and assets with low return volatility are often thought to be safe. The covariance  $\sigma_{AB}$  gives us information about the direction of any linear dependence between returns. If  $\sigma_{AB} > 0$  then the two returns tend to move in the same direction; if  $\sigma_{AB} < 0$  the returns tend to move in opposite directions; if  $\sigma_{AB} = 0$  then the returns tend to move independently. The strength of the dependence between the returns is measured by the correlation coefficient  $\rho_{AB}$ . If  $\rho_{AB}$  is close to one in absolute value then returns mimic each other extremely closely, whereas if  $\rho_{AB}$  is close to zero then the returns may show very little relationship.

### **Example 1** *Two risky asset portfolio information*

Table 1.1 gives annual return distribution parameters for two hypothetical assets A and B. Asset A is the high risk asset with an annual return of  $\mu_A = 17.5\%$  and annual standard deviation of  $\sigma_A = 25.8\%$ . Asset B is a lower risk asset with annual return  $\mu_B = 5.5\%$  and annual standard deviation of  $\sigma_B = 11.5\%$ . The assets are assumed to be slightly negatively correlated with correlation coefficient  $\rho_{AB} = -0.164$ . Given the standard deviations and the correlation, the covariance can be determined from  $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = (-0.164)(0.0258)(0.115) = -0.004875$ . In R, the example data is

```
> mu.A = 0.175
> sig.A = 0.258
> sig2.A = sig.A^2
> mu.B = 0.055
> sig.B = 0.115
> sig2.B = sig.B^2
> rho.AB = -0.164
> sig.AB = rho.AB*sig.A*sig.B
```

■

$\mu_A$	$\mu_B$	$\sigma_A^2$	$\sigma_B^2$	$\sigma_A$	$\sigma_B$	$\sigma_{AB}$	$\rho_{AB}$
0.175	0.055	0.06656	0.01323	0.258	0.115	-0.004866	-0.164

Table 1.1: Example data for two asset portfolio.

The portfolio problem is set-up as follows. We have a given amount of initial wealth  $W_0$ , and it is assumed that we will exhaust all of our wealth between investments in the two stocks. The investment problem is to decide how much wealth to put in asset A and how much to put in asset B. Let  $x_A$  denote the share of wealth invested in stock A, and  $x_B$  denote the share of wealth invested in stock B. The values of  $x_A$  and  $x_B$  can be positive or negative. Positive values denote *long* positions (purchases) in the assets. Negative values denote *short* positions (sales).<sup>1</sup> Since all wealth is put into the two investments it follows that  $x_A + x_B = 1$ . If asset A is shorted, then it is assumed that the proceeds of the short sale are used to purchase more of asset B. Therefore, to solve the investment problem we must choose the values of  $x_A$  and  $x_B$ .

Our investment in the two stocks forms a *portfolio*, and the shares  $x_A$  and  $x_B$  are referred to as *portfolio shares* or weights. The return on the portfolio over the next month is a random variable, and is given by

$$R_p = x_A R_A + x_B R_B, \quad (1.3)$$

which is a linear combination or weighted average of the random variables  $R_A$  and  $R_B$ . Since  $R_A$  and  $R_B$  are assumed to be normally distributed,  $R_p$  is also normally distributed. We use the properties of linear combinations of random variables to determine the mean and variance of this distribution.

### 1.1.1 Portfolio expected return and variance

The distribution of the return on the portfolio (1.3) is a normal with mean, variance and standard deviation given by

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<sup>1</sup>To short an asset one borrows the asset, usually from a broker, and then sells it. The proceeds from the short sale are usually kept on account with a broker and there often restrictions that prevent the use of these funds for the purchase of other assets. The short position is closed out when the asset is repurchased and then returned to original owner. If the asset drops in value then a gain is made on the short sale and if the asset increases in value a loss is made.

$$\mu_p = E[R_p] = x_A\mu_A + x_B\mu_B, \quad (1.4)$$

$$\sigma_p^2 = \text{var}(R_p) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}, \quad (1.5)$$

$$\sigma_p = \text{SD}(R_p) = \sqrt{x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}}. \quad (1.6)$$

That is,

$$R_p \sim N(\mu_p, \sigma_p^2).$$

The results (1.4) and (1.5) are so important to portfolio theory that it is worthwhile to review the derivations. For the first result (1.4), we have

$$E[R_p] = E[x_AR_A + x_BR_B] = x_AE[R_A] + x_BE[R_B] = x_A\mu_A + x_B\mu_B,$$

by the linearity of the expectation operator. For the second result (1.5), we have

$$\begin{aligned} \text{var}(R_p) &= E[(R_p - \mu_p)^2] = E[(x_A(R_A - \mu_A) + x_B(R_B - \mu_B))^2] \\ &= E[x_A^2(R_A - \mu_A)^2 + x_B^2(R_B - \mu_B)^2 + 2x_Ax_B(R_A - \mu_A)(R_B - \mu_B)] \\ &= x_A^2E[(R_A - \mu_A)^2] + x_B^2E[(R_B - \mu_B)^2] + 2x_Ax_BE[(R_A - \mu_A)(R_B - \mu_B)] \\ &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}. \end{aligned}$$

Notice that the variance of the portfolio is a weighted average of the variances of the individual assets plus two times the product of the portfolio weights times the covariance between the assets. If the portfolio weights are both positive then a positive covariance will tend to increase the portfolio variance, because both returns tend to move in the same direction, and a negative covariance will tend to reduce the portfolio variance. Thus finding assets with negatively correlated returns can be very beneficial when forming portfolios because risk, as measured by portfolio standard deviation, is reduced. What is perhaps surprising is that forming portfolios with positively correlated assets can also reduce risk as long as the correlation is not too large.

### **Example 2** *Two asset portfolios*

Consider creating some portfolios using the asset information in Table 1.1. The first portfolio is an equally weighted portfolio with  $x_A = x_B = 0.5$ . Using

(1.4)-(1.6), we have

$$\begin{aligned}\mu_p &= (0.5) \cdot (0.175) + (0.5) \cdot (0.055) = 0.115 \\ \sigma_p^2 &= (0.5)^2 \cdot (0.067) + (0.5)^2 \cdot (0.013) \\ &\quad + 2 \cdot (0.5)(0.5)(-0.004866) \\ &= 0.01751 \\ \sigma_p &= \sqrt{0.01751} = 0.1323\end{aligned}$$

This portfolio has expected return half-way between the expected returns on assets A and B, but the portfolio standard deviation is less than half-way between the asset standard deviations. This reflects risk reduction via diversification. In R, the portfolio parameters are computed using

```
> x.A = 0.5
> x.B = 0.5
> mu.p1 = x.A*mu.A + x.B*mu.B
> sig2.p1 = x.A^2 * sig2.A + x.B^2 * sig2.B + 2*x.A*x.B*sig.AB
> sig.p1 = sqrt(sig2.p)
> mu.p1
[1] 0.115
> sig2.p1
[1] 0.01751
> sig.p1
[1] 0.1323
```

Next, consider a long-short portfolio with  $x_A = 1.5$  and  $x_B = -0.5$ . In this portfolio, asset B is sold short and the proceeds of the short sale are used to leverage the investment in asset A. The portfolio characteristics are

$$\begin{aligned}\mu_p &= (1.5) \cdot (0.175) + (-0.5) \cdot (0.055) = 0.235 \\ \sigma_p^2 &= (1.5)^2 \cdot (0.067) + (-0.5)^2 \cdot (0.013) \\ &\quad + 2 \cdot (1.5)(-0.5)(-0.004866) \\ &= 0.1604 \\ \sigma_p &= \sqrt{0.1604} = 0.4005\end{aligned}$$

