1 Estimating the Single Index Model

SI model

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ t = 1, \ldots, T \\
\varepsilon_{it} &\sim \text{iid } N(0, \sigma^2_{\varepsilon,i}) \\
R_{M,t} &\sim \text{iid } N(\mu_M, \sigma^2_M) \\
\text{cov}(R_{Mt}, \varepsilon_{it}) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_i &= E[R_{it}] = \alpha_i + \beta_i \mu_M \\
\beta_i &= \frac{\text{cov}(R_{it}, R_{Mt})}{\text{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma^2_M}
\end{aligned}
$$

Main parameters to estimate: α_i, β_i and $\sigma^2_{\varepsilon,i}$

1.1 Plug-in Principle Estimators

Plug-in principle: Estimate model parameters using sample statistics

$$
\begin{aligned}
\hat{\beta}_i &= \frac{\hat{\sigma}_{iM}}{\hat{\sigma}_M^2} \\
\hat{\sigma}_{iM} &= \frac{1}{T-1}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)(R_{Mt}-\hat{\mu}_M) \\
\hat{\sigma}_M^2 &= \frac{1}{T-1}\sum_{t=1}^{T}(R_{Mt}-\hat{\mu}_M)^2 \\
\hat{\mu}_i &= \frac{1}{T}\sum_{t=1}^{T}R_{it},\ \hat{\mu}_M = \frac{1}{T}\sum_{t=1}^{T}R_{Mt}
\end{aligned}
$$

Plug-in principle estimator for α_i

$$\hat{\alpha}_i = \hat{\mu}_i - \hat{\beta}_i\hat{\mu}_M$$

1.2 Least Squares Estimation

Idea: SI model postulates a linear relationship between R_{it} and R_{Mt} with intercept α_i and slope β_i. Estimate α_i and β_i by finding the "best fitting line" to the scatterplot of data

- Problem: How to define the "best fitting line"?

- Least Squares solution: minimize the sum of squared residuals (errors)

1.2.1 Least Squares Algorithm

$$
\begin{aligned}
\hat{\alpha}_i &= \text{initial guess for } \alpha_i \\
\hat{\beta}_i &= \text{initial guess for } \beta_i \\
\hat{R}_{it} &= \hat{\alpha}_i + \hat{\beta}_i R_{Mt} = \text{fitted line} \\
\hat{\varepsilon}_{it} &= R_{it} - \hat{R}_{it} \\
&= R_{it} - (\hat{\alpha}_i + \hat{\beta}_i R_{Mt}) = \text{residual}
\end{aligned}
$$

Determine the best fitting line by minimizing the *Sum of Squared Residuals* (SSR)

$$
\begin{aligned}
\text{SSR}(\hat{\alpha}_i, \hat{\beta}_i) &= \sum_{t=1}^{T} \hat{\varepsilon}_{it}^2 \\
&= \sum_{t=1}^{T} \left(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt} \right)^2
\end{aligned}
$$

That is, the least squares estimates solve

$$\min_{\hat{\alpha}_i, \hat{\beta}_i} \mathsf{SSR}(\alpha_i, \beta_i) = \sum_{t=1}^{T} \left(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt} \right)^2$$

Note: Because $\mathsf{SSR}(\alpha_i, \beta_i)$ is a quadratic function in $\hat{\alpha}_i, \hat{\beta}_i$ there is an analytic solution to the minimization problem.

1.2.2 Calculus Solution

The first order conditions for a minimum are

$$
\begin{aligned}
0 &= \frac{\partial \mathsf{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\alpha}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt}) \\
0 &= \frac{\partial \mathsf{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\beta}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt})R_{Mt}
\end{aligned}
$$

Two linear equations in two unknowns. Solving for $\hat{\alpha}_i$ and $\hat{\beta}_i$ gives

$$
\begin{aligned}
\hat{\alpha}_i &= \hat{\mu}_i - \hat{\beta}_i \hat{\mu}_M \\
\hat{\beta}_i &= \frac{\hat{\sigma}_{iM}}{\hat{\sigma}_M^2}
\end{aligned}
$$

which are exactly the plug-in principle estimators!

