# Econ 424
# Hypothesis Testing in the CER Model

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October 28, 2014

**Hypothesis Testing**

1. Specify hypothesis to be tested

$$H_0: \text{ null hypothesis versus. } H_1: \text{ alternative hypothesis}$$

2. Specify significance level of test

$$\text{level} = \Pr(\text{Reject } H_0 | H_0 \text{ is true})$$

3. Construct test statistic, $T$, from observed data

4. Use test statistic $T$ to evaluate data evidence regarding $H_0$

$$|T| \text{ is big } \Rightarrow \text{ evidence against } H_0$$
$$|T| \text{ is small } \Rightarrow \text{ evidence in favor of } H_0$$

Decide to reject $H_0$ at specified significance level if value of $T$ falls in the rejection region

$$T \in \text{ rejection region } \Rightarrow \text{ reject } H_0$$

Usually the rejection region of $T$ is determined by a critical value, $cv$, such that

$$\begin{aligned} |T| &> cv \Rightarrow \text{reject } H_0 \\ |T| &\leq cv \Rightarrow \text{ do not reject } H_0 \end{aligned}$$

## Decision Making and Hypothesis Tests

| | Reality | |
|---|---|---|
| Decision | $H_0$ is true | $H_0$ is false |
| Reject $H_0$ | Type I error | No error |
| Do not reject $H_0$ | No error | Type II error |

Significance Level of Test

$$\text{level} = \Pr(\text{Type I error})$$
$$\Pr(\text{Reject } H_0 | H_0 \text{ is true})$$

Goal: Constuct test to have a specified small significance level

$$\text{level} = 5\% \text{ or level} = 1\%$$

Power of Test

$$\begin{aligned} & 1 - \text{Pr}(\text{Type II error}) \\ & = \text{Pr}(\text{Reject } H_0 | H_0 \text{ is false}) \end{aligned}$$

Goal: Construct test to have high power

Problem: Impossible to simultaneously have level $\approx 0$ and power $\approx 1$. As level $\to 0$ power also $\to 0$.

## Hypothesis Testing in CER Model

$$
\begin{aligned}
r_{it} &= \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1, \cdots N \\
\epsilon_{it} &\sim \text{iid } N(0, \sigma_i^2) \\
\text{cov}(\epsilon_{it},\ \epsilon_{jt}) &= \sigma_{ij},\ \text{cor}(\epsilon_{it},\ \epsilon_{jt}) = \rho_{ij} \\
\text{cov}(\epsilon_{it},\ \epsilon_{js}) &= 0 \quad t \neq s,\ \text{for all } i,\ j
\end{aligned}
$$

- Test for specific value

$$H_0 : \mu_i = \mu_i^0 \text{ vs. } H_1 : \mu_i \neq \mu_i^0$$
$$H_0 : \sigma_i = \sigma_i^0 \text{ vs. } H_1 : \sigma_i \neq \sigma_i^0$$
$$H_0 : \rho_{ij} = \rho_{ij}^0 \text{ vs. } H_1 : \rho_{ij} \neq \rho_{ij}^0$$

- Test for sign

$$H_0 : \mu_i = 0 \text{ vs. } H_1 : \mu_i > 0 \text{ or } \mu_i < 0$$
$$H_0 : \rho_{ij} = 0 \text{ vs. } H_1 : \rho_{ij} > 0 \text{ or } \rho_{ij} < 0$$

- Test for normal distribution

$$H_0 : r_{it} \sim \text{iid } N(\mu_i, \sigma_i^2)$$
$$H_1 : r_{it} \sim \text{ not normal}$$

- Test for no autocorrelation

$$H_0 : \rho_j = \text{corr}(r_{it}, r_{i,t-j}) = 0,\ j > 1$$
$$H_1 : \rho_j = \text{corr}(r_{it}, r_{i,t-j}) \neq 0 \text{ for some } j$$

- Test of constant parameters

$$H_0 : \mu_i, \sigma_i \text{ and } \rho_{ij} \text{ are constant over entire sample}$$
$$H_1 : \mu_i\ \sigma_i \text{ or } \rho_{ij} \text{ changes in some sub-sample}$$

**Definition: Chi-square random variable and distribution**

Let $Z_1, \ldots, Z_q$ be iid $N(0,1)$ random variables. Define

$$X = Z_1^2 + \cdots + Z_q^2$$

Then

$$\begin{aligned} X &\sim \chi^2(q) \\ q &= \text{degrees of freedom (d.f.)} \end{aligned}$$

