Chapter 1

The Constant Expected Return Model

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The first model of asset returns we consider is the very simple *constant expected return* (CER) model. This model is motivated by the stylized facts for monthly asset returns. The CER model assumes that an asset's return (simple or continuously compounded) over time is independent and identically normally distributed with a constant (time invariant) mean and variance. The model allows for the returns on different assets to be contemporaneously correlated but that the correlations are constant over time. The CER model is widely used in finance. For example, it is used in risk analysis (e.g., computing Value-at-Risk) for assets and portfolios, in mean-variance portfolio analysis, in the Capital Asset Pricing Model (CAPM), and in the Black-Scholes option pricing model. Although this model is very simple, it provides important intuition about the statistical behavior of asset returns and prices and serves as a benchmark against which more complicated models can be compared and evaluated. It allows us to discuss and develop several important econometric topics such as Monte Carlo simulation, estimation, bootstrapping, hypothesis testing, forecasting and model evaluation.

1.1 CER Model Assumptions

Let R_{it} denote the simple or continuously compounded (cc) return on asset i over the investment horizon between times $t-1$ and t (e.g., monthly returns).

We make the following assumptions regarding the probability distribution of R_{it} for $i = 1, \ldots, N$ assets for all times t.

Assumption 1

(i) *Covariance stationarity and ergodicity*: $\{R_{i1}, \ldots, R_{iT}\} = \{R_{it}\}_{t=1}^{T}$ is a covariance stationary and ergodic stochastic process with $E[R_{it}] = \mu_i$, $\text{var}(R_{it}) = \sigma_i^2$, $\text{cov}(R_{it}, R_{jt}) = \sigma_{ij}$ and $\text{cor}(R_{it}, R_{jt}) = \rho_{ij}$.

(ii) *Normality*: $R_{it} \sim N(\mu_i, \sigma_i^2)$ for all i and t, and all joint distributions are normal.

(iii) *No serial correlation*: $\text{cov}(R_{it}, R_{js}) = \text{cor}(R_{it}, R_{is}) = 0$ for $t \neq s$ and $i, j = 1, \ldots, N$.

Assumption 1 states that in every time period asset returns are jointly (multivariate) normally distributed, that the means and the variances of all asset returns, and all of the pairwise contemporaneous covariances and correlations between assets are constant over time. In addition, all of the asset returns are *serially uncorrelated*

$$\text{cor}(R_{it}, R_{is}) = \text{cov}(R_{it}, R_{is}) = 0 \text{ for all } i \text{ and } t \neq s,$$

and the returns on all possible pairs of assets i and j are serially uncorrelated

$$\text{cor}(R_{it}, R_{js}) = \text{cov}(R_{it}, R_{js}) = 0 \text{ for all } i \neq j \text{ and } t \neq s.$$

In addition, under the normal distribution assumption lack of serial correlation implies time indpendence of returns over time. Clearly, these are very strong assumptions. However, they allow us to develop a straightforward probabilistic model for asset returns as well as statistical tools for estimating the parameters of the model, testing hypotheses about the parameter values and assumptions.

1.1.1 Regression Model Representation

A convenient mathematical representation or *model* of asset returns can be given based on Assumption 1. This is the CER *regression model*. For assets

$i = 1, \ldots, N$ and time periods $t = 1, \ldots, T$, the CER regression model is

$$\begin{aligned} R_{it} &= \mu_i + \varepsilon_{it}, \\ \{\varepsilon_{it}\}_{t=1}^{T} &\sim \text{GWN}(0, \sigma_i^2), \\ \text{cov}(\varepsilon_{it}, \varepsilon_{js}) &= \begin{cases} \sigma_{ij} & t = s \\ 0 & t \neq s \end{cases}. \end{aligned} \tag{1.1}$$

The notation $\varepsilon_{it} \sim \text{GWN}(0, \sigma_i^2)$ stipulates that the stochastic process $\{\varepsilon_{it}\}_{t=1}^{T}$ is a Gaussian white noise process with $E[\varepsilon_{it}] = 0$ and $\text{var}(\varepsilon_{it}) = \sigma_i^2$. In addition, the random error term ε_{it} is independent of ε_{js} for all assets $i \neq j$ and all time periods $t \neq s$.

Using the basic properties of expectation, variance and covariance, we can derive the following properties of returns in the CER model:

$$\begin{aligned} E[R_{it}] &= E[\mu_i + \varepsilon_{it}] = \mu_i + E[\varepsilon_{it}] = \mu_i, \\ \text{var}(R_{it}) &= \text{var}(\mu_i + \varepsilon_{it}) = \text{var}(\varepsilon_{it}) = \sigma_i^2, \\ \text{cov}(R_{it}, R_{jt}) &= \text{cov}(\mu_i + \varepsilon_{it}, \mu_j + \varepsilon_{jt}) = \text{cov}(\varepsilon_{it}, \varepsilon_{jt}) = \sigma_{ij}, \\ \text{cov}(R_{it}, R_{js}) &= \text{cov}(\mu_i + \varepsilon_{it}, \mu_j + \varepsilon_{js}) = \text{cov}(\varepsilon_{it}, \varepsilon_{js}) = 0,\ t \neq s. \end{aligned}$$

Given that covariances and variances of returns are constant over time implies that the correlations between returns over time are also constant:

$$\begin{aligned} \text{cor}(R_{it}, R_{jt}) &= \frac{\text{cov}(R_{it}, R_{jt})}{\sqrt{\text{var}(R_{it})\text{var}(R_{jt})}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \rho_{ij}, \\ \text{cor}(R_{it}, R_{js}) &= \frac{\text{cov}(R_{it}, R_{js})}{\sqrt{\text{var}(R_{it})\text{var}(R_{js})}} = \frac{0}{\sigma_i \sigma_j} = 0,\ i \neq j, t \neq s. \end{aligned}$$

Finally, since $\{\varepsilon_{it}\}_{t=1}^{T} \sim \text{GWN}(0, \sigma_i^2)$ it follows that $\{R_{it}\}_{t=1}^{T} \sim iid\ N(\mu_i, \sigma_i^2)$. Hence, the CER regression model (1.1) for R_{it} is equivalent to the model implied by Assumption 1.

Interpretation of the CER Regression Model

The CER model has a very simple form and is identical to the *measurement error model* in the statistics literature.[1] In words, the model states that each

[1] In the measurement error model, r_{it} represents the t^{th} measurement of some physical quantity μ_i and ε_{it} represents the random measurement error associated with the measurement device. The value σ_i represents the typical size of a measurement error.

asset return is equal to a constant μ_i (the expected return) plus a normally distributed random variable ε_{it} with mean zero and constant variance. The random variable ε_{it} can be interpreted as representing the *unexpected news* concerning the value of the asset that arrives between time $t-1$ and time t. To see this, (1.1) implies that

$$\varepsilon_{it} = R_{it} - \mu_i = R_{it} - E[R_{it}],$$

so that ε_{it} is defined as the deviation of the random return from its expected value. If the news between times $t-1$ and t is good, then the realized value of ε_{it} is positive and the observed return is above its expected value μ_i. If the news is bad, then ε_{it} is negative and the observed return is less than expected. The assumption $E[\varepsilon_{it}] = 0$ means that news, on average, is neutral; neither good nor bad. The assumption that $\text{var}(\varepsilon_{it}) = \sigma_i^2$ can be interpreted as saying that volatility, or typical magnitude, of news arrival is constant over time. The random news variable affecting asset i, ε_{it}, is allowed to be contemporaneously correlated with the random news variable affecting asset j, ε_{jt}, to capture the idea that news about one asset may spill over and affect another asset. For example, if asset i is Microsoft stock and asset j is Apple Computer stock, then one interpretation of news in this context is general news about the computer industry and technology. Good news should lead to positive values of both ε_{it} and ε_{jt}. Hence these variables will be positively correlated due to a positive reaction to a common news component. Finally, the news on asset j at time s is unrelated to the news on asset i at time t for all times $t \neq s$. For example, this means that the news for Apple in January is not related to the news for Microsoft in February.