This portfolio has both a higher expected return and standard deviation than asset A. In R, the portfolio parameters are computed using

```

> x.A = 1.5
> x.B = -0.5
> mu.p2 = x.A*mu.A + x.B*mu.B
> sig2.p2 = x.A^2 * sig2.A + x.B^2 * sig2.B + 2*x.A*x.B*sig.AB
> sig.p2 = sqrt(sig2.p2)
> mu.p2
[1] 0.235
> sig2.p2
[1] 0.1604
> sig.p2
[1] 0.4005

```

■

### 1.1.2 Portfolio Value-at-Risk

Consider an initial investment of  $\$W_0$  in the portfolio of assets A and B with return given by (1.3), expected return given by (1.4) and variance given by (1.5). Then  $R_p \sim N(\mu_p, \sigma_p^2)$ . For  $\alpha \in (0, 1)$ , the  $\alpha \times 100\%$  portfolio value-at-risk is given by

$$\text{VaR}_{p,\alpha} = q_{p,\alpha}^R W_0, \quad (1.7)$$

where  $q_{p,\alpha}^R$  is the  $\alpha$  quantile of the distribution of  $R_p$  and is given by

$$q_{p,\alpha}^R = \mu_p + \sigma_p q_\alpha^z, \quad (1.8)$$

where  $q_\alpha^z$  is the  $\alpha$  quantile of the standard normal distribution.<sup>2</sup>

What is the relationship between portfolio VaR and the individual asset VaRs? Is portfolio VaR a weighted average of the individual asset VaRs? In general, portfolio VaR is not a weighted average of the asset VaRs. To see this consider the portfolio weighted average of the individual asset return quantiles

$$\begin{aligned} x_A q_{A,\alpha}^R + x_B q_{B,\alpha}^R &= x_A (\mu_A + \sigma_A q_\alpha^z) + x_B (\mu_B + \sigma_B q_\alpha^z) \\ &= x_A \mu_A + x_B \mu_B + (x_A \sigma_A + x_B \sigma_B) q_\alpha^z \\ &= \mu_p + (x_A \sigma_A + x_B \sigma_B) q_\alpha^z. \end{aligned} \quad (1.9)$$

---

<sup>2</sup>If  $R_p$  is a continuously compounded return then the implied simple return quantile is  $q_{p,\alpha}^R = \exp(\mu_p + \sigma_p q_\alpha^z) - 1$ .

The weighted asset quantile (1.9) is not equal to the portfolio quantile (1.8) unless  $\rho_{AB} = 1$ . Hence, weighted asset VaR is in general not equal to portfolio VaR because the quantile (1.9) ignores the correlation between  $R_A$  and  $R_B$ .

**Example 3** *Portfolio VaR*

Consider an initial investment of  $W_0 = \$100,000$ . Assuming that returns are simple, the 5% VaRs on assets A and B are

$$\begin{aligned}\text{VaR}_{A,0.05} &= q_{0.05}^{R_A} W_0 = (\mu_A + \sigma_A q_{0.05}^z) W_0 \\ &= (0.175 + 0.258(-1.645)) \cdot 100,000 = -24,937, \\ \text{VaR}_{B,0.05} &= q_{0.05}^{R_B} W_0 = (\mu_B + \sigma_B q_{0.05}^z) W_0 \\ &= (0.055 + 0.115(-1.645)) \cdot 100,000 = -13,416.\end{aligned}$$

The 5% VaR on the equal weighted portfolio with  $x_A = x_B = 0.5$  is

$$\begin{aligned}\text{VaR}_{p,0.05} &= q_{0.05}^{R_p} W_0 = (\mu_p + \sigma_p q_{0.05}^z) W_0 \\ &= (0.115 + 0.1323(-1.645)) \cdot 100,000 = -10,268,\end{aligned}$$

and the weighted average of the individual asset VaRs is

$$x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05} = 0.5(-24,937) + 0.5(-13,416) = -19,177.$$

The 5% VaR on the long-short portfolio with  $x_A = 1.5$  and  $x_B = -0.5$  is

$$\text{VaR}_{p,0.05} = q_{0.05}^{R_p} W_0 = (0.235 + 0.4005(-1.645)) \cdot 100,000 = -42,371,$$

and the weighted average of the individual asset VaRs is

$$x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05} = 1.5(-24,937) - 0.5(-13,416) = -30,698$$

Notice that  $\text{VaR}_{p,0.05} \neq x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05}$  because  $\rho_{AB} \neq 1$ .

Using R, these computations are

```
> w0 = 100000
> VaR.A = (mu.A + sig.A*qnorm(0.05))*w0
> VaR.A
[1] -24937
> VaR.B = (mu.B + sig.B*qnorm(0.05))*w0
> VaR.B
```

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```
[1] -13416
> VaR.p1 = (mu.p1 + sig.p1*qnorm(0.05))*w0
> VaR.p1
[1] -10268
> x.A*VaR.A + x.B*VaR.B
[1] -19177
> VaR.p2 = (mu.p2 + sig.p2*qnorm(0.05))*w0
> VaR.p2
[1] -42371
> x.A*VaR.A + x.B*VaR.B
[1] -30698
```



### **Example 4** *Create R function to compute portfolio VaR*

The previous example used repetitive R calculations to compute the 5% VaR of an investment. An alternative approach is to first create an R function to compute the VaR given  $\mu$ ,  $\sigma$ ,  $\alpha$  (VaR probability) and  $W_0$ , and then apply the function using the inputs of the different assets. A simple function to compute VaR based on normally distributed asset returns is

```
normalVaR <- function(mu, sigma, w0, tail.prob = 0.01, invert=FALSE) {
  ## compute normal VaR for collection of assets given mean and sd vector
  ## inputs:
  ## mu          n x 1 vector of expected returns
  ## sigma       n x 1 vector of standard deviations
  ## w0          scalar initial investment in $
  ## tail.prob   scalar tail probability
  ## invert      logical. If TRUE report VaR as positive number
  ## output:
  ## VaR        n x 1 vector of left tail return quantiles
  ## References:
  ## Jorian (2007) pg 111.
  if ( length(mu) != length(sigma) )
    stop("mu and sigma must have same number of elements")
  if ( tail.prob < 0 || tail.prob > 1 )
    stop("tail.prob must be between 0 and 1")
  VaR = w0*(mu + sigma*qnorm(tail.prob))
}
```



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```
    if (invert) {  
      VaR = -VaR  
    }  
    return(VaR)  
  }  
}
```

Using the `normalVaR()` function, the 5% VaR values of asset A, B and equally weighted portfolio are:

```
> normalVaR(mu=c(mu.A, mu.B, mu.p1),  
+           sigma=c(sig.A, sig.B, sig.p1),  
+           w0=100000, tail.prob=0.05)  
[1] -24937 -13416 -10268
```



## 1.2 Efficient portfolios with two risky assets

In this section we describe how mean-variance efficient portfolios are constructed. First we make the following assumptions regarding the probability distribution of asset returns and the behavior of investors:

1. Returns are covariance stationary and ergodic, and jointly normally distributed over the investment horizon. This implies that means, variances and covariances of returns are constant over the investment horizon and completely characterize the joint distribution of returns.
2. Investors know the values of asset return means, variances and covariances.
3. Investors only care about portfolio expected return and portfolio variance. Investors like portfolios with high expected return but dislike portfolios with high return variance.