Properties of $\chi^2(q)$ distribution

$$\begin{aligned} X &> 0 \\ E[X] &= q \\ \chi^2(q) &\to \text{normal as } q \to \infty \end{aligned}$$

## R functions

`rchisq()`: simulate data

`dchisq()`: compute density

`pchisq()`: compute CDF

`qchisq()`: compute quantiles

**Definition: Student's t random variable and distribution with $q$ degrees of freedom**

$$
\begin{aligned}
&Z \sim N(0,1),\ X \sim \chi^2(q) \\
&Z \text{ and } X \text{ are independent} \\
&T = \frac{Z}{\sqrt{X/q}} \sim t_q \\
&q = \text{degrees of freedom (d.f.)}
\end{aligned}
$$

Properties of $t_q$ distribution:

$$
\begin{aligned}
E[T] &= 0 \\
\text{skew}(T) &= 0 \\
\text{kurt}(T) &= \frac{3q-6}{q-4},\ q > 4 \\
T &\rightarrow N(0,1) \text{ as } q \rightarrow \infty \ (q \geq 60)
\end{aligned}
$$

## R functions

`rt()`: simulate data

`dt()`: compute density

`pt()`: compute CDF

`qt()`: compute quantiles

**Test for Specific Coefficient Value**

$$H_0 : \mu_i = \mu_i^0 \text{ vs. } H_1 : \mu_i \neq \mu_i^0$$

1. Test statistic

$$t_{\mu_i=\mu_i^0} = \frac{\hat{\mu}_i - \mu_i^0}{\widehat{\mathsf{SE}}(\hat{\mu}_i)}$$

Intuition:

- If $t_{\mu_i=\mu_i^0} \approx 0$ then $\hat{\mu}_i \approx \mu_i^0$, and $H_0 : \mu_i = \mu_i^0$ should not be rejected

- If $|t_{\mu_i=\mu_i^0}| > 2$, say, then $\hat{\mu}_i$ is more than 2 values of $\widehat{\mathsf{SE}}(\hat{\mu}_i)$ away from $\mu_i^0$. This is very unlikely if $\mu_i = \mu_i^0$, so $H_0 : \mu_i = \mu_i^0$ should be rejected.

## Distribution of t-statistic under $H_0$

Under the assumptions of the CER model, and $H_0 : \mu_i = \mu_i^0$

$$t_{\mu_i=\mu_i^0} = \frac{\hat{\mu}_i - \mu_i^0}{\widehat{\mathsf{SE}}(\hat{\mu}_i)} \sim t_{T-1}$$

where

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it},\ \widehat{\mathsf{SE}}(\hat{\mu}_i) = \frac{\hat{\sigma}_i}{\sqrt{T}},\ \hat{\sigma}_i = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \hat{\mu}_i)^2}$$

$t_{T-1} =$ Student's t distribution with

$T - 1$ degrees of freedom (d.f.)

**Remarks**:

- $t_{T-1}$ is bell-shaped and symmetric about zero (like normal) but with fatter tails than normal

- d.f. $=$ sample size - number of estimated parameters. In CER model there is one estimated parameter, $\mu_i$, so df $= T - 1$

- For $T \geq 60$, $t_{T-1} \simeq N(0,1)$. Therefore, for $T \geq 60$

$$t_{\mu_i=\mu_i^0} = \frac{\hat{\mu}_i - \mu_i^0}{\widehat{\mathsf{SE}}(\hat{\mu}_i)} \simeq N(0,1)$$

2. Set significance level and determine critical value

$$\Pr(\text{Type I error}) = 5\%$$

Test has two-sided alternative so critical value, $cv_{.025}$, is determined using

$$\Pr(|t_{T-1}| > cv_{.025}) = 0.05 \Rightarrow cv_{.025} = -q_{.025}^{t_{T-1}} = q_{.975}^{t_{T-1}}$$

where $q_{.975}^{t_{T-1}} = 97.5\%$ quantile of Student-t distribution with $T-1$ degrees of freedom.