1.2.3 Estimators for $\sigma^2_{\varepsilon,i}$ and $R - square$

Utilize plug-in principle

$$
\begin{aligned}
\hat{\varepsilon}_{it} &= R_{it} - \hat{\alpha} - \hat{\beta}_i R_{Mt} \\
\hat{\sigma}^2_{\varepsilon,i} &= \frac{1}{T-2} \sum_{t=1}^{T} \hat{\varepsilon}^2_{it} \\
\hat{\sigma}_{\varepsilon,i} &= \sqrt{\hat{\sigma}^2_{\varepsilon,i}} = \text{SER} \\
&= \text{standard error of regression}
\end{aligned}
$$

Remarks

- $\hat{\sigma}_{\varepsilon,i}$ typical magnitude of residual $=$ standard error of regression (SER)

- Divide by $T-2$ to get unbiased estimate of $\sigma^2_{\varepsilon,i}$

- $T-2 =$ degrees of freedom $=$ sample size - number of estimated parameters (α_i and β_i)

Recall

$$R_i^2 = \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2}$$
$$= 1 - \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2}$$
$$= \text{\% of variability due to market}$$

Estimate using plug-in principle

$$\begin{aligned} \hat{R}_i^2 &= \frac{\hat{\beta}_i^2 \hat{\sigma}_M^2}{\hat{\sigma}_i^2} \\ &= 1 - \frac{\hat{\sigma}_{\varepsilon,i}^2}{\hat{\sigma}_i^2} \end{aligned}$$

1.2.4 Least Squares Estimation Using R

R command

`lm` - linear model estimation

Syntax

```
lm.fit = lm(y~x,data=my.data)
```

`my.data`=data frame with variables `y` and `x`

Important method functions

```
summary()
plot()
residuals()
fitted()
coefficients()
```

1.3 Statistical Properties of Least Squares Estimates

Assuming the SI model generates the observed data, the estimators

$$\hat{\alpha}_i,\ \hat{\beta}_i \text{ and } \hat{\sigma}^2_{\varepsilon,i}$$

are random variables.

Properties

- $\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{\sigma}^2_{\varepsilon,i}$ are unbiased estimators

$$\begin{aligned} E[\hat{\alpha}_i] &= \alpha_i \\ E[\hat{\beta}_i] &= \beta_i \\ E[\hat{\sigma}^2_{\varepsilon,i}] &= \sigma^2_{\varepsilon,i} \end{aligned}$$

- Analytic standard errors are available for $\widehat{\mathsf{SE}}(\hat{\alpha}_i)$ and $\widehat{\mathsf{SE}}(\hat{\beta}_i)$

$$
\begin{aligned}
\widehat{\mathsf{SE}}(\hat{\alpha}_i) &= \frac{\hat{\sigma}_{\varepsilon,i}}{\sqrt{T \cdot \hat{\sigma}_M^2}} \cdot \sqrt{\frac{1}{T}\sum_{t=1}^{T} R_{Mt}^2} \\
\widehat{\mathsf{SE}}(\hat{\beta}_i) &= \frac{\hat{\sigma}_{\varepsilon,i}}{\sqrt{T \cdot \hat{\sigma}_M^2}}
\end{aligned}
$$

  These are routinely reported in standard regression ouput (e.g. by R `summary` command)

  - $\widehat{\mathsf{SE}}(\hat{\alpha}_i)$ and $\widehat{\mathsf{SE}}(\hat{\beta}_i)$ are smaller the smaller is $\hat{\sigma}_{\varepsilon,i}$
  - $\widehat{\mathsf{SE}}(\hat{\beta}_i)$ is smaller the larger is $\hat{\sigma}_M^2$
  - $\widehat{\mathsf{SE}}(\hat{\alpha}_i)$ and $\widehat{\mathsf{SE}}(\hat{\beta}_i) \rightarrow 0$ as T gets large $\Rightarrow$ $\hat{\alpha}_i$ and $\hat{\beta}_i$ are consistent estimators