Given the above assumptions we set out to characterize the set of *efficient portfolios*: those portfolios that have the highest expected return for a given level of risk as measured by portfolio variance. These are the portfolios that investors are most interested in holding.

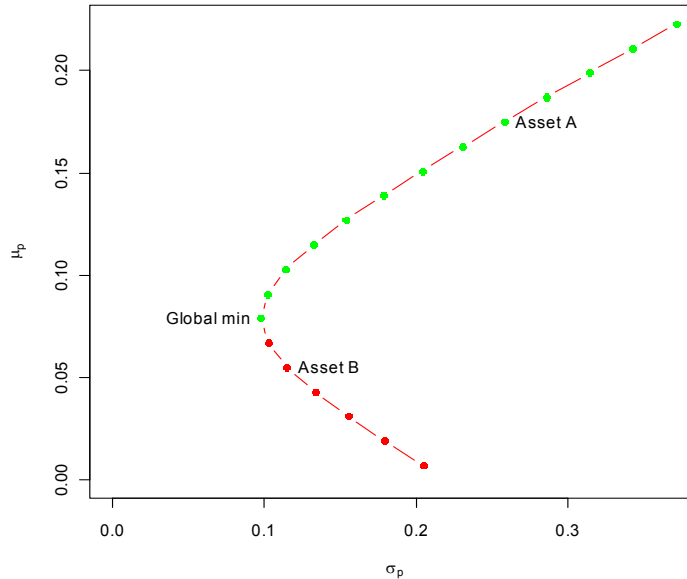


Figure 1.1: Portfolio frontier of example data.

For illustrative purposes we will show calculations using the data in the Table 1.1. The collection of all *feasible* portfolios, or the *investment possibilities set*, in the case of two assets is simply all possible portfolios that can be formed by varying the portfolio weights  $x_A$  and  $x_B$  such that the weights sum to one ( $x_A + x_B = 1$ ). We summarize the expected return-risk (mean-variance) properties of the feasible portfolios in a plot with portfolio expected return,  $\mu_p$ , on the vertical axis and portfolio standard deviation,  $\sigma_p$ , on the horizontal axis. The portfolio standard deviation is used instead of variance because standard deviation is measured in the same units as the expected value (recall, variance is the average squared deviation from the mean).

**Example 5** *Investment possibilities set for example data*

The investment possibilities set or portfolio frontier for the data in Table 1.1 is illustrated in Figure 1.1. Here the portfolio weight on asset A,  $x_A$ , is varied from -0.4 to 1.4 in increments of 0.1 and, since  $x_B = 1 - x_A$ , the weight on asset B then varies from 1.4 to -0.4. This gives us 18 portfolios with weights

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$(x_A, x_B) = (-0.4, 1.4), (-0.3, 1.3), \dots, (1.3, -0.3), (1.4, -0.4)$ . For each of these portfolios we use the formulas (1.4) and (1.6) to compute  $\mu_p$  and  $\sigma_p$ . We then plot these values. In R, the computations are

```
> x.A = seq(from=-0.4, to=1.4, by=0.1)
> x.B = 1 - x.A
> mu.p = x.A*mu.A + x.B*mu.B
> sig2.p = x.A^2 * sig2.A + x.B^2 * sig2.B + 2*x.A*x.B*sig.AB
> sig.p = sqrt(sig2.p)
> plot(sig.p, mu.p, type="b", pch=16,
+       ylim=c(0, max(mu.p)), xlim=c(0, max(sig.p)),
+       xlab=expression(sigma[p]), ylab=expression(mu[p]),
+       col=c(rep("red", 6), rep("green", 13)))
> plot(sig.p, mu.p, type="b", pch=16,
+       ylim=c(0, max(mu.p)), xlim=c(0, max(sig.p)),
+       xlab=expression(sigma[p]), ylab=expression(mu[p]),
+       col=c(rep("red", 6), rep("green", 13)))
> text(x=sig.A, y=mu.A, labels="Asset A", pos=4)
> text(x=sig.B, y=mu.B, labels="Asset B", pos=4)
```



Notice that the plot in  $(\mu_p, \sigma_p)$ -space looks like a parabola turned on its side (in fact, it is one side of a hyperbola). Since it is assumed that investors desire portfolios with the highest expected return,  $\mu_p$ , for a given level of risk,  $\sigma_p$ , combinations that are in the upper left corner are the best portfolios and those in the lower right corner are the worst. Notice that the portfolio at the bottom of the parabola has the property that it has the smallest variance among all feasible portfolios. Accordingly, this portfolio is called the *global minimum variance* portfolio.

*Efficient portfolios* are those with the highest expected return for a given level of risk. These portfolios are colored green in Figure 1.1. *Inefficient portfolios* are then portfolios such that there is another feasible portfolio that has the same risk ( $\sigma_p$ ) but a higher expected return ( $\mu_p$ ). These portfolios are colored red in Figure 1.1. From Figure 1.1 it is clear that the inefficient portfolios are the feasible portfolios that lie below the global minimum variance portfolio, and the efficient portfolios are those that lie above the global minimum variance portfolio.

### 1.2.1 Computing the Global Minimum Variance Portfolio

It is a simple exercise in calculus to find the global minimum variance portfolio. We solve the constrained optimization problem<sup>3</sup>

$$\begin{aligned} \min_{x_A, x_B} \sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \\ \text{s.t. } x_A + x_B &= 1. \end{aligned}$$

This constrained optimization problem can be solved using two methods. The first method, called the *method of substitution*, uses the constraint to substitute out one of the variables to transform the constrained optimization problem in two variables into an unconstrained optimization problem in one variable. The second method, called the *method of Lagrange multipliers*, introduces an auxiliary variable called the Lagrange multiplier and transforms the constrained optimization problem in two variables into an unconstrained optimization problem in three variables.

The substitution method is straightforward. Substituting  $x_B = 1 - x_A$  into the formula for  $\sigma_p^2$  reduces the problem to

$$\min_{x_A} \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB}.$$

The first order conditions for a minimum, via the chain rule, are

$$0 = \frac{d\sigma_p^2}{dx_A} = 2x_A^{\min} \sigma_A^2 - 2(1 - x_A^{\min})\sigma_B^2 + 2\sigma_{AB}(1 - 2x_A^{\min}),$$

and straightforward calculations yield

$$x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \quad x_B^{\min} = 1 - x_A^{\min}. \quad (1.10)$$

The method of Lagrange multipliers involves two steps. In the first step, the constraint  $x_A + x_B = 1$  is put into homogenous form  $x_A + x_B - 1 = 0$ . In the second step, the Lagrangian function is formed by adding to  $\sigma_p^2$  the homogenous constraint multiplied by an auxiliary variable  $\lambda$  (the Lagrange multiplier) giving

$$L(x_A, x_B, \lambda) = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} + \lambda(x_A + x_B - 1).$$

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<sup>3</sup>A review of optimization and constrained optimization is given in the appendix to this chapter.

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This function is then minimized with respect to  $x_A$ ,  $x_B$ , and  $\lambda$ . The first order conditions are

$$\begin{aligned}0 &= \frac{\partial L(x_A, x_B, \lambda)}{\partial x_A} = 2x_A\sigma_A^2 + 2x_B\sigma_{AB} + \lambda, \\0 &= \frac{\partial L(x_A, x_B, \lambda)}{\partial x_B} = 2x_B\sigma_B^2 + 2x_A\sigma_{AB} + \lambda, \\0 &= \frac{\partial L(x_A, x_B, \lambda)}{\partial \lambda} = x_A + x_B - 1.\end{aligned}$$

The first two equations can be rearranged to give

$$x_B = x_A \left( \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_B^2 - \sigma_{AB}} \right).$$

Substituting this value for  $x_B$  into the third equation and rearranging gives the solution (1.10).