3. Decision rule:

$$\text{reject } H_0 : \mu_i = \mu_i^0 \text{ in favor of } H_1 : \mu \neq \mu_i^0 \text{ if}$$
$$|t_{\mu_i=\mu_i^0}| > cv_{.975}$$

**Useful Rule of Thumb:**

If $T \geq 60$ then $cv_{.975} \approx 2$ and the decision rule is

$$\text{Reject } H_0 : \mu_i = \mu_i^0 \text{ at 5\% level if}$$
$$|t_{\mu_i = \mu_i^0}| > 2$$

4. P-Value of two-sided test

$$
\begin{aligned}
&\text{significance level at which test is just rejected} \\
&= \Pr(|t_{T-1}| > t_{\mu_i=\mu_i^0}) \\
&= \Pr(t_{T-1} < -t_{\mu_i=\mu_i^0}) + \Pr(t_{T-1} > t_{\mu_i=\mu_i^0}) \\
&= 2 \cdot \Pr(t_{T-1} > |t_{\mu_i=\mu_i^0}|) \\
&= 2 \times (1 - \Pr(t_{T-1} \le |t_{\mu_i=\mu_i^0}|))
\end{aligned}
$$

Decision rule based on P-Value

$$
\begin{gathered}
\text{Reject } H_0 : \mu_i = \mu_i^0 \text{ at 5\% level if} \\
\text{P-Value } < 5\%
\end{gathered}
$$

For $T \geq 60$

$$
\text{P-value} = 2 \times \Pr(z > |t_{\mu_i=\mu_i^0}|),\ z \sim N(0,1)
$$

## Tests based on CLT

Let $\hat{\theta}$ denote an estimator for $\theta$. In many cases the CLT justifies the asymptotic normal distribution

$$\hat{\theta} \sim N(\theta, \mathsf{se}(\hat{\theta})^2)$$

Consider testing

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0$$

Result: Under $H_0$,

$$t_{\theta=\theta_0} = \frac{\hat{\theta} - \theta^0}{\widehat{\mathsf{se}}(\hat{\theta})} \sim N(0,1)$$

for large sample sizes.

**Example**: In the CER model, for large enough $T$ the CLT gives

$$\hat{\sigma}_i \sim N(\sigma_i, SE(\hat{\sigma}_i)^2)$$
$$SE(\hat{\sigma}_i) = \frac{\sigma_i}{\sqrt{2T}}$$

and

$$\hat{\rho}_{ij} \sim N(\rho_{ij}, SE(\hat{\rho}_{ij})^2)$$
$$SE(\hat{\rho}_{ij}) = \frac{\sqrt{1-\rho_{ij}^2}}{\sqrt{T}}$$

**Rule-of-thumb Decision Rule**

Let Pr(Type I error)= **5%**. Then reject

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0$$

at 5% level if

$$|t_{\theta=\theta_0}| = \left| \frac{\hat{\theta} - \theta^0}{\widehat{\text{se}}(\hat{\theta})} \right| > 2$$

**Relationship Between Hypothesis Tests and Confidence Intervals**

$$H_0 : \mu_i = \mu_i^0 \text{ vs. } H_1 : \mu_i \neq \mu_i^0$$
$$\text{level } = 5\%$$
$$cv_{.975} = q_{.975}^{t_{T-1}} \approx 2 \text{ for } T > 60$$
$$t_{\mu_i=\mu_i^0} = \frac{\hat{\mu}_i - \mu_i^0}{\widehat{\mathsf{SE}}(\hat{\mu}_i)}$$
$$\text{Reject at 5\% level if } |t_{\mu_i=\mu_i^0}| > 2$$

Approximate 95% confidence interval for $\mu_i$

$$\begin{aligned}\hat{\mu}_i &= \pm 2 \cdot \widehat{\mathsf{SE}}(\hat{\mu}_i) \\ &= [\hat{\mu}_i - 2 \cdot \widehat{\mathsf{SE}}(\hat{\mu}_i),\ \hat{\mu}_i + 2 \cdot \widehat{\mathsf{SE}}(\hat{\mu}_i)]\end{aligned}$$

Decision: Reject $H_0 : \mu_i = \mu_i^0$ at 5% level if $\mu_i^0$ does not lie in 95% confidence interval.

**Test for Sign**

$$H_0 : \mu_i = 0 \text{ vs. } H_1 : \mu_i > 0$$

1. Test statistic

$$t_{\mu_i=0} = \frac{\hat{\mu}_i}{\widehat{\mathsf{SE}}(\hat{\mu}_i)}$$

Intuition:

- If $t_{\mu_i=\mu_i^0} \approx 0$ then $\hat{\mu}_i \approx 0$, and $H_0 : \mu_i = 0$ should not be rejected

- If $t_{\mu_i=\mu_i^0} >> 0$, then this is very unlikely if $\mu_i = 0$, so $H_0 : \mu_i = 0$ vs. $H_1 : \mu_i > 0$ should be rejected.