1.1.2 Location-Scale Model Representation

Sometimes it is convenient to re-express the regression form of the CER model (1.1) in *location-scale* form

$$\begin{aligned} R_{it} &= \mu_i + \varepsilon_{it} = \mu_i + \sigma_i \cdot Z_{it} \\ \{Z_{it}\}_{t=1}^{T} &\sim \text{GWN}(0,1), \end{aligned} \tag{1.2}$$

where we use the decomposition $\varepsilon_{it} = \sigma_i \cdot Z_{it}$. In this form, the random news shock is the *iid* standard normal random variable Z_{it} scaled by the "news" volatility σ_i. This form is particularly convenient for Value-at-Risk calculations because the $\alpha \times 100\%$ quantile of the return distribution has the simple form

$$q_\alpha^{R_i} = \mu_i + \sigma_i \times q_\alpha^Z$$

where q_α^Z is the $\alpha \times 100\%$ quantile of the standard normal random distribution. Let W_0 be the initial amout of wealth to be invested from time $t-1$ to t. If R_{it} is the simple return then

$$\text{VaR}_\alpha = W_0 \times q_\alpha^{R_i}$$

whereas if R_{it} is the continuously compounded return then

$$\text{VaR}_\alpha = W_0 \times \left(e^{q_\alpha^{R_i}} - 1\right)$$

1.1.3 The CER Model in Matrix Notation

Define the $N \times 1$ vectors $R_t = (R_{1t}, \ldots, R_{Nt})'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'$ and the $N \times N$ symmetric covariance matrix

$$\text{var}(\boldsymbol{\varepsilon}_t) = \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

Then the regression form of the CER model in matrix notation is

$$\begin{aligned} \mathbf{R}_t &= \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t, \\ \boldsymbol{\varepsilon}_t &\sim \text{iid } N(\mathbf{0}, \boldsymbol{\Sigma}), \end{aligned} \tag{1.3}$$

which implies that $R_t \sim iid\ N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

The location-scale form of the CER model in matrix notation makes use of the matrix square root factorization $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2\prime}$ where $\boldsymbol{\Sigma}^{1/2}$ is the lower-triangular matrix square root (usually the Cholesky factorization). Then (1.3) can be rewritten as

$$\begin{aligned} \mathbf{R}_t &= \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}_t, \\ \mathbf{Z}_t &\sim \text{iid } N(\mathbf{0}, \mathbf{I}_N), \end{aligned} \tag{1.4}$$

where $\mathbf{I}_N$ denotes the N-dimensional identity matrix.

1.1.4 The CER Model for Continuously Compounded Returns

The CER model is often used to describe cc returns defined as $R_{it} = \ln(P_{it}/P_{it-1})$ where P_{it} is the price of asset i at time t. This is particularly convenient for investment risk analysis. An advantage of the CER model for cc returns is that the model aggregates to any time horizon because multi-period cc returns are additive. The CER model for cc returns also gives rise to the random walk model for the logarithm of asset prices. The normal distribution assumption of the CER model for cc returns implies that single-period simple returns are log-normally distributed.

A disadvantage of the CER model for cc returns is that the model has some limitations for the analysis of portfolios because the cc return on a portfolio of assets is not a weighted average of the cc returns on the individual securities. As a result, for portfolio analysis the CER model is typically applied to simple returns.

Time Aggregation and the CER Model

The CER model for cc returns has the following nice aggregation property with respect to the interpretation of ε_{it} as news. For illustration purposes, suppose that t represents months so that R_{it} is the cc monthly return on asset i. Now, instead of the monthly return, suppose we are interested in the annual cc return $R_{it} = R_{it}(12)$. Since multi-period cc returns are additive, $R_{it}(12)$ is the sum of 12 monthly cc returns:

$$R_{it} = R_{it}(12) = \sum_{k=0}^{11} R_{it-k} = R_{it} + R_{it-1} + \cdots + R_{it-11}.$$

Using the CER regression model (1.1) for the monthly return R_{it}, we may express the annual return $R_{it}(12)$ as

$$R_{it}(12) = \sum_{t=0}^{11}(\mu_i + \varepsilon_{it}) = 12 \cdot \mu_i + \sum_{t=0}^{11} \varepsilon_{it} = \mu_i(12) + \varepsilon_{it}(12),$$

where $\mu_i(12) = 12 \cdot \mu_i$ is the annual expected return on asset i, and $\varepsilon_{it}(12) = \sum_{k=0}^{11} \varepsilon_{it-k}$ is the annual random news component. The annual expected return, $\mu_i(12)$, is simply 12 times the monthly expected return, μ_i. The annual random news component, $\varepsilon_{it}(12)$, is the accumulation of news over the year. As a result, the variance of the annual news component, $(\sigma_i(12))^2$, is 12 times the variance of the monthly news component:

$$\begin{aligned}
\mathrm{var}(\varepsilon_{it}(12)) &= \mathrm{var}\left(\sum_{k=0}^{11} \varepsilon_{it-k}\right) \\
&= \sum_{k=0}^{11} \mathrm{var}(\varepsilon_{it-k}) \text{ since } \varepsilon_{it} \text{ is uncorrelated over time} \\
&= \sum_{k=0}^{11} \sigma_i^2 \text{ since } \mathrm{var}(\varepsilon_{it}) \text{ is constant over time} \\
&= 12 \cdot \sigma_i^2 = \sigma_i^2(12).
\end{aligned}$$

It follows that the standard deviation of the annual news is equal to $\sqrt{12}$ times the standard deviation of monthly news:

$$\mathrm{SD}(\varepsilon_{it}(12)) = \sqrt{12} \times \mathrm{SD}(\varepsilon_{it}) = \sqrt{12} \times \sigma_i.$$

Similarly, due to the additivity of covariances, the covariance between $\varepsilon_{it}(12)$ and $\varepsilon_{jt}(12)$ is 12 times the monthly covariance:

$$\begin{aligned}
\mathrm{cov}(\varepsilon_{it}(12), \varepsilon_{jt}(12)) &= \mathrm{cov}\left(\sum_{k=0}^{11} \varepsilon_{it-k}, \sum_{k=0}^{11} \varepsilon_{jt-k}\right) \\
&= \sum_{k=0}^{11} \mathrm{cov}(\varepsilon_{it-k}, \varepsilon_{jt-k}) \text{ since } \varepsilon_{it} \text{ and } \varepsilon_{jt} \text{ are uncorrelated over time} \\
&= \sum_{k=0}^{11} \sigma_{ij} \text{ since } \mathrm{cov}(\varepsilon_{it}, \varepsilon_{jt}) \text{ is constant over time} \\
&= 12 \cdot \sigma_{ij} = \sigma_{ij}^A.
\end{aligned}$$