**Example 6** *Global minimum variance portfolio for example data*

Using the data in Table 1.1 and (1.10) we have

$$x_A^{\min} = \frac{0.01323 - (-0.004866)}{0.06656 + 0.01323 - 2(-0.004866)} = 0.2021, \quad x_B^{\min} = 0.7979.$$

The expected return, variance and standard deviation of this portfolio are

$$\begin{aligned}\mu_p &= (0.2021) \cdot (0.175) + (0.7979) \cdot (0.055) = 0.07925 \\ \sigma_p^2 &= (0.2021)^2 \cdot (0.067) + (0.7979)^2 \cdot (0.013) \\ &\quad + 2 \cdot (0.2021)(0.7979)(-0.004875) \\ &= 0.00975 \\ \sigma_p &= \sqrt{0.00975} = 0.09782\end{aligned}$$

In Figure 1.1, this portfolio is labeled “global min”. In R, the calculations to compute the global minimum variance portfolio are

```
> xA.min = (sig2.B - sig.AB)/(sig2.A + sig2.B - 2*sig.AB)
> xB.min = 1 - xA.min
> xA.min
[1] 0.2021
```

```

> xB.min
[1] 0.7979
> mu.p.min = xA.min*mu.A + xB.min*mu.B
> sig2.p.min = xA.min^2 * sig2.A + xB.min^2 * sig2.B +
+             2*xA.min*xB.min*sig.AB
> sig.p.min = sqrt(sig2.p.min)
> mu.p.min
[1] 0.07925
> sig.p.min
[1] 0.09782

```



## 1.2.2 Correlation and the Shape of the Efficient Frontier

The shape of the investment possibilities set is very sensitive to the correlation between assets A and B. If  $\rho_{AB}$  is close to 1 then the investment set approaches a straight line connecting the portfolio with all wealth invested in asset B,  $(x_A, x_B) = (0, 1)$ , to the portfolio with all wealth invested in asset A,  $(x_A, x_B) = (1, 0)$ . This case is illustrated in Figure 1.2. As  $\rho_{AB}$  approaches zero the set starts to bow toward the  $\mu_p$  axis, and the power of diversification starts to kick in. If  $\rho_{AB} = -1$  then the set actually touches the  $\mu_p$  axis. What this means is that if assets A and B are perfectly negatively correlated then there exists a portfolio of A and B that has positive expected return and zero variance! To find the portfolio with  $\sigma_p^2 = 0$  when  $\rho_{AB} = -1$  we use (1.10) and the fact that  $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B$  to give

$$x_A^{\min} = \frac{\sigma_B}{\sigma_A + \sigma_B}, \quad x_B^{\min} = 1 - x_A.$$

The case with  $\rho_{AB} = -1$  is also illustrated in Figure 1.2.

### Example 7 Portfolio frontier when $\rho = \pm 1$

Suppose  $\rho_{AB} = 1$ . Then  $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$ . The portfolio variance is then

$$\begin{aligned} \sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB} = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_A\sigma_B \\ &= (x_A\sigma_A + x_B\sigma_B)^2 \end{aligned}$$

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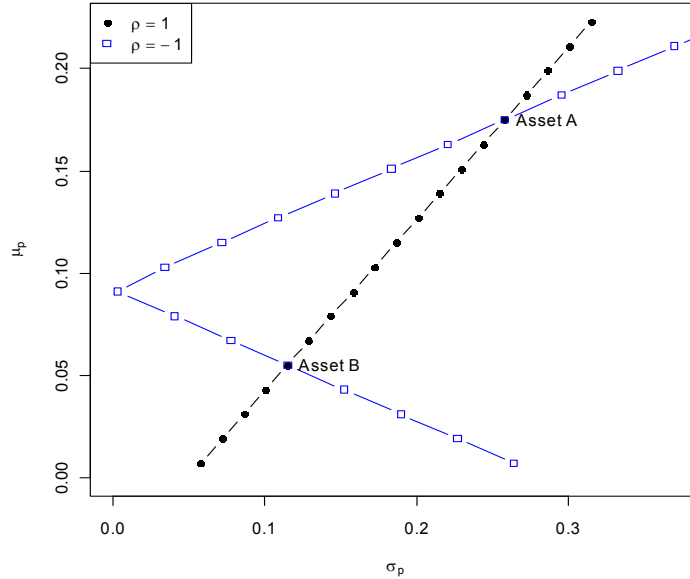


Figure 1.2: Portfolios with  $\rho_{AB} = 1$  and  $\rho_{AB} = -1$ .

Hence,  $\sigma_p = x_A\sigma_A + x_B\sigma_B = x_A\sigma_A + (1 - x_A)\sigma_B$  which shows that  $\sigma_p$  lies on a straight line connecting  $\sigma_A$  and  $\sigma_B$ . Next, suppose  $\rho_{AB} = -1$ . Then  $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = -\sigma_A\sigma_B$  which implies that  $\sigma_p^2 = (x_A\sigma_A - x_B\sigma_B)^2$  and  $\sigma_p = x_A\sigma_A - x_B\sigma_B = x_A\sigma_A + (1 - x_A)\sigma_B$ . In this case we can find a portfolio that has zero volatility. We solve for  $x_A$  such that  $\sigma_p = 0$  :

$$0 = x_A\sigma_A + (1 - x_A)\sigma_B \Rightarrow x_A = \frac{\sigma_B}{\sigma_A + \sigma_B}, x_B = 1 - x_A.$$

### 1.2.3 Optimal Portfolios

Given the efficient set of portfolios as described in Figure 1.1, which portfolio will an investor choose? Of the efficient portfolios, investors will choose the one that accords with their risk preferences. Very risk averse investors will want a portfolio that has low volatility (risk) and will choose a portfolio very close to the global minimum variance portfolio. In contrast, very risk tolerant

investors will ignore volatility and seek portfolios with high expected returns. Hence, these investors will choose portfolios with large amounts of asset A which may involve short-selling asset B.

### 1.3 Efficient portfolios with a risk-free asset

In the preceding section we constructed the efficient set of portfolios in the absence of a risk-free asset. Now we consider what happens when we introduce a risk-free asset. In the present context, a risk-free asset is equivalent to default-free pure discount bond that matures at the end of the assumed investment horizon. The risk-free rate,  $r_f$ , is then the nominal return on the bond. For example, if the investment horizon is one month then the risk-free asset is a 30-day U.S. Treasury bill (T-bill) and the risk-free rate is the nominal rate of return on the T-bill.<sup>4</sup> If our holdings of the risk-free asset is positive then we are “lending money” at the risk-free rate, and if our holdings are negative then we are “borrowing” at the risk-free rate.

