Econ 424: Introduction to Computational Finance and Financial Econometrics Constant Exected Return Model

Eric Zivot

October 26, 2009

1 Constant Expected Return Model

$$
\begin{aligned}
r_{it} &= \text{ cc return on asset } i \text{ in month } t \\
i &= 1, \cdots, N \text{ assets; } t = 1, \cdots, T \text{ months}
\end{aligned}
$$

Assumptions (normal distribution and covariance stationarity)

$$
\begin{aligned}
r_{it} &\sim N(\mu_i,\ \sigma_i^2) \\
\mu_i &= E[r_{it}] \ \text{ (constant over time)} \\
\sigma_i^2 &= \text{var}(r_{it}) \ \text{ (constant over time)} \\
\sigma_{ij} &= \text{cov}(r_{it},\ r_{jt}) \ \text{ (constant over time)} \\
\rho_{ij} &= \text{cor}(r_{it},\ r_{jt}) \text{ (constant over time)}
\end{aligned}
$$

Regression Model Representation (CER Model)

$$
\begin{aligned}
r_{it} &= \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1, \cdots N \\
\epsilon_{it} &\sim \text{iid } N(0, \sigma_i^2) \\
Cov(\epsilon_{it},\ \epsilon_{jt}) &= \sigma_{ij} \\
Cov(\epsilon_{it},\ \epsilon_{js}) &= 0 \quad t \neq s, \text{ for all } i,\ j
\end{aligned}
$$

Interpretation

- ϵ_{it} represents random news that arrives in month t
- News affecting asset i may be correlated with news affecting asset j
- News is uncorrelated over time

$$\underset{\text{unexpected news}}{\epsilon_{it}} = \underset{\text{Actual return}}{r_{it}} - \underset{\text{expected return}}{\mu_i}$$

$$\begin{aligned}
\text{No news } \epsilon_{it} &= 0 \Longrightarrow r_{it} = \mu_i \\
\text{Good news } \epsilon_{it} &> 0 \Longrightarrow r_{it} > \mu_i \\
\text{Bad news } \epsilon_{it} &< 0 \Longrightarrow r_{it} < \mu_i
\end{aligned}$$

1.1 Monte Carlo Simulation

Use computer random number generator to create simulated values from assumed model

- Reality check on proposed model
- Create "what if?" scenarios
- Study properties of statistics computed from proposed model

Simulating Random Numbers from a Distribution

Goal: simulate random number x from pdf $f(x)$ with CDF $F_X(x)$

- Generate $U \sim$ Uniform $[0, 1]$

- Generate $X \sim F_X(x)$ using inverse CDF technique:

$$\begin{aligned} x &= F_X^{-1}(u) \\ F_X^{-1} &= \text{inverse CDF function} \\ F_X^{-1}(F_X(x)) &= x \end{aligned}$$

Example - Simulate monthly returns on Microsoft from CER Model

- Specify parameters based on sample statistics (use monthly data from June 1992 - Oct 2000)

$$
\begin{aligned}
\mu &= 0.03 \text{ (monthly expected return)} \\
\sigma &= 0.11 \text{ (monthly SD)} \\
r_t &= 0.03 + \varepsilon_t,\ t = 1, \ldots, 100 \\
& \varepsilon_t \sim \text{iid } N(0, (0.11)^2)
\end{aligned}
$$

- Simulation requires generating random numbers from a normal distribution. In R use `rnorm()`.

1.2 The Random Walk Model

The CER model for cc returns is equivalent to the random walk (RW) model for log stock prices

$$\begin{aligned} r_t &= \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln P_t - \ln P_{t-1} \\ &= \ln P_t - \ln P_{t-1} \end{aligned}$$

which implies

$$\ln P_t = \ln P_{t-1} + r_t$$

Recursive substitution starting at $t = 1$ gives

$$\begin{aligned}
\ln P_1 &= \ln P_0 + r_1 \\
\ln P_2 &= \ln P_1 + r_2 \\
&= \ln P_0 + r_1 + r_2 \\
&\vdots \\
\ln P_t &= \ln P_{t-1} + r_t \\
&= \ln P_0 + \sum_{s=1}^{t} r_s
\end{aligned}$$

Interpretation: Price at t equals initial price plus accumulation of cc returns

In CER model, $r_s = \mu + \varepsilon_s$ so that

$$\begin{aligned} \ln P_t &= \ln P_0 + \sum_{s=1}^{t} (\mu + \varepsilon_s) \\ &= \ln P_0 + t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s \end{aligned}$$

Interpretation: Log price at t equals initial price $\ln P_0$, plus expected growth in prices $E[\ln P_t] = t \cdot \mu$, plus accumulation of news $\sum_{s=1}^{t} \varepsilon_s$.

