Single Index Model

Amath 462/Econ 424
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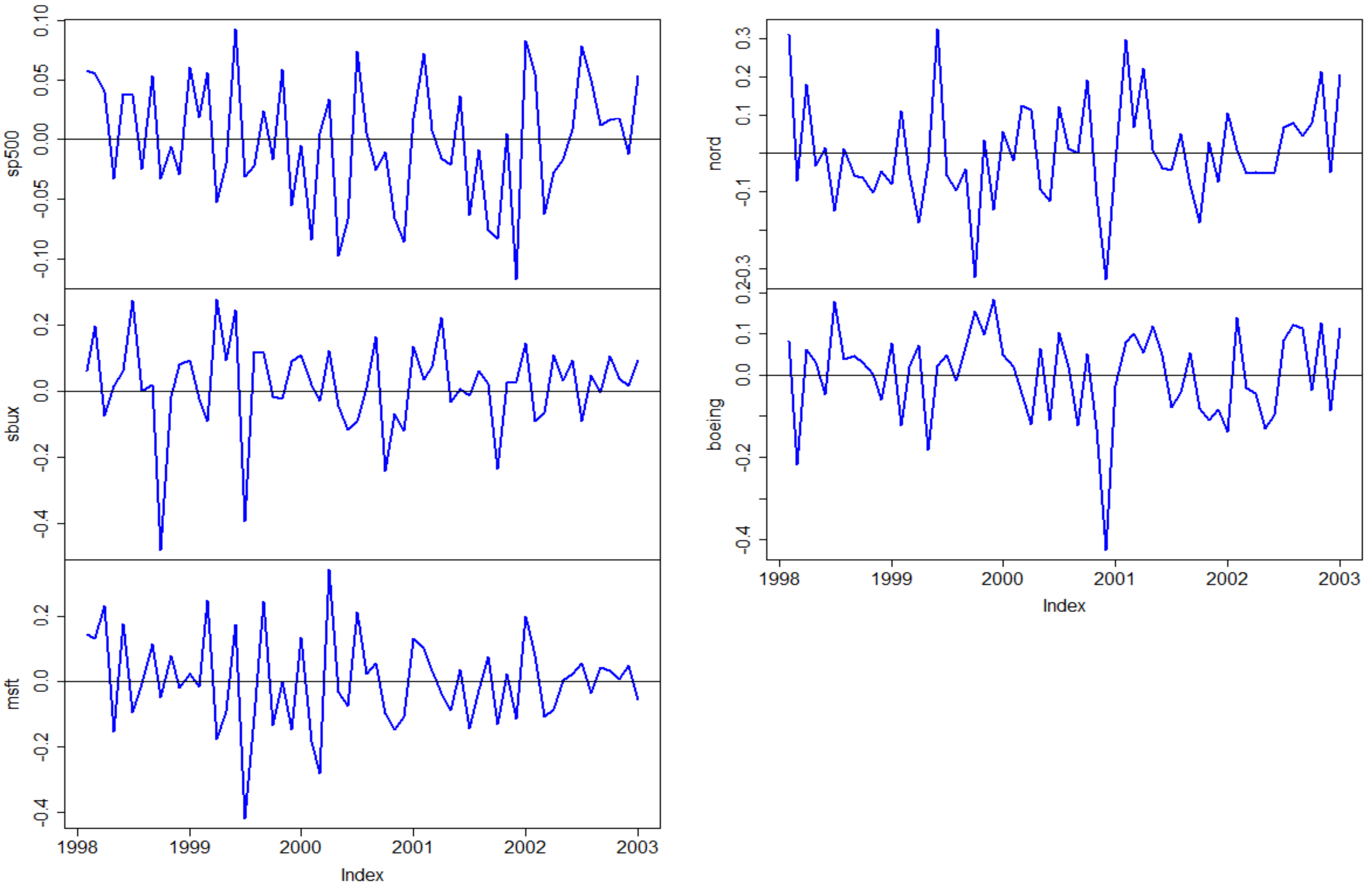
Summer 2013
Updated: August 15, 2013