#### 1.3.1 Efficient portfolios with one risky asset and one risk-free asset

Consider an investment in asset B and the risk-free asset (henceforth referred to as a T-bill). Since the risk-free rate is fixed (constant) over the investment horizon it has some special properties, namely

$$\begin{aligned}\mu_f &= E[r_f] = r_f, \\ \text{var}(r_f) &= 0, \\ \text{cov}(R_B, r_f) &= 0.\end{aligned}$$

Let  $x_B$  denote the share of wealth in asset B, and  $x_f = 1 - x_B$  denote the share of wealth in T-bills. The portfolio return is

$$R_p = (1 - x_B)r_f + x_B R_B = r_f + x_B(R_B - r_f).$$

The quantity  $R_B - r_f$  is called the *excess return* (over the return on T-bills) on asset B. The portfolio expected return is then

$$\mu_p = r_f + x_B(E[R_B] - r_f) = r_f + x_B(\mu_B - r_f), \quad (1.11)$$

---

<sup>4</sup>The default-free assumption of U.S. debt has recently been questioned due to the inability of the U.S. congress to address the long-term debt problems of the U.S. government.



### 1.3 EFFICIENT PORTFOLIOS WITH A RISK-FREE ASSET 17

where the quantity  $(\mu_B - r_f)$  is called the *expected excess return* or *risk premium* on asset B. For risky assets, the risk premium is typically positive indicating that investors expect a higher return on the risky asset than the safe asset. We may express the risk premium on the portfolio in terms of the risk premium on asset B:

$$\mu_p - r_f = x_B(\mu_B - r_f).$$

The more we invest in asset B the higher the risk premium on the portfolio.

Because the risk-free rate is constant, the portfolio variance only depends on the variability of asset B and is given by

$$\sigma_p^2 = x_B^2 \sigma_B^2.$$

The portfolio standard deviation is therefore proportional to the standard deviation on asset B

$$\sigma_p = x_B \sigma_B, \tag{1.12}$$

which we can use to solve for  $x_B$

$$x_B = \frac{\sigma_p}{\sigma_B}.$$

Using the last result, the feasible (and efficient) set of portfolios follows the equation

$$\mu_p = r_f + \frac{\mu_B - r_f}{\sigma_B} \cdot \sigma_p, \tag{1.13}$$

which is simply straight line in  $(\mu_p, \sigma_p)$ - space with intercept  $r_f$  and slope  $\frac{\mu_B - r_f}{\sigma_B}$ . This line is often called the *capital allocation line* (CAL). The slope of the CAL is called the *Sharpe ratio* (SR) or *Sharpe's slope* (named after the economist William Sharpe), and it measures the risk premium on the asset per unit of risk (as measured by the standard deviation of the asset).

#### **Example 8** *Portfolios of T-Bills and risky assets*

The portfolios which are combinations of asset A and T-bills and combinations of asset B and T-bills, using the data in Table 1.1 with  $r_f = 0.03$ , are illustrated in Figure 1.3 which is created using the R code

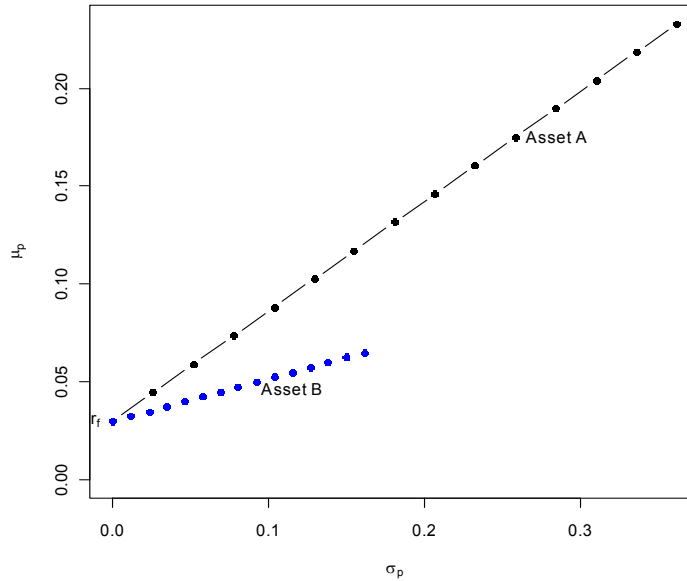


Figure 1.3: Portfolios of T-Bills and risky assets.

```

> r.f = 0.03
# T-bills + asset A
> x.A = seq(from=0, to=1.4, by=0.1)
> mu.p.A = r.f + x.A*(mu.A - r.f)
> sig.p.A = x.A*sig.A
# T-bills + asset B
> x.B = seq(from=0, to=1.4, by=0.1)
> mu.p.B = r.f + x.B*(mu.B - r.f)
> sig.p.B = x.B*sig.B
# plot portfolios of T-Bills and assets A and B
> plot(sig.p.A, mu.p.A, type="b", col="black", ylim=c(0, max(mu.p.A)),
+      xlim=c(0, max(sig.p.A, sig.p.B)), pch=16,
+      xlab=expression(sigma[p]), ylab=expression(mu[p]))
> points(sig.p.B, mu.p.B, type="b", col="blue", pch=16)
> text(x=sig.A, y=mu.A, labels="Asset A", pos=4)
> text(x=sig.B, y=mu.B, labels="Asset B", pos=1)

```

## 1.4 EFFICIENT PORTFOLIOS WITH TWO RISKY ASSETS AND A RISK-FREE ASSET

```
> text(x=0, y=r.f, labels=expression(r[f]), pos=2)
```

Notice that expected return-risk trade off of these portfolios is linear. Also, notice that the portfolios which are combinations of asset A and T-bills have expected returns uniformly higher than the portfolios consisting of asset B and T-bills. This occurs because the Sharpe ratio for asset A is higher than the ratio for asset B:

$$\begin{aligned} \text{SR}_A &= \frac{\mu_A - r_f}{\sigma_A} = \frac{0.175 - 0.03}{0.258} = 0.562, \\ \text{SR}_B &= \frac{\mu_B - r_f}{\sigma_B} = \frac{0.055 - 0.03}{0.115} = 0.217. \end{aligned}$$

Hence, portfolios of asset A and T-bills are efficient relative to portfolios of asset B and T-bills. ■

The previous example shows that the Sharpe ratio can be used to rank the risk return properties of individual assets. Assets with a high Sharpe ratio have a better risk-return tradeoff than assets with a low Sharpe ratio. Accordingly, investment analysts routinely rank assets based on their Sharpe ratios.

### 1.4 Efficient portfolios with two risky assets and a risk-free asset

Now we expand on the previous results by allowing our investor to form portfolios of assets A, B and T-bills. The efficient set in this case will still be a straight line in  $(\mu_p, \sigma_p)$ -space with intercept  $r_f$ . The slope of the efficient set, the maximum Sharpe ratio, is such that it is tangent to the efficient set constructed just using the two risky assets A and B. Figure 1.4 illustrates why this is so.