2. Set significance level and determine critical value

$$\text{Pr(Type I error)} = 5\%$$

One-sided critical value $cv$ is determined using

$$\Pr(t_{T-1} > cv_{.05}) = 0.05$$
$$\Rightarrow cv_{.05} = q_{.95}^{t_{T-1}}$$

where $q_{.95}^{t_{T-1}} = 95\%$ quantile of Student-t distribution with $T - 1$ degrees of freedom.

3. Decision rule:

$$\text{Reject } H_0 : \mu_i = 0 \text{ vs. } H_1 : \mu_i > 0 \text{ at } 5\% \text{ level if}$$
$$t_{\mu_i=0} > q_{.95}^{t_{T-1}}$$

**Useful Rule of Thumb:**

If $T \geq 60$ then $q_{.95}^{t_{T-1}} \approx q_{.95}^{z} = 1.645$ and the decision rule is

$$\text{Reject } H_0 : \mu_i = 0 \text{ vs. } H_1 : \mu_i > 0 \text{ at 5\% level if}$$
$$t_{\mu_i=0} > 1.645$$

4. P-Value of test

$$\begin{aligned} &\text{significance level at which test is just rejected} \\ &= \Pr(t_{T-1} > t_{\mu_i=0}) \\ &= \Pr(Z > t_{\mu_i=0}) \text{ for } T \geq 60 \end{aligned}$$

## Test for Normal Distribution

$$H_0 : r_t \sim \text{iid } N(\mu, \sigma^2)$$
$$H_1 : r_t \sim \text{ not normal}$$

1. Test statistic (Jarque-Bera statistic)

$$\text{JB} = \frac{T}{6}\left(\widehat{\text{skew}}^2 + \frac{(\widehat{\text{kurt}} - 3)^2}{4}\right)$$

See R package tseries function `jarque.bera.test`

Intuition

- If $r_t \sim$ iid $N(\mu, \sigma^2)$ then $\widehat{\text{skew}}(r_t) \approx 0$ and $\widehat{\text{kurt}}(r_t) \approx 3$ so that JB $\approx 0$.

- If $r_t$ is not normally distributed then $\widehat{\text{skew}}(r_t) \neq 0$ and/or $\widehat{\text{kurt}}(r_t) \neq 3$ so that JB $>> 0$

## Distribution of JB under $H_0$

If $H_0 : r_t \sim$ iid $N(\mu, \sigma^2)$ is true then

$$\text{JB} \sim \chi^2(2)$$

where $\chi^2(2)$ denotes a chi-square distribution with 2 degrees of freedom (d.f.).

2. Set significance level and determine critical value

$$\Pr(\text{Type I error}) = 5\%$$

Critical value $cv$ is determined using

$$\Pr(\chi^2(2) > cv) = 0.05$$
$$\Rightarrow cv = q_{.95}^{\chi^2(2)} \approx 6$$

where $q_{.95}^{\chi^2(2)} \approx 6 \approx 95\%$ quantile of chi-square distribution with 2 degrees of freedom.

3. Decision rule:

$$\text{Reject } H_0 : r_t \sim \text{iid } N(\mu, \sigma^2)$$
$$\text{at } 5\% \text{ level if JB} > 6$$

4. P-Value of test

$$\text{significance level at which test is just rejected}$$
$$= \mathsf{Pr}(\chi^2(2) > \mathsf{JB})$$

## Test for No Autocorrelation

Recall, the j$^{th}$ lag autocorrelation for $r_t$ is

$$\begin{aligned}\rho_j &= \text{cor}(r_t, r_{t-j}) \\ &= \frac{\text{cov}(r_t, r_{t-j})}{\text{var}(r_t)}\end{aligned}$$

Hypotheses to be tested

$$\begin{aligned}H_0 &: \rho_j = 0, \text{ for all } j = 1, \ldots, q \\ H_1 &: \rho_j \neq 0 \text{ for some } j\end{aligned}$$

1. Estimate $\rho_j$ using sample autocorrelation

$$\hat{\rho}_j = \frac{\frac{1}{T-1}\sum_{t=j+1}^{T}(r_t - \hat{\mu})(r_{t-j} - \hat{\mu})}{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^2}$$