Example Data

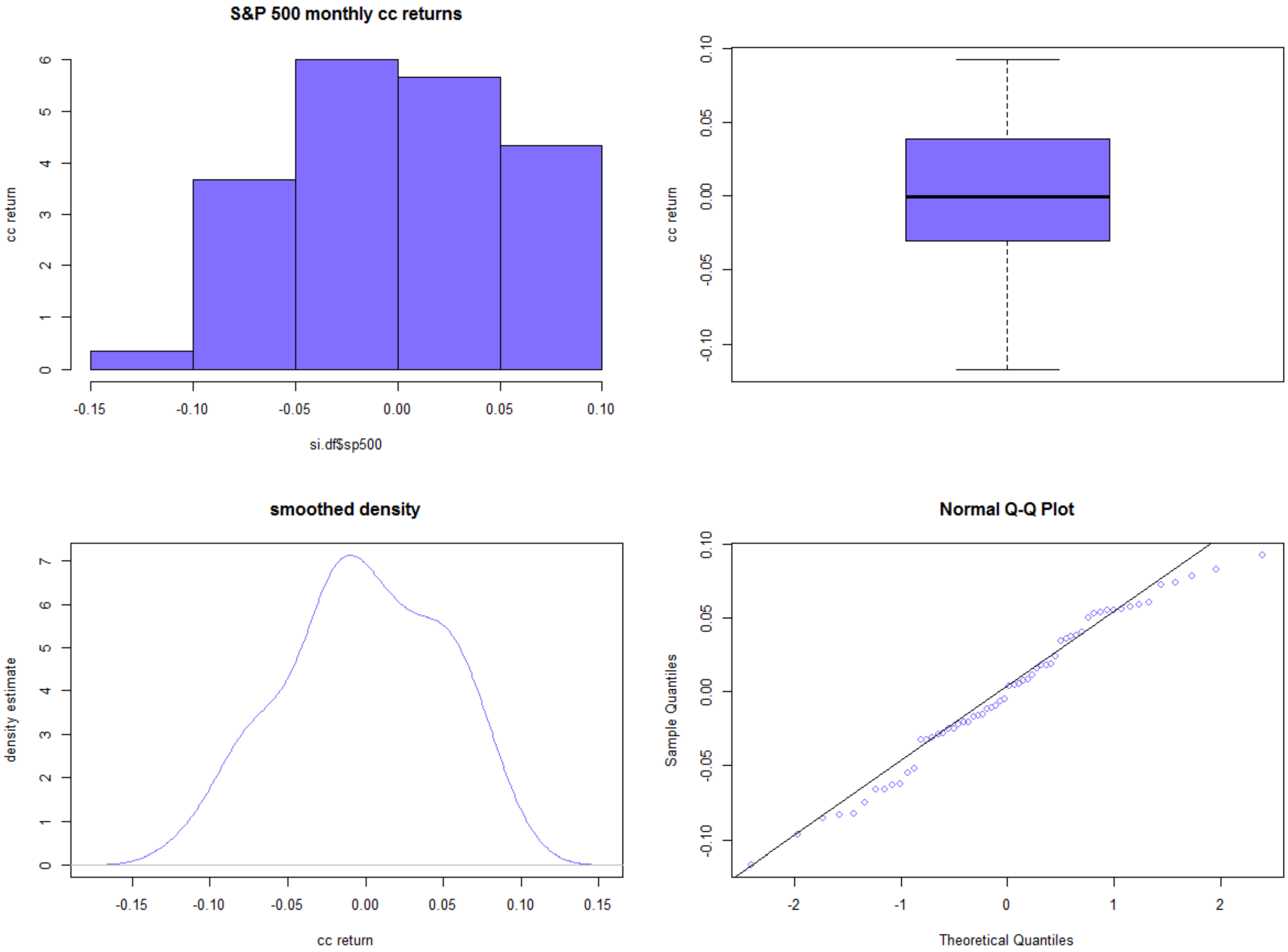
Monthly continuously compounded returns on S&P 500, Starbucks, Microsoft, Nordstrom and Boeing from 1/1/98 – 1/1/2003 (5 years of monthly data)

```
> head(si.df)
             sp500     sbux     msft     nord   boeing
1998-02-01  0.05744  0.06079  0.14155  0.31280  0.08202
1998-03-01  0.05484  0.19625  0.12835 -0.07138 -0.21923
1998-04-01  0.04019 -0.07471  0.23258  0.18243  0.06103
1998-05-01 -0.03282  0.01524 -0.15346 -0.03172  0.03069
1998-06-01  0.03806  0.05947  0.17738  0.01545 -0.04702
1998-07-01  0.03724  0.27495 -0.09734 -0.14975  0.17825
```