If we invest only in asset B and T-bills then the Sharpe ratio is  $\text{SR}_B = \frac{\mu_B - r_f}{\sigma_B} = 0.217$  and the CAL intersects the parabola at point B. This is clearly not the efficient set of portfolios. For example, we could do uniformly better if we instead invest only in asset A and T-bills. This gives us a Sharpe ratio of  $\text{SR}_A = \frac{\mu_A - r_f}{\sigma_A} = 0.562$ , and the new CAL intersects the parabola at point A. However, we could do better still if we invest in T-bills and some combination of assets A and B. Geometrically, it is easy to see that the best we can do is obtained for the combination of assets A and B such that the CAL is just

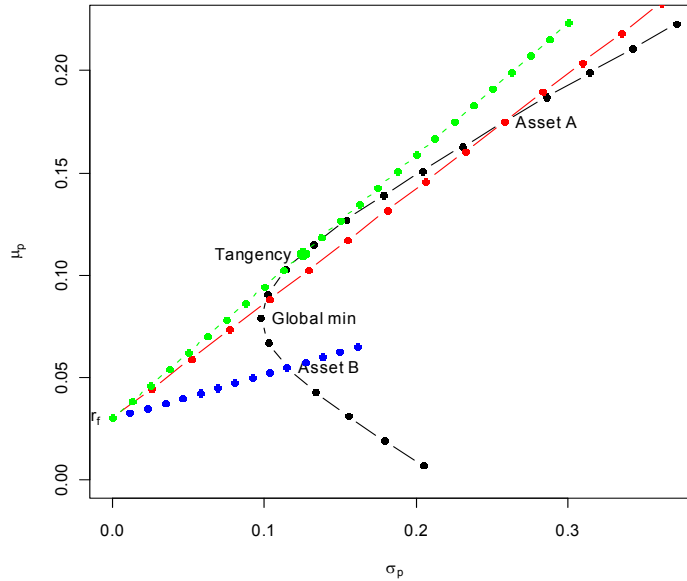


Figure 1.4: Efficient portfolios of two risky assets and T-Bills.

tangent to the parabola. This point is labeled “Tangency” on the graph and represents the *tangency portfolio* of assets A and B. Portfolios of T-Bills and the tangency portfolio are the set of efficient portfolios consisting of T-Bills, asset A and asset B.

### 1.4.1 Solving for the Tangency Portfolio

We can determine the proportions of each asset in the tangency portfolio by finding the values of  $x_A$  and  $x_B$  that maximize the Sharpe ratio of a portfolio that is on the envelope of the parabola. Formally, we solve the constrained

## 1.4 EFFICIENT PORTFOLIOS WITH TWO RISKY ASSETS AND A RISK-FREE ASSET

maximization problem

$$\begin{aligned} \max_{x_A, x_B} \text{SR}_p &= \frac{\mu_p - r_f}{\sigma_p} \quad s.t. \\ \mu_p &= x_A \mu_A + x_B \mu_B, \\ \sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB}, \\ 1 &= x_A + x_B. \end{aligned}$$

After various substitutions, the above problem can be reduced to

$$\max_{x_A} \frac{x_A(\mu_A - r_f) + (1 - x_A)(\mu_B - r_f)}{(x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB})^{1/2}}.$$

This is a straightforward, albeit very tedious, calculus problem and the solution can be shown to be

$$\begin{aligned} x_T^A &= \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}, \quad (1.14) \\ x_T^B &= 1 - x_T^A. \end{aligned}$$

**Example 9** *Tangency portfolio for example data*

For the example data in Table 1.1 using (1.14) with  $r_f = 0.03$ , we get  $x_T^A = 0.4625$  and  $x_T^B = 0.5375$ . The expected return, variance and standard deviation on the tangency portfolio are

$$\begin{aligned} \mu_T &= x_T^A \mu_A + x_T^B \mu_B \\ &= (0.4625)(0.175) + (0.5375)(0.055) = 0.1105, \\ \sigma_T^2 &= (x_T^A)^2 \sigma_A^2 + (x_T^B)^2 \sigma_B^2 + 2x_T^A x_T^B \sigma_{AB} \\ &= (0.4625)^2 (0.06656) + (0.5375)^2 (0.01323) + \\ &\quad 2(0.4625)(0.5375)(-0.004866) = 0.01564, \\ \sigma_T &= \sqrt{0.01564} = 0.1251. \end{aligned}$$

In R, the computations to compute the tangency portfolio are

```
> top = (mu.A - r.f)*sig2.B - (mu.B - r.f)*sig.AB
> bot = (mu.A - r.f)*sig2.B + (mu.B - r.f)*sig2.A -
+      (mu.A - r.f + mu.B - r.f)*sig.AB
```

```

> x.A.tan = top/bot
> x.B.tan = 1 - x.A.tan
> x.A.tan
[1] 0.4625
> x.B.tan
[1] 0.5375
> mu.p.tan = x.A.tan*mu.A + x.B.tan*mu.B
> sig2.p.tan = x.A.tan^2 * sig2.A + x.B.tan^2 * sig2.B +
+           2*x.A.tan*x.B.tan*sig.AB
> sig.p.tan = sqrt(sig2.p.tan)
> mu.p.tan
[1] 0.1105
> sig.p.tan
[1] 0.1251

```

■

## 1.4.2 Mutual Fund Separation

The efficient portfolios are combinations of the tangency portfolio and the T-bill. Accordingly, using (1.11) and (1.12) the expected return and standard deviation of any efficient portfolio are given by

$$\mu_p^e = r_f + x_T(\mu_T - r_f), \quad (1.15)$$

$$\sigma_p^e = x_T\sigma_T, \quad (1.16)$$

where  $x_T$  represents the fraction of wealth invested in the tangency portfolio (and  $1 - x_T$  represents the fraction of wealth invested in T-Bills), and  $\mu_T$  and  $\sigma_T$  are the expected return and standard deviation of the tangency portfolio, respectively. This important result is known as the *mutual fund separation theorem*. The tangency portfolio can be considered as a mutual fund of the two risky assets, where the shares of the two assets in the mutual fund are determined by the tangency portfolio weights ( $x_T^A$  and  $x_T^B$  determined from (1.14)), and the T-bill can be considered as a mutual fund of risk-free assets. The expected return-risk trade-off of these portfolios is given by the line connecting the risk-free rate to the tangency point on the efficient frontier of risky asset only portfolios. Which combination of the tangency portfolio and the T-bill an investor will choose depends on the investor's risk preferences. If the investor is very risk averse, then she will choose a portfolio with low

## 1.4 EFFICIENT PORTFOLIOS WITH TWO RISKY ASSETS AND A RISK-FREE ASSET

volatility which will be a portfolio with very little weight in the tangency portfolio and a lot of weight in the T-bill. This will produce a portfolio with an expected return close to the risk-free rate and a variance that is close to zero. If the investor can tolerate a large amount of risk, then she would prefer a portfolio with highest expected return regardless of the volatility. This portfolio may involve borrowing at the risk-free rate (leveraging) and investing the proceeds in the tangency portfolio to achieve a high expected return.

**Example 10** *Efficient portfolios chosen by risk averse and risk tolerant investors*

A highly risk averse investor may choose to put 10% of her wealth in the tangency portfolio and 90% in the T-bill. Then she will hold  $(10\%) \times (46.25\%) = 4.625\%$  of her wealth in asset *A*,  $(10\%) \times (53.75\%) = 5.375\%$  of her wealth in asset *B*, and 90% of her wealth in the T-bill. The expected return on this portfolio is

$$\mu_p^e = r_f + 0.10(\mu_T - r_f) = 0.03 + 0.10(0.1105 - 0.03) = 0.03805,$$

and the standard deviation is

$$\sigma_p^e = 0.10\sigma_T = 0.10(0.1251) = 0.01251.$$

In Figure 1.5, this efficient portfolio is labeled “Safe”. A very risk tolerant investor may actually borrow at the risk-free rate and use these funds to leverage her investment in the tangency portfolio. For example, suppose the risk tolerant investor borrows 100% of her wealth at the risk-free rate and uses the proceed to purchase 200% of her wealth in the tangency portfolio. Then she would hold  $(200\%) \times (46.25\%) = 92.50\%$  of her wealth in asset *A*,  $(200\%) \times (53.75\%) = 107.5\%$  in asset *B*, and she would owe 100% of her wealth to her lender. The expected return and standard deviation on this portfolio is

$$\begin{aligned}\mu_p^e &= 0.03 + 2(0.1105 - 0.03) = 0.1910, \\ \sigma_p^e &= 2(0.1251) = 0.2501.\end{aligned}$$