The above results imply that the correlation between the annual errors $\varepsilon_{it}(12)$ and $\varepsilon_{jt}(12)$ is the same as the correlation between the monthly errors ε_{it} and

ε_{jt} :

$$\begin{aligned}
\mathrm{cor}(\varepsilon_{it}(12), \varepsilon_{jt}(12)) &= \frac{\mathrm{cov}(\varepsilon_{it}(12), \varepsilon_{jt}(12))}{\sqrt{\mathrm{var}(\varepsilon_{it}(12)) \cdot \mathrm{var}(\varepsilon_{jt}(12))}} \\
&= \frac{12 \cdot \sigma_{ij}}{\sqrt{12\sigma_i^2 \cdot 12\sigma_j^2}} \\
&= \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \rho_{ij} = \mathrm{cor}(\varepsilon_{it}, \varepsilon_{jt}).
\end{aligned}$$

The above results generalize to aggregating returns to arbitrary time horizons. Let R_{it} denote the cc return between times $t-1$ and t, where t represents the general investment horizon, and let $R_{it}(k) = \sum_{j=0}^{k-1} R_{it-j}$ denote the k-period cc return. Then the CER model for $R_{it}(k)$ has the form

$$\begin{aligned}
R_{it}(k) &= \mu_i(k) + \varepsilon_{it}(k), \\
\varepsilon_{it}(k) &\sim N(0, \sigma_i^2(k)),
\end{aligned}$$

where $\mu_i(k) = k \times \mu_i$ is the k-period expected return, $\varepsilon_{it}(k) = \sum_{j=0}^{k-1} \varepsilon_{it-j}$ is the k-period error term, and $\sigma_i^2(k) = k \times \sigma_i^2$ is the k-period variance. The k-period volatility follows the *square-root-of-time rule*: $\sigma_i(k) = \sqrt{k} \times \sigma_i$. This aggregation result is exact for cc returns but it is often used as an approximation for simple returns.

The Random Walk Model of Asset Prices

The CER model for cc returns (1.1) gives rise to the so-called *random walk* (RW) model for the *logarithm* of asset prices. To see this, recall that the cc return, R_{it}, is defined from asset prices via $R_{it} = \ln\left(\frac{P_{it}}{P_{it-1}}\right) = \ln(P_{it}) - \ln(P_{it-1})$. Letting $p_{it} = \ln(P_{it})$ and using the representation of R_{it} in the CER model (1.1), we can express the log-price as:

$$p_{it} = p_{it-1} + \mu_i + \varepsilon_{it}. \tag{1.5}$$

The representation in (1.5) is known as the RW model for log-prices.[2] It is a representation of the CER model in terms of log-prices.

[2] The model (1.5) is technically a random walk with drift μ_i. A pure random walk has zero drift ($\mu_i = 0$).

In the RW model (1.5), μ_i represents the expected change in the log-price (cc return) between months $t-1$ and t, and ε_{it} represents the unexpected change in the log-price. That is,

$$\begin{aligned} E[\Delta p_{it}] &= E[R_{it}] = \mu_i, \\ \varepsilon_{it} &= \Delta p_{it} - E[\Delta p_{it}]. \end{aligned}$$

where $\Delta p_{it} = p_{it} - p_{it-1}$. Further, in the RW model, the unexpected changes in log-price, ε_{it}, are uncorrelated over time ($\text{cov}(\varepsilon_{it}, \varepsilon_{is}) = 0$ for $t \neq s$) so that future changes in log-price cannot be predicted from past changes in the log-price.[3]

The RW model gives the following interpretation for the evolution of log prices. Let p_{i0} denote the initial log price of asset i. The RW model says that the log-price at time $t = 1$ is

$$p_{i1} = p_{i0} + \mu_i + \varepsilon_{i1},$$

where ε_{i1} is the value of random news that arrives between times 0 and 1. At time $t = 0$ the expected log-price at time $t = 1$ is

$$E[p_{i1}] = p_{i0} + \mu_i + E[\varepsilon_{i1}] = p_{i0} + \mu_i,$$

which is the initial price plus the expected return between times 0 and 1. Similarly, by recursive substitution the log-price at time $t = 2$ is

$$\begin{aligned} p_{i2} &= p_{i1} + \mu_i + \varepsilon_{i2} \\ &= p_{i0} + \mu_i + \mu_i + \varepsilon_{i1} + \varepsilon_{i2} \\ &= p_{i0} + 2 \cdot \mu_i + \sum_{t=1}^{2} \varepsilon_{it}, \end{aligned}$$

which is equal to the initial log-price, p_{i0}, plus the two period expected return, $2 \cdot \mu_i$, plus the accumulated random news over the two periods, $\sum_{t=1}^{2} \varepsilon_{it}$. By repeated recursive substitution, the log price at time $t = T$ is

$$p_{iT} = p_{i0} + T \cdot \mu_i + \sum_{t=1}^{T} \varepsilon_{it}.$$

[3] The notion that future changes in asset prices cannot be predicted from past changes in asset prices is often referred to as the weak form of the efficient markets hypothesis.

At time $t = 0$, the expected log-price at time $t = T$ is

$$E[p_{iT}] = p_{i0} + T \cdot \mu_i,$$

which is the initial price plus the expected growth in prices over T periods. The actual price, p_{iT}, deviates from the expected price by the accumulated random news:

$$p_{iT} - E[p_{iT}] = \sum_{t=1}^{T} \varepsilon_{it}.$$

At time $t = 0$, the variance of the log-price at time T is

$$\text{var}(p_{iT}) = \text{var}\left(\sum_{t=1}^{T} \varepsilon_{it}\right) = T \cdot \sigma_i^2$$

Hence, the RW model implies that the stochastic process of log-prices $\{p_{it}\}$ is non-stationary because the variance of p_{it} increases with t. Finally, because $\varepsilon_{it} \sim iid\ N(0, \sigma_i^2)$ it follows that (conditional on p_{i0}) $p_{iT} \sim N(p_{i0} + T\mu_i, T\sigma_i^2)$.

The term *random walk* was originally used to describe the unpredictable movements of a drunken sailor staggering down the street. The sailor starts at an initial position, p_0, outside the bar. The sailor generally moves in the direction described by μ but randomly deviates from this direction after each step t by an amount equal to ε_t. After T steps the sailor ends up at position $p_T = p_0 + \mu \cdot T + \sum_{t=1}^{T} \varepsilon_t$. The sailor is expected to be at location μT, but where he actually ends up depends on the accumulation of the random changes in direction $\sum_{t=1}^{T} \varepsilon_t$. Because $\text{var}(p_T) = \sigma^2 T$, the uncertainty about where the sailor will be increases with each step.

The RW model for log-prices implies the following model for prices:

$$P_{it} = e^{p_{it}} = P_{i0}e^{\mu_i \cdot t + \sum_{s=1}^{t} \varepsilon_{is}} = P_{i0}e^{\mu_i t}e^{\sum_{s=1}^{t} \varepsilon_{is}},$$

where $p_{it} = p_{i0} + \mu_i t + \sum_{s=1}^{t} \varepsilon_s$. The term $e^{\mu_i t}$ represents the expected exponential growth rate in prices between times 0 and time t, and the term $e^{\sum_{s=1}^{t} \varepsilon_{is}}$ represents the unexpected exponential growth in prices. Here, conditional on P_{i0}, P_{it} is log-normally distributed because $p_{it} = \ln P_{it}$ is normally distributed.

