Econ 424/Amath 462
Capital Asset Pricing Model

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SI Model and Efficient Portfolios

N assets with returns $R_{it} \sim$ iid $N(\mu_i, \sigma_i^2)$

$$\mathbf{R} = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{1N} & \cdots & \sigma_N^2 \end{pmatrix}$$

Assume risk-free asset with return r_f

Compute tangency portfolio allowing short sales

$$\max_{\mathbf{t}} \frac{\mu_{p,\mathsf{tan}} - r_f}{\sigma_{p,\mathsf{tan}}}$$
$$\mu_{p,\mathsf{tan}} = \mathbf{t}'\mu$$
$$\sigma^2_{p,\mathsf{tan}} = \mathbf{t}'\mathbf{\Sigma}\mathbf{t}$$
$$\mathbf{1} = \mathbf{t}'\mathbf{1}$$

For each asset compute "beta" with respect to tangency portfolio using SI model

$$\begin{aligned} R_{it} &= \alpha_i + \beta_{i,\mathsf{tan}} R_{\mathsf{tan},t} + \varepsilon_{it} \\ R_{\mathsf{tan},t} &= \mathbf{t}'\mathbf{R} \\ \beta_{i,\mathsf{tan}} &= \frac{\mathsf{cov}(R_{it}, R_{\mathsf{tan},t})}{\mathsf{var}(R_{\mathsf{tan},t})} \end{aligned}$$

Result: For any asset i

$$\mu_i = r_f + \beta_{i,\mathsf{tan}}(\mu_{p,\mathsf{tan}} - r_f)$$

That is, there is an exact linear relationship between μ_i and $\beta_{i,\mathsf{tan}}$

Recall, for an efficient portfolio that is a combination of T-Bills and the tangency portfolio that has the same expected return as asset i

$$\mu_i = \mu_p^e = r_f + x_{i,\mathsf{tan}}(\mu_{p,\mathsf{tan}} - r_f)$$
$$x_{i,\mathsf{tan}} + x_{i,f} = 1$$

Therefore

$$\beta_{i,\mathsf{tan}} = x_{i,\mathsf{tan}}$$
$$1 - \beta_{i,\mathsf{tan}} = x_f$$

Interpretation: The efficient portfolio with

- $\beta_{i,\mathsf{tan}}$ invested in tangency portfolio

- $1 - \beta_{i,\mathsf{tan}}$ invested in T-Bills

has the same expected return as an investment in asset i, but has lower SD (risk).

Verifying the Proposition with Data

1. Collect data on N assets for sample $t = 1, \ldots, T$

2. Compute tangency portfolio from N assets assuming value for r_f

3. Estimate $\beta_{i,\text{tan}}$ for $i = 1, \ldots, N$ assets using linear regression

$$\hat{R}_{it} = \hat{\alpha}_i + \hat{\beta}_{i,\text{tan}} R_{\text{tan},t}$$

4. Compute average returns for $i = 1, \ldots, N$

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it}$$
$$\hat{\mu}_{p,\mathsf{tan}} = \frac{1}{T}\sum_{t=1}^{T} R_{\mathsf{tan},t}$$

5. Plot $\hat{\mu}_i$ vs. $\hat{\beta}_{i,\mathsf{tan}}$.

$$\begin{aligned} \text{intercept } &= r_f \\ \text{slope } &= (\hat{\mu}_{p,\mathsf{tan}} - r_f) \end{aligned}$$

6a. Estimate the linear regression

$$\hat{\mu}_i = \gamma_0 + \gamma_1 \hat{\beta}_{i,\mathsf{tan}} + \eta_{it}$$

and we should see

$$\begin{aligned}\hat{\gamma}_0 &= r_f \\ \hat{\gamma}_1 &= (\hat{\mu}_{p,\mathsf{tan}} - r_f) \\ R^2 &= 1\end{aligned}$$

6b. Alternatively, estimate the linear regression

$$\hat{\mu}_i - r_f = \delta_0 + \delta_1 \hat{\beta}_{i,\mathsf{tan}} + \eta_{it}$$

and we should see

$$\begin{aligned}\hat{\delta}_0 &= 0 \\ \hat{\delta}_1 &= (\hat{\mu}_{p,\mathsf{tan}} - r_f) \\ R^2 &= 1\end{aligned}$$

Capital Asset Pricing Model (CAPM)

Assumptions

1. Many identical investors who are price takers

2. All investors plan to invest over the same time horizon

3. No taxes or transaction costs

4. Can borrow and lend at risk-free rate, r_f

5. Investors only care about portfolio expected return and variance

   (a) like high μ_p but low σ_p^2

6. Market consists of all publicly traded assets

   (a) market portfolio of assets $=$ value weighted index of all publicly traded assets

$$w_i = \frac{P_i S_i}{\sum_{j=1}^{N} P_j S_j}$$

$$P_i = \text{price of asset } i$$

$$S_i = \text{total shares outstanding}$$

CAPM Conclusions

1. All investors use the Markowitz algorithm and hold 2 portfolios

   - T-bills
   - Tangency portfolio

2. Risk averse investors hold mostly T-Bills (lend at r_f), risk tolerant investors hold mostly tangency portfolio (borrow at r_f to leverage tangency portfolio).