Example Data: Returns



Distribution of Market Returns

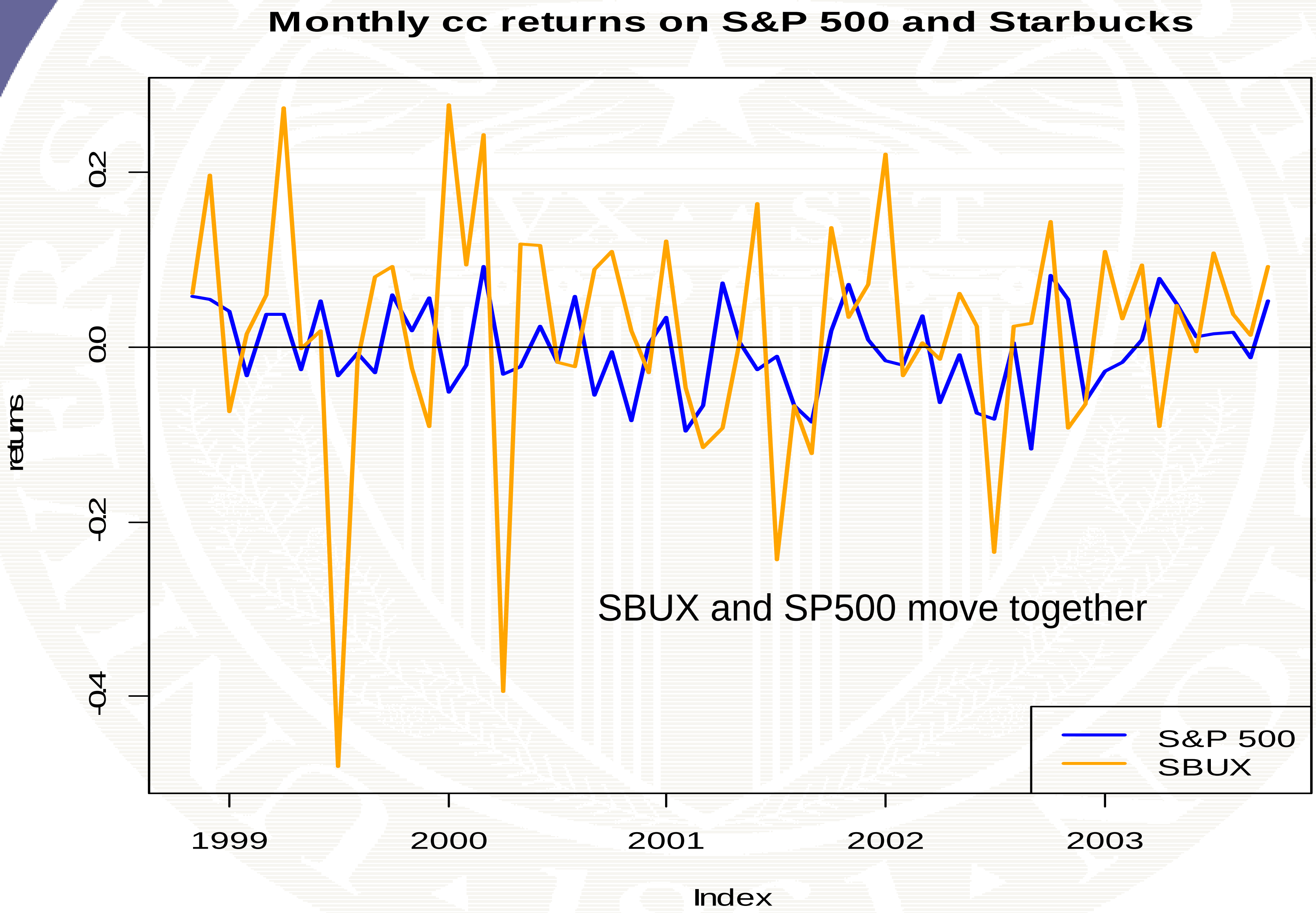


S&P 500 monthly cc returns

smoothed density