1.1.5 CER Model for Simple Returns

For simple returns, defined as $R_{it} = \frac{P_{it} - P_{it-1}}{P_{it-1}}$, the CER model is often used for the analysis of portfolios as discussed in Chapters xxx and xxx. The reason is that the simple return on a portfolio of N assets is weighted average of the simple returns on the individual assets. Hence, the CER model for simple returns extends naturally to portfolios of assets.

CER Model and Portfolios

Consider the CER model in matrix form (1.3) for the $N \times 1$ vector of simple returns $R_t = (R_{1t}, \ldots, R_{Nt})'$. For a vector of portfolio weights $\mathbf{w} = (w_1, \ldots, w_N)$ such that $\mathbf{w}'\mathbf{1} = \sum w_i = 1$, the simple return on the portfolio is

$$R_{pt} = \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{N} w_i R_{it}$$

Substituting in (1.1) gives the CER model for the portfolio returns

$$R_{pt} = \mathbf{w}'\left(\boldsymbol{\mu} + \boldsymbol{\varepsilon}_t\right) = \mathbf{w}'\boldsymbol{\mu} + \mathbf{w}'\boldsymbol{\varepsilon}_t = \mu_p + \varepsilon_{pt} \tag{1.6}$$

where $\mu_p = \mathbf{w}'\boldsymbol{\mu} = \sum_{i=1}^{N} w_i\mu_i$ is the portfolio expected return, and $\varepsilon_{pt} = \mathbf{w}'\boldsymbol{\varepsilon}_t = \sum_{i=1}^{N} w_i\varepsilon_{it}$ is the portfolio error. The variance of R_{pt} is given by

$$\text{var}(R_{pt}) = \text{var}(\mathbf{w}'\mathbf{R}_t) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \sigma_p^2.$$

Therefore, the distribution of portfolio returns is normal

$$R_{pt} \sim N(\mu_p, \sigma_p^2).$$

This result is exact for simple returns but is often used as an approximation for cc returns.

CER Model for Multi-Period Simple Returns

The CER model for single period simple returns does not extend exactly to multi-period simple returns because multi-period simple returns are not additive. Recall, the k-period simple return has a multiplicative relationship to single period returns

$$\begin{aligned}
R_t(k) &= (1 + R_t)(1 + R_{t-1}) \times \cdots \times (1 + R_{t-k+1}) - 1 \\
&= R_t + R_{t-1} + \cdots + R_{t-k+1} \\
&\quad + R_t R_{t-1} + R_t R_{t-2} + \cdots + R_{t-k+2} R_{t-k+1}.
\end{aligned}$$

Even though single period returns are normally distributed in the CER model, multi-period returns are not normally distributed because the product of two normally distributed random variables is not normally distributed. Hence, the CER model does not exactly generalize to multi-period simple returns. However, if single period returns are small then all of the cross products of returns are approximately zero ($R_t R_{t-1} \approx \cdots \approx R_{t-k+2} R_{t-k+1} \approx 0$) and

$$\begin{aligned} R_t(k) &\approx R_t + R_{t-1} + \cdots + R_{t-k+1} \\ &\approx \mu(k) + \varepsilon_t(k). \end{aligned}$$

where $\mu(k) = k\mu$ and $\varepsilon_t(k) = \sum_{j=0}^{k-1} \varepsilon_{t-j}$. Hence, the CER model is approximately true for multi-period simple returns when single period simple returns are not too big.

Some exact returns can be derived for the mean and variance of multi-period simple returns. For simplicity, let $k = 2$ so that

$$R_t(2) = (1 + R_t)(1 + R_{t-1}) - 1 = R_t + R_{t-1} + R_t R_{t-1}.$$

Substituting in (1.1) then gives

$$\begin{aligned} R_t(2) &= (\mu + \varepsilon_t) + (\mu + \varepsilon_{t-1}) + (\mu + \varepsilon_t)(\mu + \varepsilon_{t-1}) \\ &= 2\mu + \varepsilon_t + \varepsilon_{t-1} + \mu^2 + \mu\varepsilon_t + \mu\varepsilon_{t-1} + \varepsilon_t\varepsilon_{t-1} \\ &= 2\mu + \mu^2 + \varepsilon_t(1 + \mu) + \varepsilon_{t-1}(1 + \mu) + \varepsilon_t\varepsilon_{t-1}. \end{aligned}$$

The result for the expected return is easy

$$\begin{aligned} E[R_t(2)] &= 2\mu + \mu^2 + (1 + \mu)E[\varepsilon_t] + (1 + \mu)E[\varepsilon_{t-1}] + E[\varepsilon_t\varepsilon_{t-1}] \\ &= 2\mu + \mu^2 = (1 + \mu)^2 - 1, \end{aligned}$$

The result uses the independence of ε_t and ε_{t-1} to get $E[\varepsilon_t\varepsilon_{t-1}] = E[\varepsilon_t]E[\varepsilon_{t-1}] = 0$. The result for the variance, however, is more work

$$\begin{aligned} \text{var}(R_t(2)) &= \text{var}(\varepsilon_t(1 + \mu) + \varepsilon_{t-1}(1 + \mu) + \varepsilon_t\varepsilon_{t-1}) \\ &= (1 + \mu)^2\text{var}(\varepsilon_t) + (1 + \mu)^2\text{var}(\varepsilon_{t-1}) + \text{var}(\varepsilon_t\varepsilon_{t-1}) \\ &\quad +2(1 + \mu)^2\text{cov}(\varepsilon_t, \varepsilon_{t-1}) + 2(1 + \mu)\text{cov}(\varepsilon_t, \varepsilon_t\varepsilon_{t-1}) \\ &\quad +2(1 + \mu)\text{cov}(\varepsilon_{t-1}, \varepsilon_t\varepsilon_{t-1}) \end{aligned}$$

Now, $\text{var}(\varepsilon_t) = \text{var}(\varepsilon_{t-1}) = \sigma^2$ and $\text{cov}(\varepsilon_t, \varepsilon_{t-1}) = 0$. Next, note that

$\text{var}(\varepsilon_t\varepsilon_{t-1}) = E[\varepsilon_t^2\varepsilon_{t-1}^2] - (E[\varepsilon_t\varepsilon_{t-1}])^2 = E[\varepsilon_t^2]E[\varepsilon_{t-1}^2] - (E[\varepsilon_t]E[\varepsilon_{t-1}])^2 = 2\sigma^2.$

Finally,

$$\begin{aligned}\mathrm{cov}(\varepsilon_t, \varepsilon_t\varepsilon_{t-1}) &= E[\varepsilon_t(\varepsilon_t\varepsilon_{t-1})] - E[\varepsilon_t]E[\varepsilon_t\varepsilon_{t-1}] \\ &= E[\varepsilon_t^2]E[\varepsilon_{t-1}] - E[\varepsilon_t]E[\varepsilon_t]E[\varepsilon_{t-1}] \\ &= 0.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{var}(R_t(2)) &= (1+\mu)^2\sigma^2 + (1+\mu)^2\sigma^2 + 2\sigma^2 \\ &= 2\sigma^2[(1+\mu)^2 + 1].\end{aligned}$$

If μ is close to zero then $E[R_t(2)] \approx 2\mu$ and $\mathrm{var}(R_t(2)) \approx 2\sigma^2$ and so the square-root-of-time rule holds approximately.