In Figure 1.5, this efficient portfolio is labeled “Risky”. ■

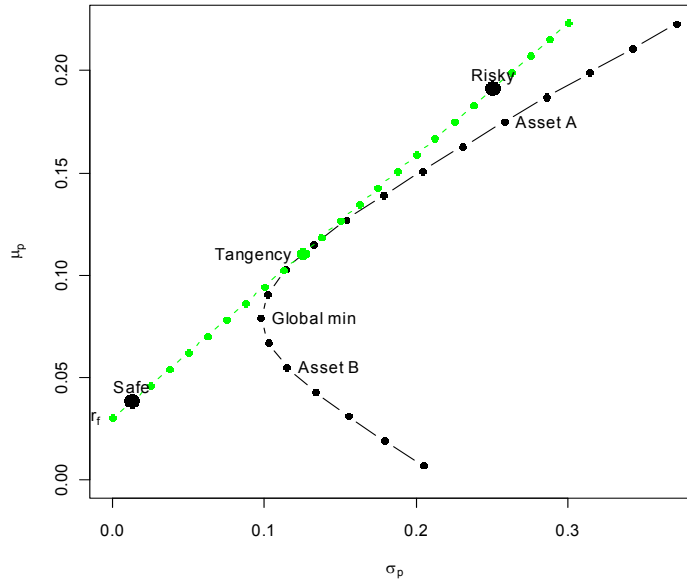


Figure 1.5: The efficient portfolio labeled “safe” has 10% invested in the tangency portfolio and 90% invested in T-Bills; the efficient portfolio labeled “risky” has 200% invested in the tangency portfolio and -100% invested in T-Bills.

### 1.4.3 Interpreting Efficient Portfolios

As we have seen, efficient portfolios are those portfolios that have the highest expected return for a given level of risk as measured by portfolio standard deviation. For portfolios with expected returns above the T-bill rate, efficient portfolios can also be characterized as those portfolios that have minimum risk (as measured by portfolio standard deviation) for a given target expected return.

To illustrate, consider Figure 1.6 which shows the portfolio frontier for two risky assets and the efficient frontier for two risky assets plus T-Bills. Suppose an investor initially holds all of his wealth in asset B. The expected return on this portfolio is  $\mu_B = 0.055$ , and the standard deviation (risk) is  $\sigma_B = 0.115$ . An efficient portfolio (combinations of the tangency portfolio and T-bills) that has the same standard deviation (risk) as asset B is given by the



## 1.4 EFFICIENT PORTFOLIOS WITH TWO RISKY ASSETS AND A RISK-FREE ASSET

portfolio on the efficient frontier that is directly above  $\sigma_B = 0.115$ . To find the shares in the tangency portfolio and T-bills in this portfolio recall from (1.16) that the standard deviation of an efficient portfolio with  $x_T$  invested in the tangency portfolio and  $1 - x_T$  invested in T-bills is  $\sigma_p^e = x_T \sigma_T$ . Since we want to find the efficient portfolio with  $\sigma_p^e = \sigma_B = 0.115$ , we solve

$$x_T = \frac{\sigma_B}{\sigma_T} = \frac{0.115}{0.1251} = 0.9195, \quad x_f = 1 - x_T = 0.08049.$$

That is, if we invest 91.95% of our wealth in the tangency portfolio and 8.049% in T-bills we will have a portfolio with the same standard deviation as asset B. Since this is an efficient portfolio, the expected return should be higher than the expected return on asset B. Indeed it is since

$$\mu_p^e = r_f + x_T(\mu_T - r_f) = 0.03 + 0.9195(0.1105 - 0.03) = 0.1040.$$

Notice that by diversifying our holding into assets A, B and T-bills we can obtain a portfolio with the same risk as asset B but with almost twice the expected return!

Next, consider finding an efficient portfolio that has the same expected return as asset B. Visually, this involves finding the combination of the tangency portfolio and T-bills that corresponds with the intersection of a horizontal line with intercept  $\mu_B = 0.055$  and the line representing efficient combinations of T-bills and the tangency portfolio. To find the shares in the tangency portfolio and T-bills in this portfolio recall from (1.15) that the expected return of an efficient portfolio with  $x_T$  invested in the tangency portfolio and  $1 - x_T$  invested in T-bills has expected return equal to  $\mu_p^e = r_f + x_T(\mu_T - r_f)$ . Since we want to find the efficient portfolio with  $\mu_p^e = \mu_B = 0.055$  we solve

$$x_T = \frac{\mu_p^e - r_f}{\mu_T - r_f} = \frac{0.055 - 0.03}{0.1105 - 0.03} = 0.3105, \quad x_f = 1 - x_T = 0.6895.$$

That is, if we invest 31.05% of wealth in the tangency portfolio and 68.95% of our wealth in T-bills we have a portfolio with the same expected return as asset B. Since this is an efficient portfolio, the standard deviation (risk) of this portfolio should be lower than the standard deviation on asset B. Indeed it is since

$$\sigma_p^e = x_T \sigma_T = 0.3105(0.124) = 0.03884.$$

Notice how large the risk reduction is by forming an efficient portfolio. The standard deviation on the efficient portfolio is almost three times smaller than the standard deviation of asset B!

The above example illustrates two ways to interpret the benefits from forming efficient portfolios. Starting from some benchmark portfolio, we can fix standard deviation (risk) at the value for the benchmark and then determine the gain in expected return from forming a diversified portfolio<sup>5</sup>. The gain in expected return has concrete meaning. Alternatively, we can fix expected return at the value for the benchmark and then determine the reduction in standard deviation (risk) from forming a diversified portfolio. The meaning to an investor of the reduction in standard deviation is not as clear as the meaning to an investor of the increase in expected return. It would be helpful if the risk reduction benefit can be translated into a number that is more interpretable than the standard deviation. The concept of Value-at-Risk (VaR) provides such a translation.

#### 1.4.4 Efficient Portfolios and Value-at-Risk

Recall, the VaR of an investment is the (lower bound of) loss in investment value over a given horizon with a stated probability. For example, consider an investor who invests  $W_0 = \$100,000$  in asset B over the next year. Assuming that  $R_B \sim N(0.055, (0.115)^2)$  represents the annual simple return on asset B, the 5% VaR is

$$\text{VaR}_{B,0.05} = q_{0.05}^{R_B} W_0 = (0.055 + 0.115(-1.645)) \cdot \$100,000 = -\$13,416.$$

If an investor holds \$100,000 in asset B over the next year, then there is a 5% probability that he will lose \$13,416 or more.

Now suppose the investor chooses to hold an efficient portfolio with the same expected return as asset B. This portfolio consists of 31.05% in the tangency portfolio and 68.95% in T-bills and has a standard deviation equal to 0.03884. Then  $R_p \sim N(0.055, 0.03884)$  and the 5% VaR on the portfolio is

$$\text{VaR}_{p,0.05} = q_{0.05}^{R_p} W_0 = (0.055 + 0.03884(-1.645)) \cdot \$100,000 = -\$884.$$

---

<sup>5</sup>The gain in expected return by investing in an efficient portfolio abstracts from the costs associated with selling the benchmark portfolio and buying the efficient portfolio.