3. In capital market equilibrium

$$\begin{aligned} \text{demand} &= \text{supply} \Rightarrow \\ \text{total borrowing} &= \text{total lending} \end{aligned}$$

4. In equilibrium

$$\begin{aligned} \text{demand for risky assets} &= \text{tangency portfolio} \\ \text{supply of assets} &= \text{market portfolio} \end{aligned}$$

which implies

$$\begin{aligned} \text{tangency portfolio} &= \text{market portfolio} \\ &\Rightarrow \text{market portfolio is efficient} \end{aligned}$$

All efficient portfolios are combinations of T-Bills and the market portfolio

5. In equilibrium

$$E[R_{Mt}] - r_f = (\mu_M - r_f) > 0$$

6. Security Market Line (SML) pricing relationship hold for all assets

$$E[R_{it}] = r_f + \beta_i(E[R_{Mt}] - r_f)$$
$$\beta_i = \frac{\text{cov}(R_{it}, R_{Mt})}{\text{var}(R_{Mt})}$$

or

$$\mu_i = r_f + \beta_i(\mu_M - r_f)$$

Since $(\mu_M - r_f) > 0$

$$\text{high } \beta_i \Rightarrow \text{high } \mu_i$$
$$\text{low } \beta_i \Rightarrow \text{low } \mu_i$$

7. Alternative representation of SML

$$\mu_i - r_f = \beta_i(\mu_M - r_f)$$

Relationship between SI Model and CAPM

SI model

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$
$$R_{Mt} = \text{return on market index}$$

Steps to convert SI model to CAPM regression

1. Set $R_{Mt} =$ true market portfolio

2. Subtract r_f from both sides of SI model

$$R_{it} - r_f = \alpha_i - r_f + \beta_i R_{Mt} + \varepsilon_{it}$$

3. Add and subtract $\beta_i r_f$ from right-hand-side sides of SI model

$$\begin{aligned} R_{it} - r_f &= \alpha_i - r_f + \beta_i r_f - \beta_i r_f + \beta_i R_{Mt} + \varepsilon_{it} \\ &= \alpha_i - r_f(1 - \beta_i) + \beta_i(R_{Mt} - r_f) + \varepsilon_{it} \\ &= \alpha_i^* + \beta_i(R_{Mt} - r_f) + \varepsilon_{it} \end{aligned}$$

where

$$\alpha_i^* = \alpha_i - r_f(1 - \beta_i)$$

4. CAPM security market line (SML) relationship

$$E[R_{it}] - r_f = \beta_i(E[R_{Mt}] - r_f)$$

implies that

$$\alpha_i^* = 0$$

for every asset i.

Regression Test of the CAPM

Use linear regression to estimate the excess returns SI model

$$R_{it} - r_f = \alpha_i^* + \beta_i(R_{Mt} - r_f) + \varepsilon_{it}$$
$$i = 1, \ldots, N \text{ assets}$$

Test the hypotheses

$$H_0 : \alpha_i^* = 0 \text{ (CAPM holds)}$$
$$H_1 : \alpha_i^* \neq 0 \text{ (CAPM does not hold)}$$

for all assets $i = 1, \ldots, N$ assets

Q: What happens if you reject $H_0 : \alpha_i^* = 0$?

Suppose $\alpha_i^* > 0$ (positive "alpha"). Then

$$\alpha_i^* = \left(E[R_{it}] - r_f\right) - \beta_i(E[R_{Mt}] - r_f) > 0$$

so that expected excess return on asset i is greater than what CAPM predicts.

- Asset is underpriced relative to CAPM (expected return too high $\Rightarrow$ current price too low)

- If CAPM is true, then expected return should fall soon which implies that current price should rise soon.