The price level at time t is

$$\begin{aligned} P_t &= P_0 \exp\left(t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s\right) = P_0 \exp\left(t \cdot \mu +\right) \exp\left(\sum_{s=1}^{t} \varepsilon_s\right) \\ \exp\left(t \cdot \mu\right) &= \text{expected growth in price} \\ \exp\left(\sum_{s=1}^{t} \varepsilon_s\right) &= \text{unexpected growth in price} \end{aligned}$$

1.3 Estimating Parameters of CER model

Parameters of CER Model

$$\begin{aligned} \mu_i &= E[r_{it}] \\ \sigma_i^2 &= \text{var}(r_{it}) \\ \sigma_{ij} &= \text{cov}(r_{it}, r_{jt}) \\ \rho_{ij} &= \text{cor}(r_{it}, r_{jt}) \end{aligned}$$

are not known with certainty

First Econometric Task

- Estimate μ_i, σ_i^2, σ_{ij}, ρ_{ij} using observed sample of historical monthly returns

1.3.1 Estimators and Estimates

Definition: An estimator is a rule or algorithm for computing an estimate of a parameter based on a random sample

Example: Sample mean as estimator of $E[r_{it}]$

$$\begin{aligned} \{r_{i1}, \ldots, r_{iT}\} &= \text{random sample} \\ \bar{r}_i &= \frac{1}{T}\sum_{t=1}^{T} r_{it} = \text{sample mean} \\ &= \text{random variable} \end{aligned}$$

Definition: An estimate of a parameter is simply the value of an estimator based on observed data

Example: Sample mean from an observed sample

$$\begin{aligned}\bar{r}_i &= \frac{1}{T}(.02 + 0.01 - .01 + \cdots) \\ &= \text{number} = 0.01 \text{ (say)}\end{aligned}$$

Estimators of CER Model Parameters

Use plug-in principle: Estimate model parameters using sample statistics

$$
\begin{aligned}
\hat{\mu}_i &= \frac{1}{T}\sum_{t=1}^{T} r_{it} \\
\hat{\sigma}_i^2 &= \frac{1}{T-1}\sum_{t=1}^{T}(r_{it}-\hat{\mu}_i)^2 \\
\hat{\sigma}_i &= \sqrt{\hat{\sigma}_i^2} \\
\hat{\sigma}_{ij} &= \frac{1}{T-1}\sum_{t=1}^{T}(r_{it}-\hat{\mu}_i)(r_{jt}-\hat{\mu}_j) \\
\hat{\rho}_{ij} &= \frac{\hat{\sigma}_{ij}}{\hat{\sigma}_i \cdot \hat{\sigma}_j}
\end{aligned}
$$

1.3.2 Properties of Estimators

$$
\begin{aligned}
\theta &= \text{parameter to be estimated} \\
\hat{\theta} &= \text{estimator of } \theta \text{ from random sample}
\end{aligned}
$$

KEY POINTS

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample

- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample

- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Bias

$$\text{bias}(\hat{\theta}, \theta) = E\left[\hat{\theta}\right] - \theta$$
$$\hat{\theta} \text{ is unbiased if } E[\hat{\theta}] = \theta$$

Precision

$$\begin{aligned} \text{SE}(\hat{\theta}) &= \text{Standard error of } \hat{\theta} \\ &= \sqrt{\text{var}(\hat{\theta})} = \sqrt{E[(\hat{\theta} - E[\hat{\theta}])^2]} \\ &= \sigma_{\hat{\theta}} \end{aligned}$$

Results

- $\hat{\mu}_i, \hat{\sigma}_i^2$ and $\hat{\sigma}_{ij}$ are unbiased estimators:

$$\begin{aligned} E\left[\hat{\mu}_i\right] &= \mu_i \\ E\left[\hat{\sigma}_i^2\right] &= \sigma_i^2 \\ E\left[\hat{\sigma}_{ij}\right] &= \sigma_{ij} \end{aligned}$$

- $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are biased estimators

$$\begin{aligned} E[\hat{\sigma}_i] &\neq \sigma_i \\ E[\hat{\rho}_{ij}] &\neq \rho_{ij} \end{aligned}$$

but bias disappears as sample size T gets large

Remarks

- “On average” being correct doesn’t mean the estimate is any good for your sample!