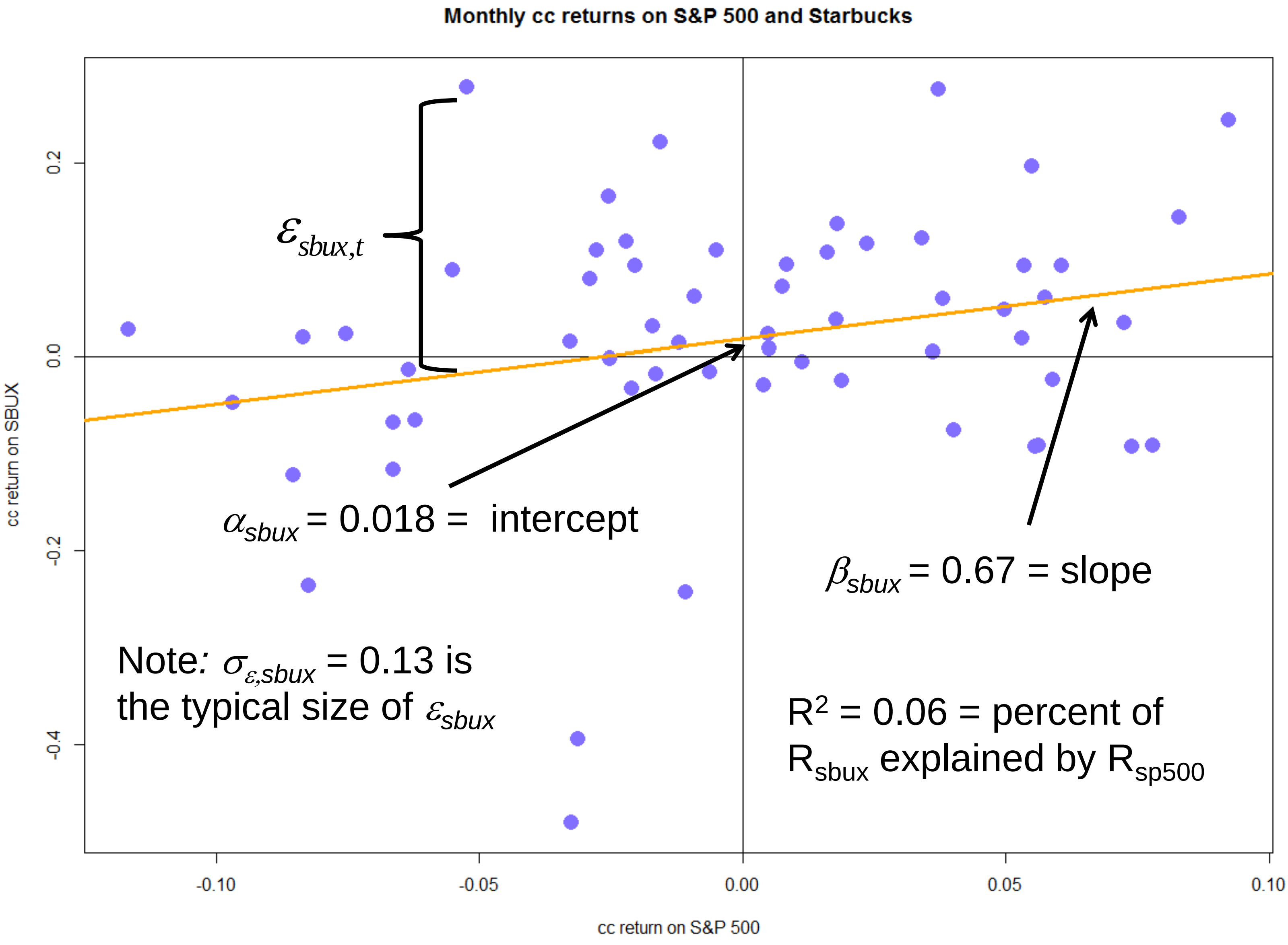
Normal Q-Q Plot

$$R_{sbux} = \alpha_{sbux} + \beta_{sbux} R_M + \varepsilon_{sbux} \qquad \beta_{sbux} = 0.67$$