- $\alpha_i^* > 0 \Rightarrow$ buy asset today; $\alpha_i^* < 0 \Rightarrow$ sell asset today

Prediction Test of CAPM

Security Market Line (SML) says

$$\mu_i - r_f = \beta_i(\mu_{Mt} - r_f)$$
$$\mu_{Mt} - r_f > 0$$

Implication:

- High β stocks should have high average returns μ_i
- Low β stocks should have low average returns μ_i

Simple prediction test

1. Compute estimates $\hat{\beta}_i$ for a bunch of assets using excess returns SI model

2. Compute average excess returns $\hat{\mu}_i - r_f$ using sample means

3. Plot $\hat{\mu}_i - r_f$ against $\hat{\beta}_i$

4. Estimate SML using regression

$$\hat{\mu}_i - r_f = \gamma_0 + \gamma_1 \hat{\beta}_i + error_i$$

and test hypotheses

$$H_0 : \gamma_0 = 0 \text{ and } \gamma_1 = \mu_{Mt} - r_f$$
$$H_1 : \gamma_0 \neq 0 \text{ or } \gamma_1 \neq \mu_{Mt} - r_f \text{ or both}$$

Prediction Test II

Idea: A true prediction test would use $\beta' s$ estimated during one period to predict average returns in another period.

Example: 10 years of monthly data

1. Split sample into 2 non-overlapping 5 year sub-samples
2. Estimate β_i over first 5 years
3. Estimate $\mu_i - r_f$ over second 5 years
4. Perform prediction test as described above

Survey of Classical Papers that Test CAPM

1. Litner (1965), "Security Prices, Risk and Maximal Gains from Diversification," *Journal of Finance*.

- Uses annual data on 631 NYSE stocks for 10 years: 1954 - 1963

- Uses In-Sample Prediction Test (2 step process)

  – Compute $\hat{\beta}_1, \ldots, \hat{\beta}_{631}$ and $\hat{\mu}_1 - r_f, \ldots, \hat{\mu}_{631} - r_f$ using full 10 year sample

  – Estimate SML

$$\hat{\mu}_i - r_f = \gamma_0 + \gamma_1 \hat{\beta}_i + error_i$$

– Test

$$H_0 : \gamma_0 = 0 \text{ and } \gamma_1 = \mu_{Mt} - r_f$$
$$H_1 : \gamma_0 \neq 0 \text{ or } \gamma_1 \neq \mu_{Mt} - r_f \text{ or both}$$

Results:

$$\begin{aligned}
\hat{\mu}_i - r_f &= \underset{(0.006)}{0.124} + \underset{(0.006)}{0.042} \cdot \hat{\beta}_i \\
\hat{\mu}_{Mt} - r_f &= 0.165 \\
t_{\gamma_0=0} &= \frac{0.124}{0.006} = 21.16 \\
t_{\gamma_1=0.165} &= \frac{0.042 - 0.165}{0.006} = -20.5
\end{aligned}$$

Conclusion: Estimated SML is too flat!

Problem: Measurement error in $\hat{\beta}_i$ causes downward bias (toward zero) in $\hat{\gamma}_1$

2. Black, Jensen and Scholes, 1972. "The Capital Asset Pricing Model: Some Empirical Tests," in *Studies in the Theory of Capital Markets*.

3. Fama and MacBeth, 1973"Risk, Return and Equilibrium: Empirical Tests," *Journal of Political Economy*.

Contributions: Developed method to get around measurement error in $\hat{\beta}_i$

Main idea: Estimate β_i for portfolios

- Diversification $\Rightarrow$ higher % market risk $\Rightarrow$ more precise $\hat{\beta}_i$

- Construct portfolios with broad range of β_i values (low to high) and estimate SML using portfolios

Three step technique: perform Prediction Test II using 3 non-overlapping sub-samples:

1. 1st sample - estimate β_i for individual assets. Sort assets into 10 portfolios based on β_i

2. 2nd sample - estimate β_i for 10 portfolios

3. 3rd sample - estimate $\mu_i - r_f$ for 10 portfolios, estimate SML

Result: Measurement error in β_i for 10 portfolios is small

- – measurement errors for portfolios are independent from measurement error for individual assets

Black et. al. results

$$\hat{\mu}_i - r_f = \underset{(0.0006)}{0.0036} + \underset{(0.00052)}{0.0108} \cdot \hat{\beta}_i$$
$$\hat{\mu}_M - r_f = 0.0142$$
$$t_{\gamma_0=0} = \frac{0.0036}{0.0006} = 6$$
$$t_{\gamma_1=0.0142} = \frac{0.0108 - 0.0142}{0.006} = 6.54$$

Conclusion: Estimated SML is still too flat, but results look better.