1.2 Monte Carlo Simulation of the CER Model

A simple technique that can be used to understand the probabilistic behavior of a model involves using computer simulation methods to create pseudo data from the model. The process of creating such pseudo data is called *Monte Carlo simulation*.[4] Monte Carlo simulation of a model can be used as a first pass "reality check" of the model. If simulated data from the model do not look like the data that the model is supposed to describe, then serious doubt is cast on the model. However, if simulated data look reasonably close to the actual data then the first step reality check is passed. Ideally, one should consider many simulated samples from the model because it is possible for a given simulated sample to look strange simply because of an unusual set of random numbers. Monte Carlo simulation can also be used to create "what if?" type scenarios for a model. Different scenarios typically correspond with different model parameter values. Finally, Monte Carlo simulation can be used to study properties of statistics computed from the pseudo data from the model. For example, Monte Carlo simulation can be used to illustrate the concepts of estimator bias and confidence interval coverage probabilities.

To illustrate the use of Monte Carlo simulation, consider creating pseudo return data from the CER model (1.1) for a single asset. The steps to create a Monte Carlo simulation from the CER model are:

1. Fix values for the CER model parameters μ and σ.

[4] Monte Carlo refers to the famous city in Monaco where gambling is legal.

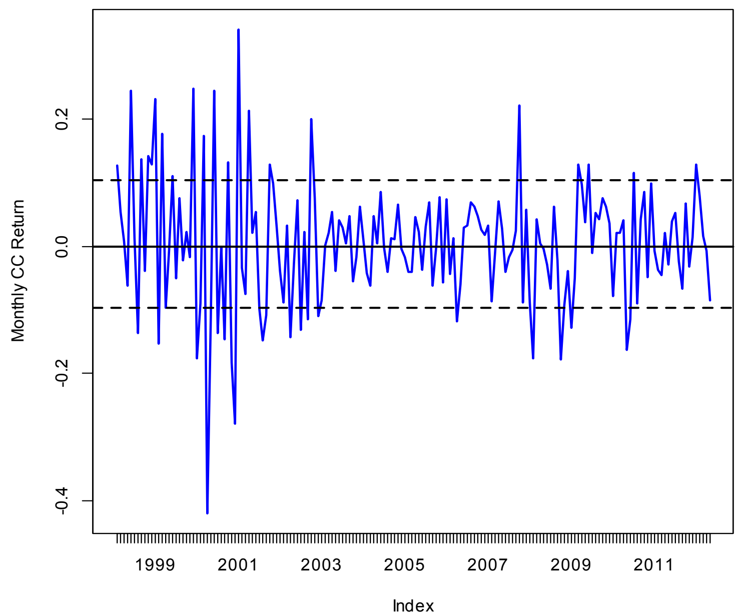

Figure 1.1: Monthly continuously compounded returns on Microsoft. Dashed lines indicate $\hat{\mu} \pm \hat{\sigma}$.

2. Determine the number of simulated values, T, to create.

3. Use a computer random number generator to simulate T *iid* values of ε_t from a $N(0, \sigma^2)$ distribution. Denote these simulated values as $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_T$.

4. Create the simulated return data $\tilde{R}_t = \mu + \tilde{\varepsilon}_t$ for $t = 1, \ldots, T$.

Example 1 *Microsoft data to calibrate univariate Monte Carlo simulation of CER model*

To motivate plausible values for μ and σ in the simulation, Figure 1.1 shows the monthly cc returns on Microsoft stock over the period January 1998 through May 2012. The data is the same as that used in Chapter xxx

(Descriptive Statistics for Finance Data) and is retrieved from Yahoo! using the **tseries** function `get.hist.quote()` as follows

```
> msftPrices = get.hist.quote(instrument="msft", start="1998-01-01",
+                             end="2012-05-31", quote="AdjClose",
+                             provider="yahoo", origin="1970-01-01",
+                             compression="m", retclass="zoo")
> colnames(msftPrices) = "MSFT"
> index(msftPrices) = as.yearmon(index(msftPrices))
> msftRetS = Return.calculate(msftPrices, method="simple")
> msftRetS = msftRetS[-1]
> msftRetC = log(1 + msftRetS)
```

The parameter $\mu = E[R_t]$ in the CER model is the expected monthly return, and σ represents the typical size of a deviation about μ. In Figure 1.1, the returns seem to fluctuate up and down about a central value near 0 and the typical size of a return deviation about 0 is roughly 0.10, or 10% (see dashed lines in figure). The sample mean turns out to be $\hat{\mu} = 0.004$ (0.4%) and the sample standard deviation is $\hat{\sigma} = 0.100$ (10%). Figure 1.2 shows three distribution summaries (histogram, boxplot and normal qq-plot) and the SACF. The returns look to have slightly fatter tails than the normal distribution and show little evidence of linear time dependence (autocorrelation). ■

Example 2 *Simulating observations from the CER model*

To mimic the monthly return data on Microsoft in the Monte Carlo simulation, the values $\mu = 0.004$ and $\sigma = 0.10$ are used as the model's true parameters and $T = 172$ is the number of simulated values (sample size of actual data). Let $\{\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{172}\}$ denote the 172 simulated values of the news variable $\varepsilon_t \sim \text{GWN}(0, (0.10)^2)$.The simulated returns are then computed using[5]

$$\tilde{R}_t = 0.004 + \tilde{\varepsilon}_t, \; t = 1, \ldots, 172 \tag{1.7}$$

To create and plot the simulated returns from (1.7) use

```
> mu = 0.004
> sd.e = 0.10
```

[5] Alternatively, the returns can be simulated directly by simulating observations from a normal distribution with mean 0.0 and standard deviation 0.10.

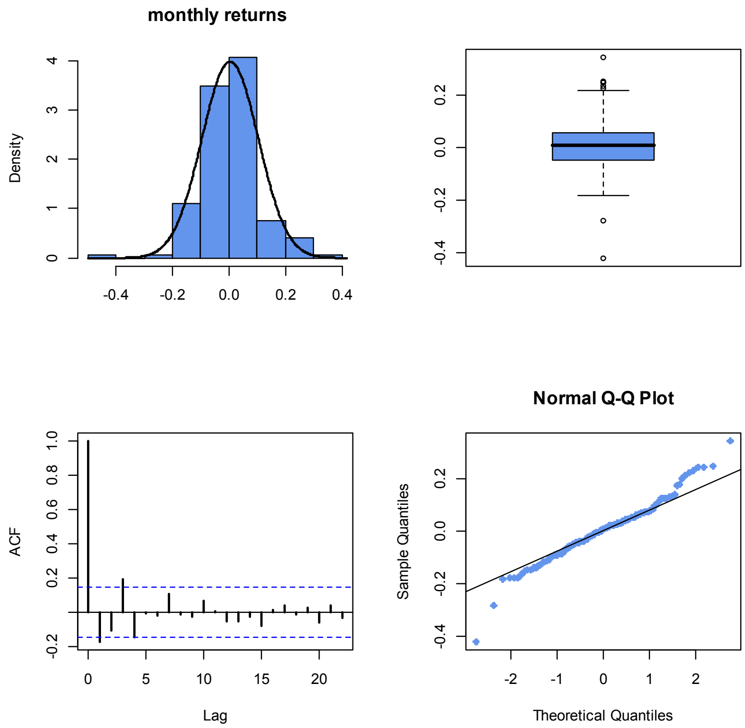

Figure 1.2: Graphical descriptive statistics for the monthly cc returns on Microsoft.

```
> nobs = 172
> set.seed(111)
> sim.e = rnorm(nobs, mean=0, sd=sd.e)
> sim.ret = mu + sim.e
> sim.ret = zoo(sim.ret, index(msftRetC))
> plot(sim.ret, main="",
+       lwd=2, col="blue", ylab="Monthly CC Return")
> abline(h=0, lwd=2)
> abline(h=(mu+sd.e), lty="dashed", lwd=2)
> abline(h=(mu-sd.e), lty="dashed", lwd=2)
```