SBUX and SP500 move together

$$R_{sbux} = \alpha_{sbux} + \beta_{sbux} R_M + \varepsilon_{sbux},\ \ \varepsilon_{sbux} \sim N(0, \sigma^2_{\varepsilon,sbux})$$



$\varepsilon_{sbux,t}$

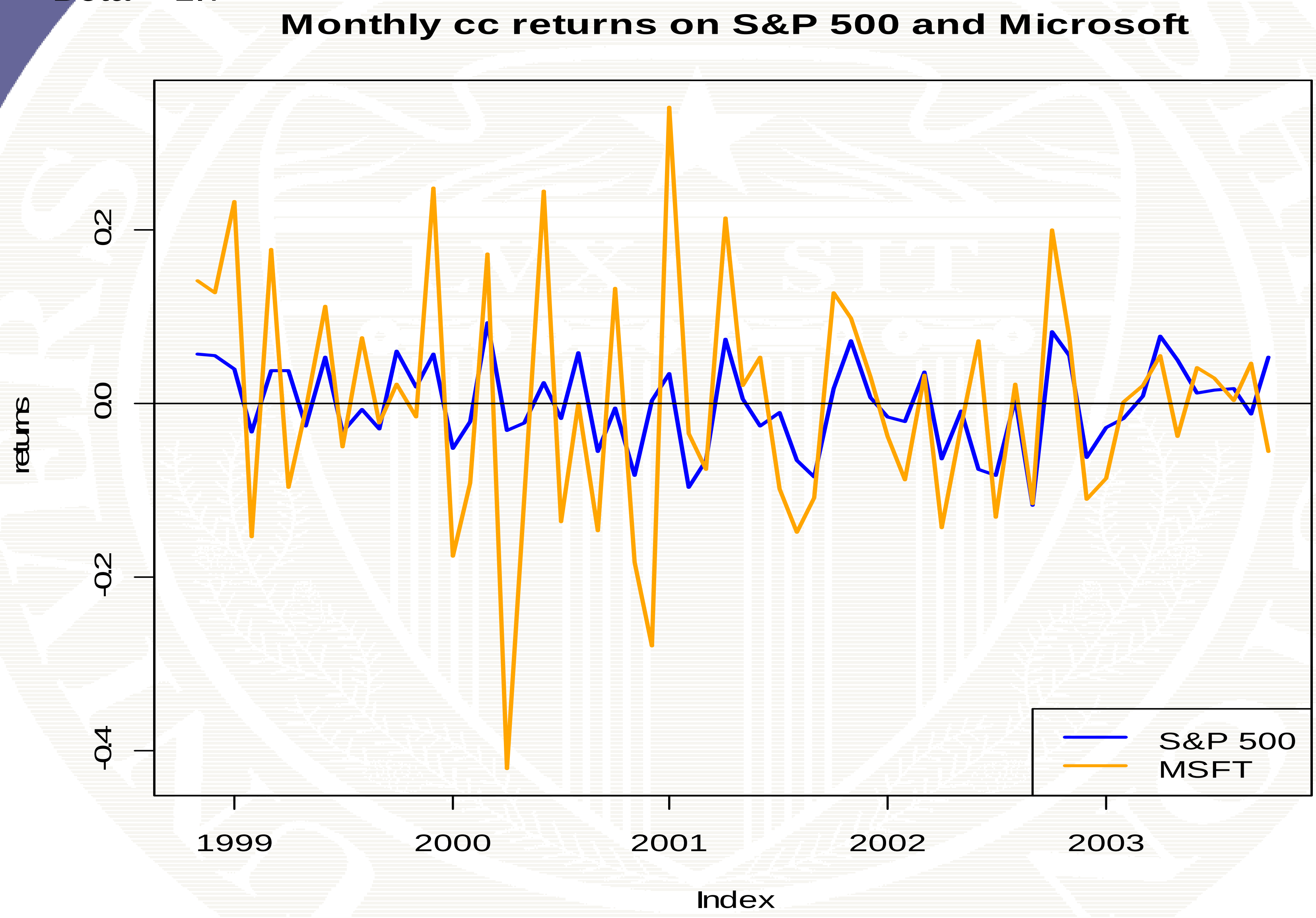
α_{sbux} = 0.018 = intercept

β_{sbux} = 0.67 = slope

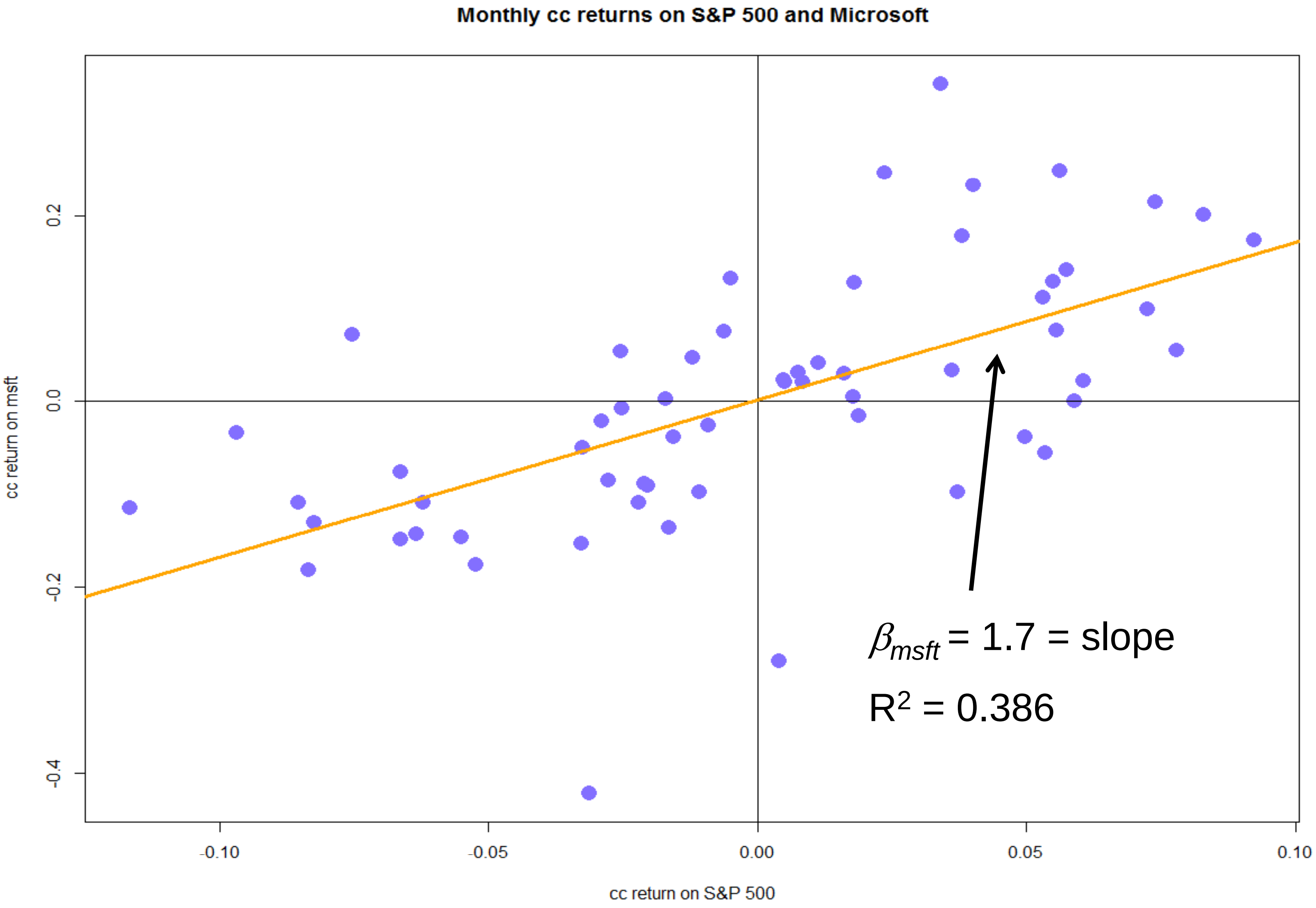
Note: $\sigma_{\varepsilon,sbux}$ = 0.13 is the typical size of ε_{sbux}