- Standard errors for $\hat{\sigma}^2_{\varepsilon,i}$, $\hat{\sigma}_{\varepsilon,i}$ or $R-$square can be computed using the bootstrap

- For T large enough, the central limit theorem (CLT) tells us that

$$\hat{\alpha}_i \sim N(\alpha_i, \widehat{\mathsf{SE}}(\hat{\alpha}_i)^2)$$
$$\hat{\beta}_i \sim N(\beta_i, \widehat{\mathsf{SE}}(\hat{\beta}_i)^2)$$

- Approximate 95% confidence intervals

$$\hat{\alpha}_i \pm 2 \cdot \widehat{\mathsf{SE}}(\hat{\alpha}_i)$$
$$\hat{\beta}_i \pm 2 \cdot \widehat{\mathsf{SE}}(\hat{\beta}_i)$$

1.4 SI Model Using Matrix Algebra

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ t = 1, \ldots, T$$

Stack over observations $t = 1, \ldots, T$

$$\begin{pmatrix} R_{i1} \\ \vdots \\ R_{iT} \end{pmatrix} = \alpha_i \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \beta_i \begin{pmatrix} R_{M1} \\ \vdots \\ R_{MT} \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

or

$$\begin{aligned} \mathbf{R}_i &= \alpha_i \cdot \mathbf{1} + \beta_i \cdot \mathbf{R}_M + \boldsymbol{\varepsilon}_i = \begin{pmatrix} 1 & R_M \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} + \varepsilon_i \\ &= \mathbf{X}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i \end{aligned}$$

Recall the least squares normal equations

$$
\begin{aligned}
0 &= \frac{\partial \mathsf{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\alpha}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt}) \\
0 &= \frac{\partial \mathsf{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\beta}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt})R_{Mt}
\end{aligned}
$$

Using matrix algebra these equations are

$$
\begin{pmatrix} \sum_{t=1}^{T} R_{it} \\ \sum_{t=1}^{T} R_{it}R_{Mt} \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} R_{Mt} \\ \sum_{t=1}^{T} R_{Mt} & \sum_{t=1}^{T} R_{Mt}^2 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{pmatrix}
$$

Equivalently,

$$\begin{pmatrix} \mathbf{1}'\mathbf{R}_i \\ \mathbf{R}_M'\mathbf{R}_i \end{pmatrix} = \begin{pmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{R}_M \\ \mathbf{1}'\mathbf{R}_M & \mathbf{R}_M'\mathbf{R}_M \end{pmatrix} \begin{pmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{pmatrix}$$

or

$$\mathbf{X}'\mathbf{R}_i = \mathbf{X}'\mathbf{X}\hat{\boldsymbol{\gamma}}_i$$

Solving for $\hat{\boldsymbol{\gamma}}_i$ gives the least squares estimates

$$\hat{\boldsymbol{\gamma}}_i = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{R}_i$$

1.5 Estimating SI Model Covariance Matrix

Recall, in the SI model

$$\begin{aligned}
\mathbf{\Sigma} &= \sigma_M^2 \boldsymbol{\beta}\boldsymbol{\beta}' + \mathbf{D} \\
\boldsymbol{\beta} &= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{\varepsilon,n}^2 \end{pmatrix}
\end{aligned}$$

Estimate $\mathbf{\Sigma}$ using plug-in principle

$$\hat{\mathbf{\Sigma}} = \hat{\sigma}_M^2 \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}' + \hat{\mathbf{D}}$$

where

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_n \end{pmatrix},\ \hat{\mathbf{D}} = \begin{pmatrix} \hat{\sigma}_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \hat{\sigma}_{\varepsilon,n}^2 \end{pmatrix}$$