The simulated returns $\{\tilde{R}_t\}_{t=1}^{172}$ (with the same time index as the Microsoft returns) are shown in Figure **??**. The simulated return data fluctuate ran-

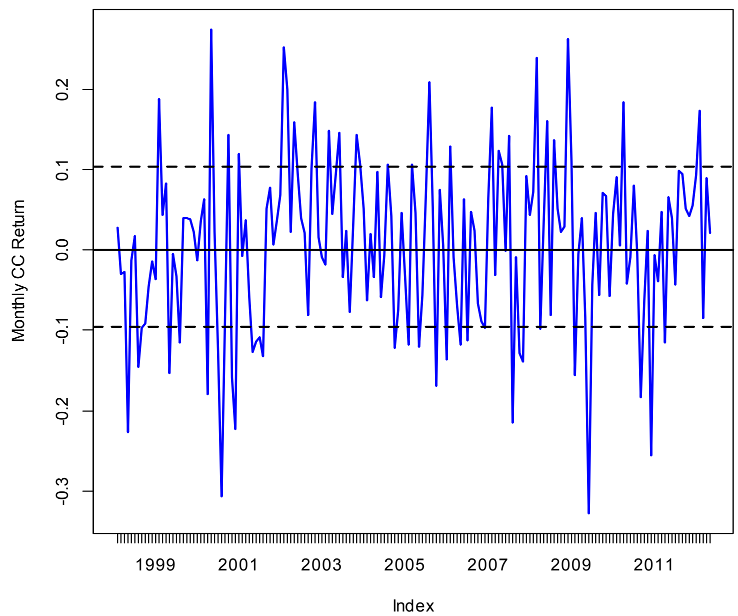

Figure 1.3: Monte Carlo simulated returns from the CER model for Microsoft.

domly about $\mu = 0.004$, and the typical size of the fluctuation is approximately equal to $\sigma = 0.10$. The simulated return data look somewhat like the actual monthly return data for Microsoft. The main difference is that the return volatility for Microsoft appears to have decreased in the latter part of the sample whereas the simulated data has constant volatility over the entire sample. Figure 1.4 shows the distribution summaries (histogram, boxplot and normal qq-plot) and the SACF for the simulated returns. The simulated returns are normally distributed and show thinner tails than the actual returns. The simulated returns also show no evidence of linear time dependence (autocorrelation). ■

Example 3 *Simulating log-prices from the RW model*

The RW model for log-price based on the CER model (1.7) calibrated to

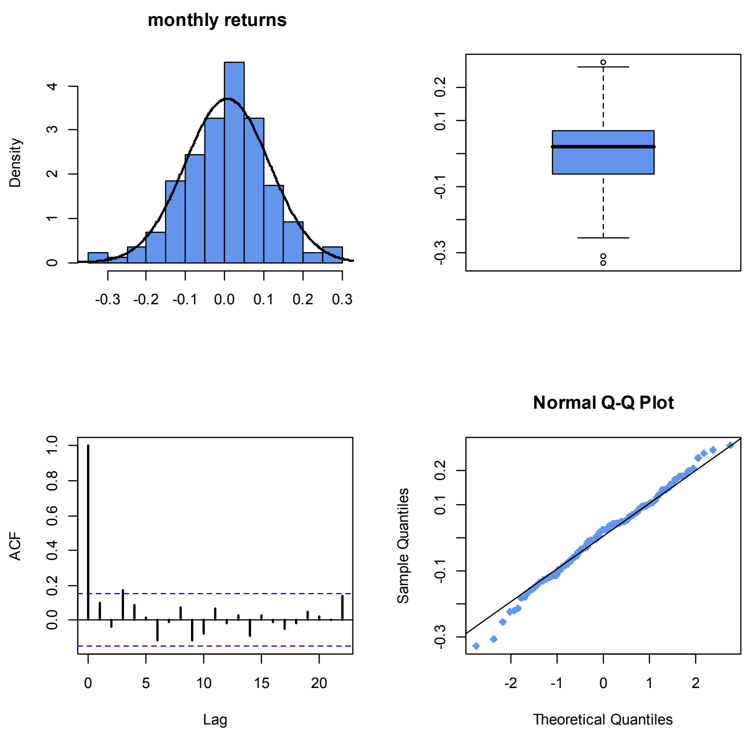

Figure 1.4: Graphical descriptive statistics for the Monte Carlo simulated returns on Microsoft.

Microsoft log prices is

$$p_t = 2.592 + 0.004 \cdot t + \sum_{j=1}^{t} \varepsilon_j, \varepsilon_t \sim \text{GWN}(0, (0.10)^2).$$

where $p_0 = 2.592 = \ln(13.36)$ is the log of first Microsoft Price. A Monte Carlo simulation of this RW model with can be created in R using

```
> sim.p = 2.592 + mu*seq(nobs) + cumsum(sim.e)
> sim.P = exp(sim.p)
```

Figure 1.5 shows the simulated values. The top panel shows the simulated log price, $\tilde{p}_t$ (blue solid line), the expected price $E[\tilde{p}_t] = 2.592 + 0.004 \cdot t$ (green dashed line) and the accumulated random news $\tilde{p}_t - E[\tilde{p}_t] = \sum_{s=1}^{t} \tilde{\varepsilon}_s$

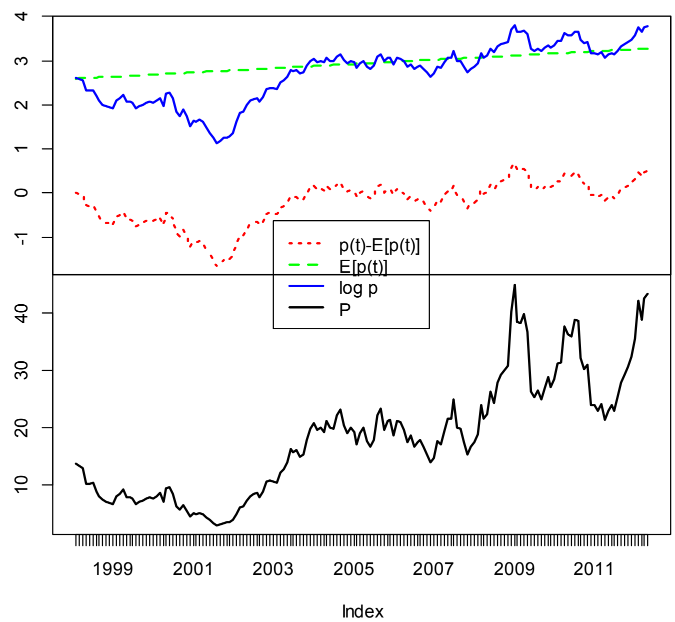

Figure 1.5:

(dotted red line). The bottom panel shows the simulated price levels $\tilde{P}_t = e^{\tilde{p}_t}$ (solid black line). Figure 1.6 shows the actual log prices and price levels for Microsoft stock. Notice the similarity between the simulated random walk data and the actual data. ■

1.2.1 Simulating Returns on More than One Asset

Creating a Monte Carlo simulation of more than one return from the CER model requires simulating observations from a multivariate normal distribution. This follows from the matrix representation of the CER model given in (1.3). The steps required to create a multivariate Monte Carlo simulation are:

1. Fix values for $N \times 1$ mean vector $\boldsymbol{\mu}$ and the $N \times N$ covariance matrix $\boldsymbol{\Sigma}$.