R^2 = 0.06 = percent of R_{sbux} explained by R_{sp500}

Beta = 1.7


Monthly cc returns on S&P 500 and Microsoft
returns
0.2
0.0
-0.2
-0.4
1999
2000
2001
2002
2003
Index
S&P 500
MSFT

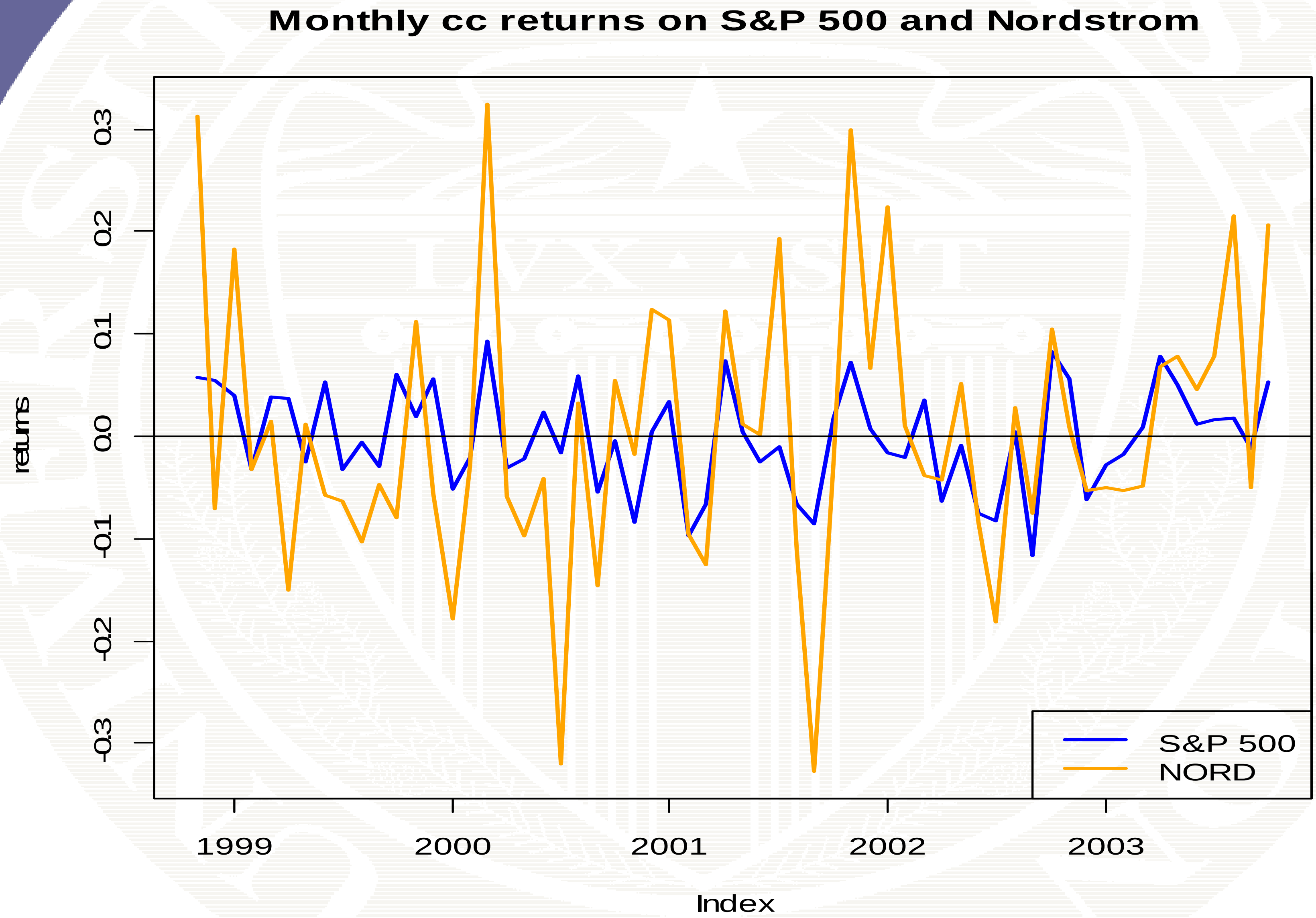
$$R_{msft} = \alpha_{msft} + \beta_{msft} R_M + \varepsilon_{msft}$$