- The value of SE($\hat{\theta}$) will tell you how far from θ the estimate $\hat{\theta}$ typically will be.

Proof that $E\left[\hat{\mu}_i\right] = \mu_i$

Recall,

$$\begin{aligned} \hat{\mu}_i &= \frac{1}{T}\sum_{t=1}^{T} r_{it} \\ r_{it} &= \mu_i + \epsilon_{it},\ \epsilon_{it} \sim \text{iid } N(0, \sigma^2) \end{aligned}$$

Now

$$E[r_{it}] = \mu_i + E[\epsilon_{it}] = \mu_i$$

since $E[\epsilon_{it}] = 0$.

Therefore,

$$
\begin{aligned}
E[\hat{\mu}_i] &= \frac{1}{T}\sum_{t=1}^{T} E[r_{it}] \\
&= \frac{1}{T}\sum_{t=1}^{T} \mu_i \\
&= \frac{1}{T} T\mu_i = \mu_i
\end{aligned}
$$

Standard Error formulas for $\hat{\mu}_i,\ \hat{\sigma}_i,\ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$

$$
\begin{aligned}
\mathsf{SE}(\hat{\mu}_i) &= \frac{\sigma_i}{\sqrt{T}} \\
\mathsf{SE}(\hat{\sigma}_i^2) &\approx \frac{\sigma_i^2}{\sqrt{T/2}} \\
\mathsf{SE}(\hat{\sigma}_i) &\approx \frac{\sigma_i}{\sqrt{2T}} \\
\mathsf{SE}(\hat{\sigma}_{ij}) &: \quad \text{no easy formula!} \\
\mathsf{SE}(\hat{\rho}_{ij}) &\approx \frac{(1-\rho_{ij}^2)}{\sqrt{T}}
\end{aligned}
$$

Note: "$\approx$" denotes "approximately equal to", where approximation error $\longrightarrow 0$ as $T \longrightarrow \infty$ for normally distributed data.

Remarks

- Large SE$\Longrightarrow$ imprecise estimate; Small SE$\Longrightarrow$ precise estimate
- Precision increases with sample size: SE$\longrightarrow 0$ as $T \longrightarrow \infty$
- $\hat{\sigma}_i$ is generally a more precise estimate than $\hat{\mu}_i$ or $\hat{\rho}_{ij}$
- SE formulas for $\hat{\sigma}_i,\ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ are approximations. Monte Carlo simulation and bootstrapping can be used to get better approximations
- SE formulas depend on unknown values of parameters $\Rightarrow$ formulas are not practically useful

- Practically useful formulas replace unknown values with estimated values:

$$\begin{aligned}
\widehat{\mathsf{SE}}(\hat{\mu}_i) &= \frac{\hat{\sigma}_i}{\sqrt{T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i \\
\widehat{\mathsf{SE}}(\hat{\sigma}_i^2) &\approx \frac{\hat{\sigma}_i^2}{\sqrt{T/2}},\ \hat{\sigma}_i^2 \text{ replaces } \sigma_i^2 \\
\widehat{\mathsf{SE}}(\hat{\sigma}_i) &\approx \frac{\hat{\sigma}_i}{\sqrt{2T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i \\
\widehat{\mathsf{SE}}(\hat{\rho}_{ij}) &\approx \frac{(1-\hat{\rho}_{ij}^2)}{\sqrt{T}},\ \hat{\rho}_{ij} \text{ replaces } \rho_{ij}
\end{aligned}$$

Deriving SE($\hat{\mu}_i$)

$$\begin{aligned}
\text{var}(\hat{\mu}_i) &= \text{var}\left(\frac{1}{T}\sum_{t=1}^{T} r_{it}\right) \\
&= \frac{1}{T^2}\sum_{t=1}^{T} \text{var}(r_{it}) \text{ (since } r_{it} \text{ are independent)} \\
&= \frac{1}{T^2}\sum_{t=1}^{T} \sigma_i^2 = \frac{\sigma_i^2}{T} \text{ (since var}(r_{it}) = \sigma^2\text{)} \\
\text{SE}(\hat{\mu}_i) &= \sqrt{\text{var}(\hat{\mu}_i)} = \frac{\sigma_i}{\sqrt{T}}
\end{aligned}$$

Consistency

Definition: An estimator $\hat{\theta}$ is consistent for θ (converges in probability to θ) if for any $\varepsilon > 0$

$$\lim_{T \to \infty} \Pr(|\hat{\theta} - \theta| > \varepsilon) = 0$$

Intuitively, as we get enough data then $\hat{\theta}$ will eventually equal θ.