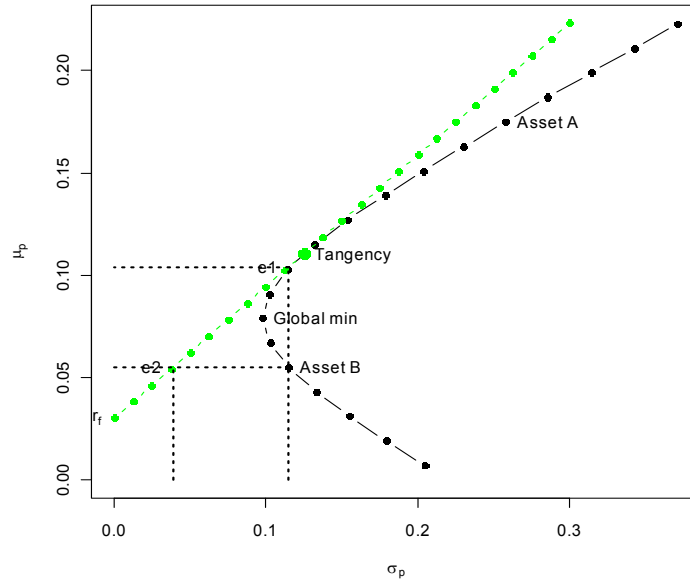


Figure 1.6: The point e1 represents an efficient portfolio with the same standard deviation as asset B; the point e2 represents an efficient portfolio with the same expected returns as asset B.

Notice that the 5% VaR for the efficient portfolio is almost fifteen times smaller than the 5% VaR for the investment in asset B. Since VaR translates risk into a dollar figure, it is more interpretable than standard deviation.

## 1.5 Further Reading

The classic text on portfolio optimization is Markowitz (1954). Good intermediate level treatments are given in Benninga (2000), Bodie, Kane and Marcus (1999) and Elton and Gruber (1995). An interesting recent treatment with an emphasis on statistical properties is Michaud (1998). Many practical results can be found in the *Financial Analysts Journal* and the *Journal of Portfolio Management*. An excellent treatment of value-at-risk and risk budgeting is given in Jorion (1997).

## 1.6 Appendix Review of Optimization and Constrained Optimization

Consider the function of a single variable

$$y = f(x) = x^2.$$

Clearly the minimum of this function occurs at the point  $x = 0$ . Using calculus, we find the minimum by solving

$$\min_x y = x^2.$$

The first order (necessary) condition for a minimum is

$$0 = \frac{d}{dx}f(x) = \frac{d}{dx}x^2 = 2x$$

and solving for  $x$  gives  $x = 0$ . The second order condition for a minimum is

$$0 < \frac{d^2}{dx^2}f(x),$$

and this condition is clearly satisfied for  $f(x) = x^2$ .

Next, consider the function of two variables

$$y = f(x, z) = x^2 + z^2 \tag{1.17}$$

This function looks like a salad bowl whose bottom is at  $x = 0$  and  $z = 0$ . To find the minimum of (1.17), we solve

$$\min_{x,z} y = x^2 + z^2,$$

and the first order necessary conditions are

$$0 = \frac{\partial y}{\partial x} = 2x,$$

$$0 = \frac{\partial y}{\partial z} = 2z.$$

Solving these two linear equations gives  $x = 0$  and  $z = 0$ .

## 1.6 APPENDIX REVIEW OF OPTIMIZATION AND CONSTRAINED OPTIMIZATION

Now suppose we want to minimize (1.17) subject to the linear constraint

$$x + z = 1. \quad (1.18)$$

The minimization problem is now a *constrained minimization*

$$\begin{aligned} \min_{x,z} y &= x^2 + z^2 \text{ subject to (s.t.)} \\ x + z &= 1 \end{aligned}$$

Given the constraint  $x + z = 1$ , the function (1.17) is no longer minimized at the point  $(x, z) = (0, 0)$  because this point does not satisfy  $x + z = 1$ . One simple way to solve this problem is to substitute the restriction (1.18) into the function (1.17) and reduce the problem to a minimization over one variable. To illustrate, use the restriction (1.18) to solve for  $z$  as

$$z = 1 - x. \quad (1.19)$$

Now substitute (1.19) into (1.17) giving

$$y = f(x, z) = f(x, 1 - x) = x^2 + (1 - x)^2. \quad (1.20)$$

The function (1.20) satisfies the restriction (1.18) by construction. The constrained minimization problem now becomes

$$\min_x y = x^2 + (1 - x)^2.$$

The first order conditions for a minimum are

$$0 = \frac{d}{dx}(x^2 + (1 - x)^2) = 2x - 2(1 - x) = 4x - 2,$$

and solving for  $x$  gives  $x = 1/2$ . To solve for  $z$ , use (1.19) to give  $z = 1 - (1/2) = 1/2$ . Hence, the solution to the constrained minimization problem is  $(x, z) = (1/2, 1/2)$ .

Another way to solve the constrained minimization is to use the method of *Lagrange multipliers*. This method augments the function to be minimized with a linear function of the constraint in homogeneous form. The constraint (1.18) in homogeneous form is

$$x + z - 1 = 0.$$

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The augmented function to be minimized is called the *Lagrangian* and is given by

$$L(x, z, \lambda) = x^2 + z^2 - \lambda(x + z - 1).$$

The coefficient on the constraint in homogeneous form,  $\lambda$ , is called the *Lagrange multiplier*. It measures the cost, or shadow price, of imposing the constraint relative to the unconstrained problem. The constrained minimization problem to be solved is now

$$\min_{x, z, \lambda} L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 1).$$

The first order conditions for a minimum are

$$\begin{aligned} 0 &= \frac{\partial L(x, z, \lambda)}{\partial x} = 2x + \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial z} = 2z + \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial \lambda} = x + z - 1 \end{aligned}$$

The first order conditions give three linear equations in three unknowns. Notice that the first order condition with respect to  $\lambda$  imposes the constraint. The first two conditions give

$$2x = 2z = -\lambda,$$

or

$$x = z.$$

Substituting  $x = z$  into the third condition gives

$$2z - 1 = 0$$

or

$$z = 1/2.$$

The final solution is  $(x, y, \lambda) = (1/2, 1/2, -1)$ .

The Lagrange multiplier,  $\lambda$ , measures the marginal cost, in terms of the value of the objective function, of imposing the constraint. Here,  $\lambda = -1$  which indicates that imposing the constraint  $x + z = 1$  reduces the objective function. To understand the roll of the Lagrange multiplier better, consider

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imposing the constraint  $x + z = 0$ . Notice that the unconstrained minimum achieved at  $x = 0, z = 0$  satisfies this constraint. Hence, imposing  $x + z = 0$  does not cost anything and so the Lagrange multiplier associated with this constraint should be zero. To confirm this, we solve the problem

$$\min_{x,z,\lambda} L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 0).$$

The first order conditions for a minimum are

$$\begin{aligned} 0 &= \frac{\partial L(x, z, \lambda)}{\partial x} = 2x - \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial z} = 2z - \lambda \\ 0 &= \frac{\partial L(x, z, \lambda)}{\partial \lambda} = x + z \end{aligned}$$

The first two conditions give

$$2x = 2z = -\lambda$$

or

$$x = z.$$

Substituting  $x = z$  into the third condition gives

$$2z = 0,$$

or

$$z = 0.$$

The final solution is  $(x, y, \lambda) = (0, 0, 0)$ . Notice that the Lagrange multiplier,  $\lambda$ , is equal to zero in this case.





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