Result: Under $H_0 : \rho_j = 0$ for all $j = 1, \ldots, q$, if $T$ is large then

$$\hat{\rho}_j \sim N\left(0, \frac{1}{T}\right) \text{ for all } j \geq 1$$

$$\text{SE}(\hat{\rho}_j) = \frac{1}{\sqrt{T}}$$

2. Test Statistic

$$t_{\rho_{j=0}} = \frac{\hat{\rho}_j}{\text{SE}(\hat{\rho}_j)} = \frac{\hat{\rho}_j}{1/\sqrt{T}} = \sqrt{T}\hat{\rho}_j$$

and 95% confidence interval

$$\hat{\rho}_j \pm 2 \cdot \frac{1}{\sqrt{T}}$$

3. Decision rule

$$\text{Reject } H_0 : \rho_j = 0 \text{ at 5\% level}$$
$$\text{if } |t_{\rho_{j=0}}| = \left|\sqrt{T}\hat{\rho}_j\right| > 2$$

That is, reject if

$$\hat{\rho}_j > \frac{2}{\sqrt{T}} \text{ or } \hat{\rho}_j < \frac{-2}{\sqrt{T}}$$

Remark:

The dotted lines on the sample ACF are at the points $\pm 2 \cdot \frac{1}{\sqrt{T}}$

## Diagnostics for Constant Parameters

$$H_0 : \mu_i \text{ is constant over time vs. } H_1 : \mu_i \text{ changes over time}$$
$$H_0 : \sigma_i \text{ is constant over time vs. } H_1 : \sigma_i \text{ changes over time}$$
$$H_0 : \rho_{ij} \text{ is constant over time vs. } H_1 : \rho_{ij} \text{ changes over time}$$

Remarks

- Formal test statistics are available but require advanced statistics
  - See R package strucchange
- Informal graphical diagnostics: Rolling estimates of $\mu_i$, $\sigma_i$ and $\rho_{ij}$

## Rolling Means

Idea: compute estimate of $\mu_i$ over rolling windows of length $n < T$

$$\begin{aligned}\hat{\mu}_{it}(n) &= \frac{1}{n}\sum_{j=0}^{n-1} r_{it-j} \\ &= \frac{1}{n}(r_{it} + r_{it-1} + \cdots + r_{it-n+1})\end{aligned}$$

R function (package zoo)

`rollapply`

If $H_0 : \mu_i$ is constant is true, then $\hat{\mu}_{it}(n)$ should stay fairly constant over different windows.

If $H_0 : \mu_i$ is constant is false, then $\hat{\mu}_{it}(n)$ should fluctuate across different windows

**Rolling Variances and Standard Deviations**

Idea: Compute estimates of $\sigma_i^2$ and $\sigma_i$ over rolling windows of length $n < T$

$$\hat{\sigma}_{it}^2(n) = \frac{1}{n-1}\sum_{j=0}^{n-1}(r_{it-j} - \hat{\mu}_{it}(n))^2$$
$$\hat{\sigma}_{it}(n) = \sqrt{\hat{\sigma}_{it}^2(n)}$$

If $H_0 : \sigma_i$ is constant is true, then $\hat{\sigma}_{it}(n)$ should stay fairly constant over different windows.

If $H_0 : \sigma_i$ is constant is false, then $\hat{\sigma}_{it}(n)$ should fluctuate across different windows

## Rolling Covariances and Correlations

Idea: Compute estimates of $\sigma_{jk}$ and $\rho_{jk}$ over rolling windows of length $n < T$

$$\hat{\sigma}_{jk,t}(n) = \frac{1}{n-1}\sum_{i=0}^{n-1}(r_{jt-i} - \hat{\mu}_j(n))(r_{kt-i} - \hat{\mu}_k(n))$$

$$\hat{\rho}_{jk,t}(n) = \frac{\hat{\sigma}_{jk,t}(n)}{\hat{\sigma}_{jt}(n)\hat{\sigma}_{kt}(n)}$$

If $H_0 : \rho_{jk}$ is constant is true, then $\hat{\rho}_{jk,t}(n)$ should stay fairly constant over different windows.

If $H_0 : \rho_{jk}$ is constant is false, then $\hat{\rho}_{jk,t}(n)$ should fluctuate across different windows