Result: An estimator $\hat{\theta}$ is consistent for θ if

- bias$(\hat{\theta}, \theta) = 0$ as $T \rightarrow \infty$
- SE$\left(\hat{\theta}\right) = 0$ as $T \rightarrow \infty$

Result: In the CER model, the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are consistent.

Distribution of CER Model Estimators

$$\begin{aligned} \theta &= \text{parameter to be estimated} \\ \hat{\theta} &= \text{estimator of } \theta \text{ from random sample} \end{aligned}$$

KEY POINTS

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample
- $f(\hat{\theta})$ = pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample
- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Example: Distribution of $\hat{\mu}$ in CER Model

$$\begin{aligned}\hat{\mu}_i &= \frac{1}{T}\sum_{t=1}^{T} r_{it} \\ r_{it} &= \mu_i + \epsilon_{it},\ \epsilon_{it} \sim \text{iid } N(0,\sigma^2)\end{aligned}$$

Result:

$\hat{\mu}_i$ is $\frac{1}{T}$ times the sum of T normally distributed random variables $\Rightarrow$ $\hat{\mu}_i$ is also normally distributed with

$$\begin{aligned}E[\hat{\mu}_i] &= \mu_i \\ \text{var}(\hat{\mu}_i) &= \frac{\sigma^2}{T}\end{aligned}$$

That is,

$$\hat{\mu}_i \sim N\left(\mu_i, \frac{\sigma^2}{T}\right)$$

Distribution of $\hat{\sigma}_i, \ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$

Result: The exact distributions of $\hat{\sigma}_i, \ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ are not normal.

However, as the sample size gets large the exact distributions of $\hat{\sigma}_i, \ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ get closer and closer to the normal distribution. This is the due to the famous Central Limit Theorem.

Central Limit Theorem (CLT)

Let $X_1, \ldots, X_T$ be a iid random variables with $E[X_t] = \mu$ and $\text{var}(X_t) = \sigma^2$. Then

$$\frac{\bar{X} - \mu}{\text{SE}(\bar{X})} = \frac{\bar{X} - \mu}{\sigma/\sqrt{T}} \sim N(0, 1) \text{ as } T \to \infty$$

Equivalently,

$$\bar{X} \sim N\left(\mu, \text{SE}(\bar{X})^2\right) \sim N\left(\mu, \frac{\sigma^2}{T}\right)$$
$$\text{for large enough } T$$

We say that $\bar{X}$ is asymptotically normally distributed with mean μ and variance $\text{SE}(\bar{X})^2$.

Definition: An estimator $\hat{\theta}$ is asymptotically normally distributed if

$$\hat{\theta} \sim N(\theta, \mathsf{SE}(\hat{\theta})^2)$$
$$\text{for large enough } T$$

Result: An implication of the CLT is that the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are asymptotically normally distributed under the CER model.

Confidence Intervals

$$\begin{aligned}\hat{\theta} &= \text{estimate of } \theta \\ &= \text{best guess for unknown value of } \theta\end{aligned}$$

Idea: A confidence interval for θ is an interval estimate of θ that covers θ with a stated probability

Result: Let $\hat{\theta}$ be an asymptotically normal estimator for θ. Then

- An approximate 95% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right]$$
$$\hat{\theta} \pm 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)$$

  that covers θ with probability approximately equal to 95. That is

$$\Pr\left\{\hat{\theta} - 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right) \leq \theta \leq \hat{\theta} + 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right\} \approx 0.95$$

- An approximate 99% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right]$$

$$\hat{\theta} \pm 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 99%

Remarks

- 99% confidence intervals are wider than 95% confidence intervals
- For a given confidence level the width of a confidence interval depends on the size of $\widehat{\mathsf{SE}}(\hat{\theta})$

In the CER model, 95% Confidence Intervals for $\mu_i,\ \sigma_i,$ and ρ_{ij} are:

$$\hat{\mu}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{T}}$$
$$\hat{\sigma}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{2T}}$$
$$\hat{\rho}_{ij} \pm 2 \cdot \frac{(1 - \hat{\rho}_{ij}^2)}{\sqrt{T}}$$