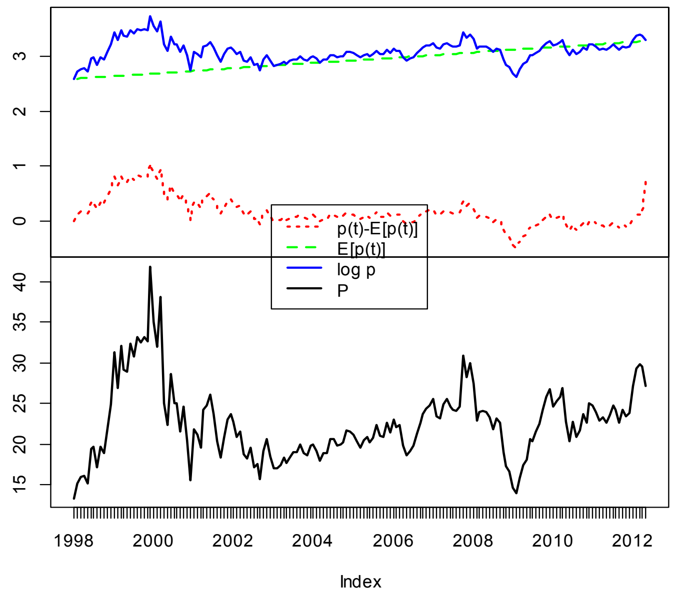

Figure 1.6:

2. Determine the number of simulated values, T, to create.

3. Use a computer random number generator to simulate T *iid* values of the $N \times 1$ random vector $\boldsymbol{\varepsilon}_t$ from the multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$. Denote these simulated vectors as $\tilde{\boldsymbol{\varepsilon}}_1, \ldots, \tilde{\boldsymbol{\varepsilon}}_T$.

4. Create the $N \times 1$ simulated return vector $\tilde{R}_t = \boldsymbol{\mu} + \tilde{\boldsymbol{\varepsilon}}_t$ for $t = 1, \ldots, T$.

Example 4 *Microsoft, Starbucks and S&P 500 data to calibrate multivariate Monte Carlo simulation of CER model*

To motivate the parameters for a multivariate simulation of the CER model, consider the monthly cc returns for Microsoft, Starbucks and the S&P 500 index over the period January 1998 through May 2012 illustrated in Figures 1.7 and 1.8. The data is assembled using the R commands

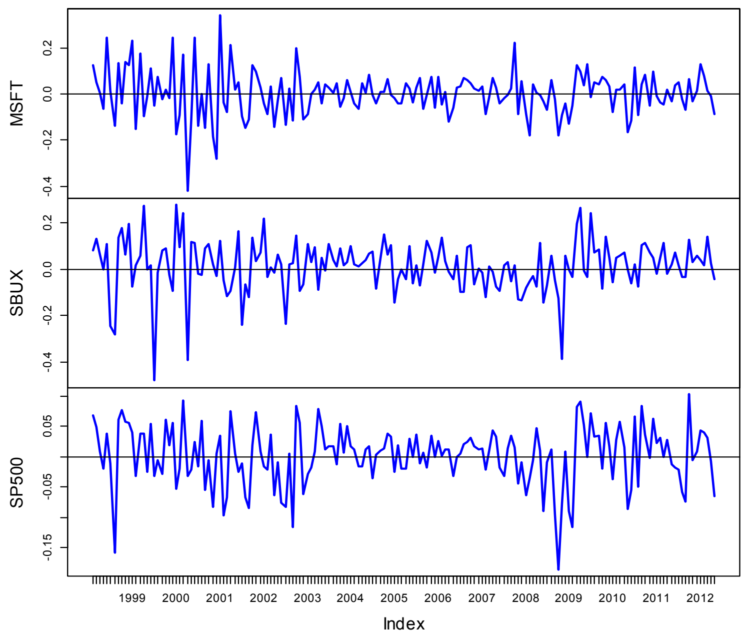

Figure 1.7:

```
> sbuxPrices = get.hist.quote(instrument="sbux", start="1998-01-01",
+                             end="2012-05-31", quote="AdjClose",
+                             provider="yahoo", origin="1970-01-01",
+                             compression="m", retclass="zoo")
> sp500Prices = get.hist.quote(instrument="^gspc", start="1998-01-01",
+                              end="2012-05-31", quote="AdjClose",
+                              provider="yahoo", origin="1970-01-01",
+                              compression="m", retclass="zoo")
> colnames(sbuxPrices) = "SBUX"
> colnames(sp500Prices) = "SP500"
> index(sbuxPrices) = as.yearmon(index(sbuxPrices))
> index(sp500Prices) = as.yearmon(index(sp500Prices))
> cerPrices = merge(msftPrices, sbuxPrices, sp500Prices)
> sbuxRetS = Return.calculate(sbuxPrices, method="simple")
> sp500RetS = Return.calculate(sp500Prices, method="simple")
```

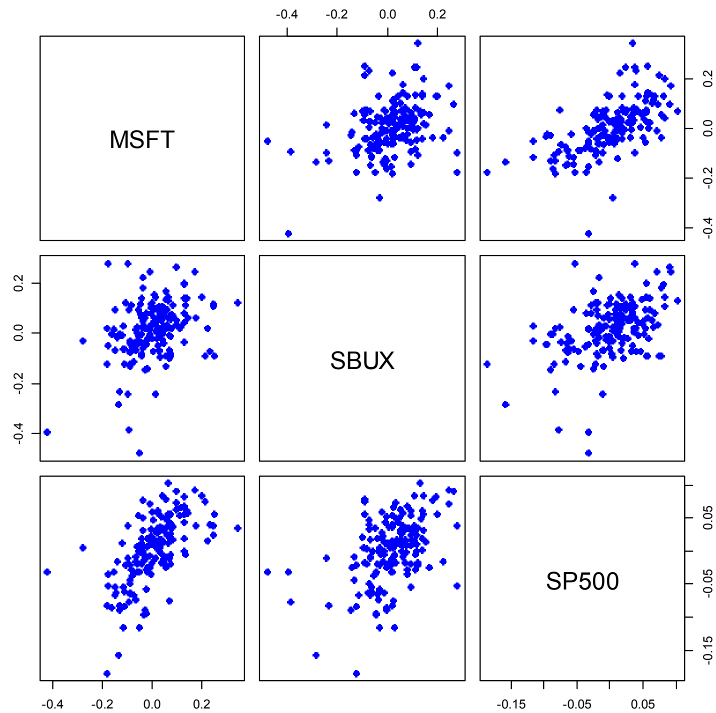

Figure 1.8:

```
> cerRetS = Return.calculate(cerPrices, method="simple")
> sbuxRetS = sbuxRetS[-1]
> sp500RetS = sp500RetS[-1]
> cerRetS = cerRetS[-1]
> sbuxRetC = log(1 + sbuxRetS)
> sp500RetC = log(1 + sp500RetS)
> cerRetC = merge(msftRetC, sbuxRetC, sp500RetC)
```