Problems with Prediction Tests

- SML is a relationship between expected returns, $E[R_{it}]$, and true β_i. Both values cannot be observed without error
  - Test uses estimates $\hat{\beta}_i$ and $\hat{\mu}_i$:

$$\hat{\beta}_i = \beta_i + error_i$$
$$\hat{\mu}_i = \mu_i + error_i$$
$$E[error_i] = 0$$

since $\hat{\beta}_i$ and $\hat{\mu}_i$ are unbiased estimates

- β_i is not estimated very precisely for individual assets
  - measurement error in $\hat{\beta}_i$ is large
- Measurement error in $\hat{\beta}_i$ creates bias in estimates of γ_0 and γ_1
- β_i and μ_i may change over time

Measurement Error in Regression

Simple regression

$$y_i = \beta x_i + \varepsilon_i$$
$$\text{cov}(x_i, \varepsilon_i) = 0$$

Then

$$\begin{aligned}
\frac{\text{cov}(y_i, x_i)}{\text{var}(x_i)} &= \frac{\text{cov}(\beta x_i + \varepsilon_i, x_i)}{\text{var}(x_i)} \\
&= \frac{\text{cov}(\beta x_i, x_i) + \text{cov}(\varepsilon_i, x_i)}{\text{var}(x_i)} \\
&= \frac{\beta var(x_i) + 0}{\text{var}(x_i)} = \beta
\end{aligned}$$

Least square estimation: as T gets large

$$\hat{\beta} = \frac{\widehat{\mathsf{cov}}(y_i, x_i)}{\widehat{\mathsf{var}}(x_i)} \rightarrow \frac{\mathsf{cov}(y_i, x_i)}{\mathsf{var}(x_i)} = \beta$$

so that

$$\hat{\beta} \text{ is consistent for } \beta$$

Measurement Error in y_i

Suppose y_i is measured with error

$$\begin{aligned} y_i^* &= y_i + u_i \\ &= \text{ mis-measured } y_i \\ u_i &= \text{measurement error} \\ \text{cov}(x_i, u_i) &= 0 \end{aligned}$$

Regression model with mis-measured y_i^*

$$y_i^* - u_i = \beta x_i + \varepsilon_i$$

Add u_i to both sides

$$\begin{aligned} y_i^* &= \beta x_i + (\varepsilon_i + u_i) \\ &= \beta x_i + v_i \end{aligned}$$

Result: As long as $\text{cov}(u_i, x_i) = 0$, measurement error in y_i^* does not cause any bias in least squares estimate of β :

$$\begin{aligned}
\text{cov}(x_i, v_i) &= \text{cov}(x_i, \varepsilon_i + u_i) \\
&= \text{cov}(x_i, \varepsilon_i) + \text{cov}(x_i, u_i) \\
&= 0
\end{aligned}$$

Measurement Error in x_i

Suppose x_i is measured with error

$$
\begin{aligned}
x_i^* &= x_i + u_i \\
&= \text{ mis-measured } x_i \\
u_i &= \text{measurement error} \\
\text{cov}(x_i^*, u_i) &= 0
\end{aligned}
$$

Regression model with mis-measured x_i^*

$$
\begin{aligned}
y_i &= (x_i^* - u_i)\beta + \varepsilon_i \\
&= x_i^*\beta + \varepsilon_i - \beta u_i \\
&= x_i^*\beta + \varepsilon_i^* \\
\varepsilon_i^* &= \varepsilon_i - \beta u_i
\end{aligned}
$$

Note:

$$
\begin{aligned}
\text{cov}(x_i^*, \varepsilon_i^*) &= \text{cov}(x_i + u_i, \varepsilon_i - \beta u_i) \\
&= \text{cov}(-\beta u_i, u_i) \\
&= -\beta var(u_i)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\frac{\text{cov}(y_i, x_i^*)}{\text{var}(x_i^*)} &= \frac{\text{cov}(\beta x_i^* + \varepsilon_i^*, x_i^*)}{\text{var}(x_i^*)} \\
&= \frac{\text{cov}(\beta x_i^*, x_i^*) + \text{cov}(\varepsilon_i^*, x_i^*)}{\text{var}(x_i^*)} \\
&= \frac{\beta var(x_i^*)}{\text{var}(x_i^*)} + \frac{-\beta var(u_i)}{\text{var}(x_i^*)} \\
&= \beta - \beta \cdot \frac{\text{var}(u_i)}{\text{var}(x_i^*)} \neq \beta
\end{aligned}
$$

Result: If x_i is measured with error then as T gets large

$$\hat{\beta} = \frac{\widehat{\text{cov}}(y_i, x_i)}{\widehat{\text{var}}(x_i)} \rightarrow \beta - \beta \cdot \frac{\text{var}(u_i)}{\text{var}(x_i^*)} \neq \beta$$

so that

$$\hat{\beta} \text{ is not consistent for } \beta$$

Note that

$$\beta - \beta \cdot \frac{\text{var}(u_i)}{\text{var}(x_i^*)} = \beta \left(1 - \frac{\text{var}(u_i)}{\text{var}(x_i^*)}\right)$$

$$\frac{\text{var}(u_i)}{\text{var}(x_i^*)} < 1$$

so that $\hat{\beta}$ is downward biased (toward zero)