Using Monte Carlo Simulation to Evaluate Bias, Standard Error and Confidence Interval Coverage

- Create many simulated samples from CER model
- Compute parameter estimates for each simulated sample
- Compute mean and sd of estimates over simulated samples
- Compute 95% confidence interval for each sample
- Count number of intervals that cover true parameter

1.4 Value-at-Risk in the CER Model

In the CER model

$$r_{it} \sim N(\mu_i, \sigma_i^2)$$

The $\alpha \cdot 100\%$ quantile q_α may be expressed as

$$\begin{aligned} q_\alpha &= \mu_i + \sigma_i q_\alpha^Z \\ q_\alpha^Z &= \text{standard Normal quantile} \end{aligned}$$

To see this, note

$$\begin{aligned} \Pr(r_{it} &\le \mu_i + \sigma_i q_\alpha^Z) = \Pr\left(\frac{r_{it} - \mu_i}{\sigma_i} \le q_\alpha^Z\right) \\ &= \Pr(Z \le q_\alpha^Z),\ Z \sim N(0,1) \end{aligned}$$

Estimating Quantiles from CER Model

$$\begin{aligned}\hat{q}_\alpha &= \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z \\ q_\alpha^Z &= \text{standard Normal quantile}\end{aligned}$$

Estimating Value-at-Risk from CER Model

$$\begin{aligned}\widehat{\mathsf{VaR}}_\alpha &= (\exp(\hat{q}_\alpha) - 1) \cdot W_0 \\ \hat{q}_\alpha &= \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z \\ W_0 &= \text{initial investment in \$}\end{aligned}$$

Example: $r_t \sim N(0.02, (0.10)^2)$ and $W_0 = \$10,000$

$$
\begin{aligned}
q_{.05}^Z &= -1.645 \\
q_{.05} &= 0.02 + (0.10)(-1.645) = -0.1445 \\
\widehat{\text{VaR}}_\alpha &= (\exp(-0.1145) - 1) \cdot 10,000 = -1,345
\end{aligned}
$$

Computing Standard Errors for VaR

- We can compute $\mathsf{SE}(\hat{q}_\alpha)$ using

$$\begin{aligned}
\mathsf{var}(\hat{q}_\alpha) &= \mathsf{var}(\hat{\mu}_i) + \left(q_\alpha^Z\right)^2 \mathsf{var}(\hat{\sigma}_i) + 2\mathsf{cov}(\hat{\mu}_i, \hat{\sigma}_i) \\
&= \mathsf{var}(\hat{\mu}_i) + \left(q_\alpha^Z\right)^2 \mathsf{var}(\hat{\sigma}_i), \text{ since } \mathsf{cov}(\hat{\mu}_i, \hat{\sigma}_i) = 0 \\
&\Rightarrow \mathsf{SE}(\hat{q}_\alpha) \neq \mathsf{SE}(\hat{\mu}_i) + \mathsf{SE}(\hat{\sigma}_i) \cdot q_\alpha^Z
\end{aligned}$$

  Then

$$\mathsf{SE}(\hat{q}_\alpha) = \sqrt{\mathsf{var}(\hat{\mu}_i) + \left(q_\alpha^Z\right)^2 \mathsf{var}(\hat{\sigma}_i)}$$

- However, computing $\mathsf{SE}(\widehat{\mathsf{VaR}}_\alpha)$ is difficult since

$$\mathsf{var}\left(\widehat{\mathsf{VaR}}_\alpha\right) = \mathsf{var}\left((\exp(\hat{q}_\alpha) - 1) \cdot W_0\right)$$

1.5 Monte Carlo Simulation: Multivariate Returns

Example: Simulating observations from CER model for three assets

- Specify parameters based on sample statistics (use monthly data from June 1992 - Oct 2000)

$$\mu_{SBUX} = .028,\ \mu_{MSFT} = .028,\ \mu_{SP500} = .012$$

$$\Sigma = \begin{pmatrix} .018 & .004 & .002 \\ & .011 & .002 \\ & & .001 \end{pmatrix}$$

$$r_{it} = \mu_i + \varepsilon_{it},\ t = 1, \ldots, 100$$

$$\varepsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

$$cov(\varepsilon_{it}, \varepsilon_{jt}) = \sigma_{ij}$$

- Simulation requires generating random numbers from a multivariate normal distribution.

- R package `mvtnorm` has function `mvnorm()` for simulating data from a multivariate normal distribution.