The multivariate sample descriptive statistics (mean vector, standard deviation vector, covariance matrix and correlation matrix) are

```
> apply(cerRetC, 2, mean)
    MSFT      SBUX     SP500
0.004127 0.014657 0.001687
```

```
> apply(cerRetC, 2, sd)
   MSFT    SBUX   SP500
0.10026 0.11164 0.04847
> cov(cerRetC)
          MSFT     SBUX    SP500
MSFT  0.010051 0.003819 0.003000
SBUX  0.003819 0.012465 0.002476
SP500 0.003000 0.002476 0.002349
> cor(cerRetC)
        MSFT   SBUX  SP500
MSFT  1.0000 0.3412 0.6173
SBUX  0.3412 1.0000 0.4575
SP500 0.6173 0.4575 1.0000
```

All returns fluctuate around mean values close to zero. The volatilities of Microsoft and Starbucks are similar with typical magnitudes around 0.10, or 10%. The volatility of the S&P 500 index is considerably smaller at about 0.05, or 5%. The pairwise scatterplots show that all returns are positively related. The pairs (MSFT, SP500) and (SBUX, SP500) are the most correlated with sample correlation values around 0.5. The pair (MSFT, SBUX) has a moderate positive correlation around 0.3. ■

Example 5 *Monte Carlo simulation of CER model for three assets*

Simulating values from the multivariate CER model (1.3) requires simulating multivariate normal random variables. In R, this can be done using the function `rmvnorm()` from the package **mvtnorm**. The function `rmvnorm()` requires a vector of mean values and a covariance matrix. Define

$$\mathbf{R}_t = \begin{pmatrix} R_{msft,t} \\ R_{sbux,t} \\ R_{sp500,t} \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu_{msft,t} \\ \mu_{sbux,t} \\ \mu_{sp500,t} \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2_{msft} & \sigma_{msft,sbux} & \sigma_{msft,sp500} \\ \sigma_{msft,sbux} & \sigma^2_{sbux} & \sigma_{sbux,sp500} \\ \sigma_{msft,sp500} & \sigma_{sbux,sp500} & \sigma^2_{sp500} \end{pmatrix}$$

The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ of the multivariate CER model are set equal to the sample mean vector $\boldsymbol{\mu}$ and sample covariance matrix $\boldsymbol{\Sigma}$

```
> muVec = apply(cerRetC, 2, mean)
> covMat = cov(cerRetC)
```

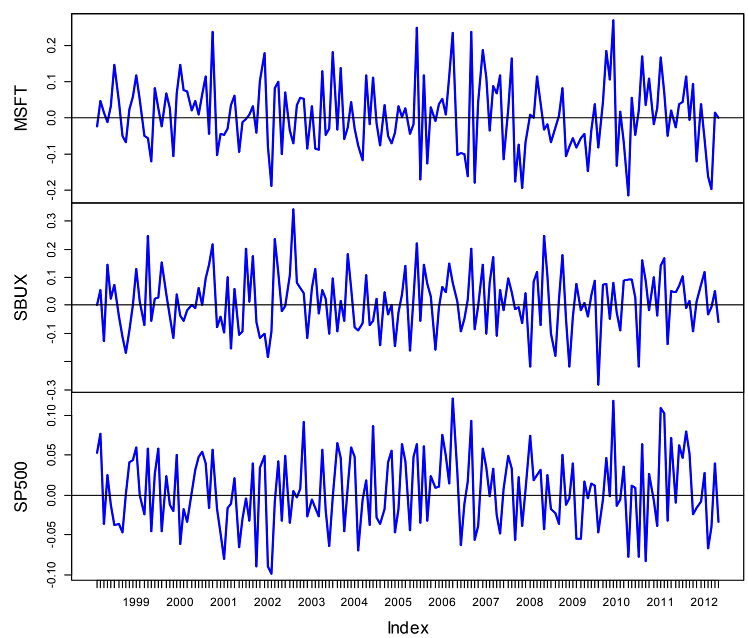

Figure 1.9:

To create a Monte Carlo simulation from the CER model calibrated to the month continuously returns on Microsoft, Starbucks and the S&P 500 index use

```
> set.seed(123)
> returns.sim = rmvnorm(n.obs, mean=muVec, sigma=covMat)
> colnames(returns.sim) = colnames(cerRetC)
> returns.sim = zoo(returns.sim, index(cerRetC))
```

The simulated returns are shown in Figures 1.9 and 1.10. They look similar to the actual returns shown in Figures 1.7 and 1.8. The actual returns show periods of high and low volatility that the simulated returns do not. The sample statistics from the simulated returns, however, are close to the sample statistics of the actual data

```
> apply(returns.sim, 2, mean)
```

```
    MSFT     SBUX    SP500
0.006709 0.013812 0.005080
> apply(returns.sim, 2, sd)
   MSFT    SBUX   SP500
0.09513 0.10512 0.04601
> cov(returns.sim)
          MSFT     SBUX    SP500
MSFT  0.009051 0.003539 0.002464
SBUX  0.003539 0.011050 0.001942
SP500 0.002464 0.001942 0.002117
> cor(returns.sim)
        MSFT   SBUX  SP500
MSFT  1.0000 0.3539 0.5630
SBUX  0.3539 1.0000 0.4015
SP500 0.5630 0.4015 1.0000
```

■

1.3 Conclusions

- Next chapters discuss estimation, hypothesis testing and model validation.

1.4 Further Reading

To be completed

1.5 Problems

To be completed

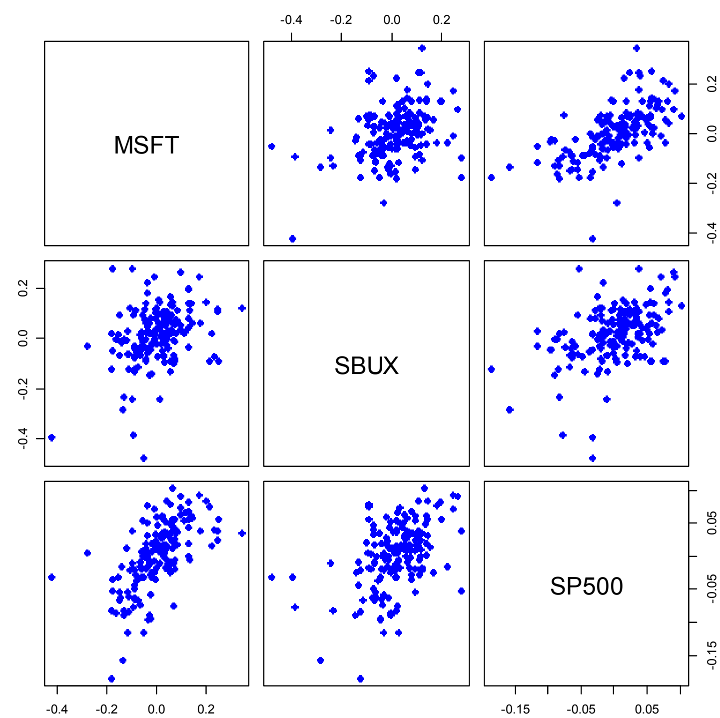

Figure 1.10:

Bibliography

[1] Campbell, Lo and MacKinley (1998). *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.