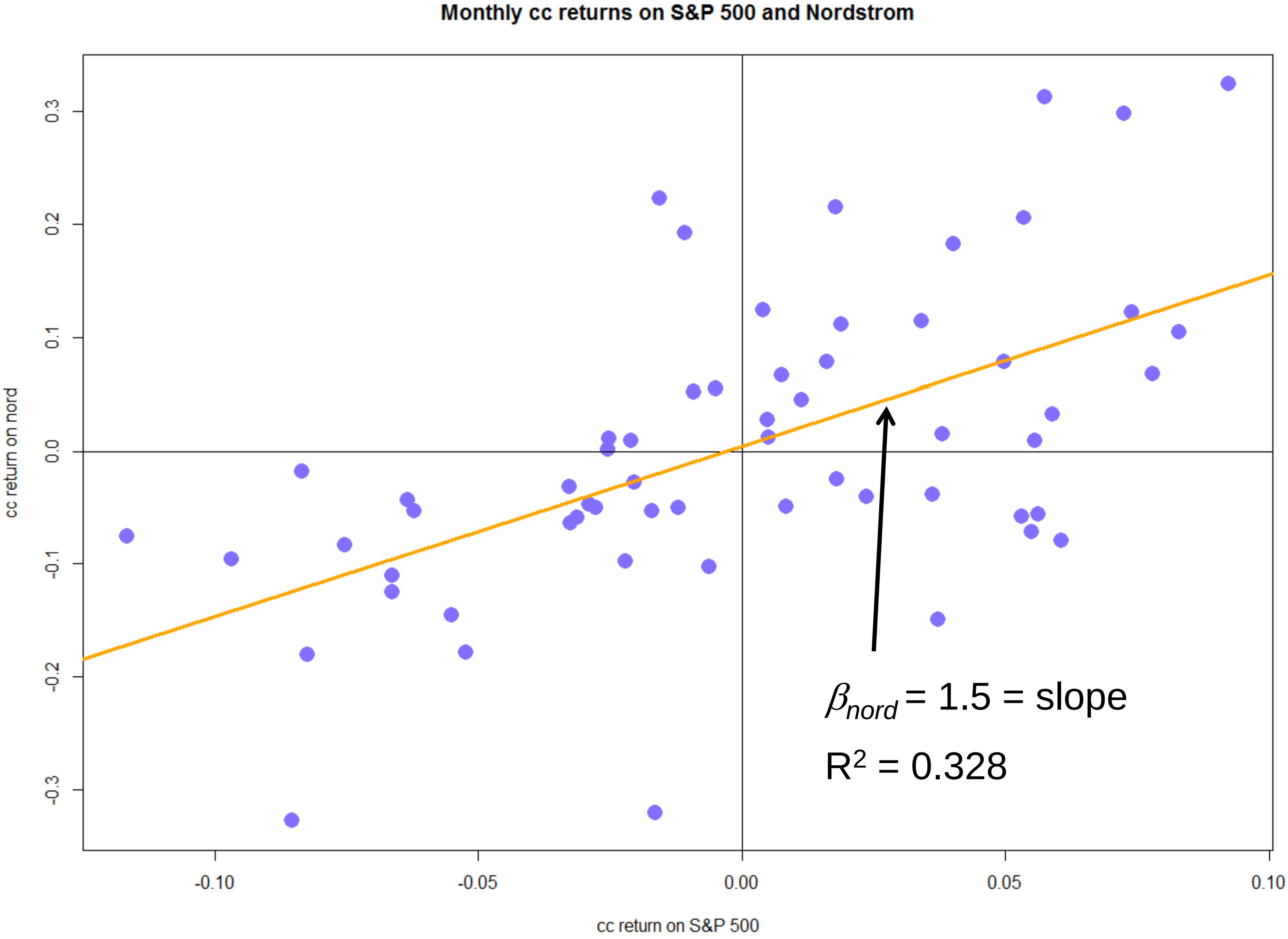
β_{msft} = 1.7 = slope

R^2 = 0.386

Beta = 1.5


Monthly cc returns on S&P 500 and Nordstrom
returns
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
1999
2000
2001
2002
2003
Index
S&P 500
NORD

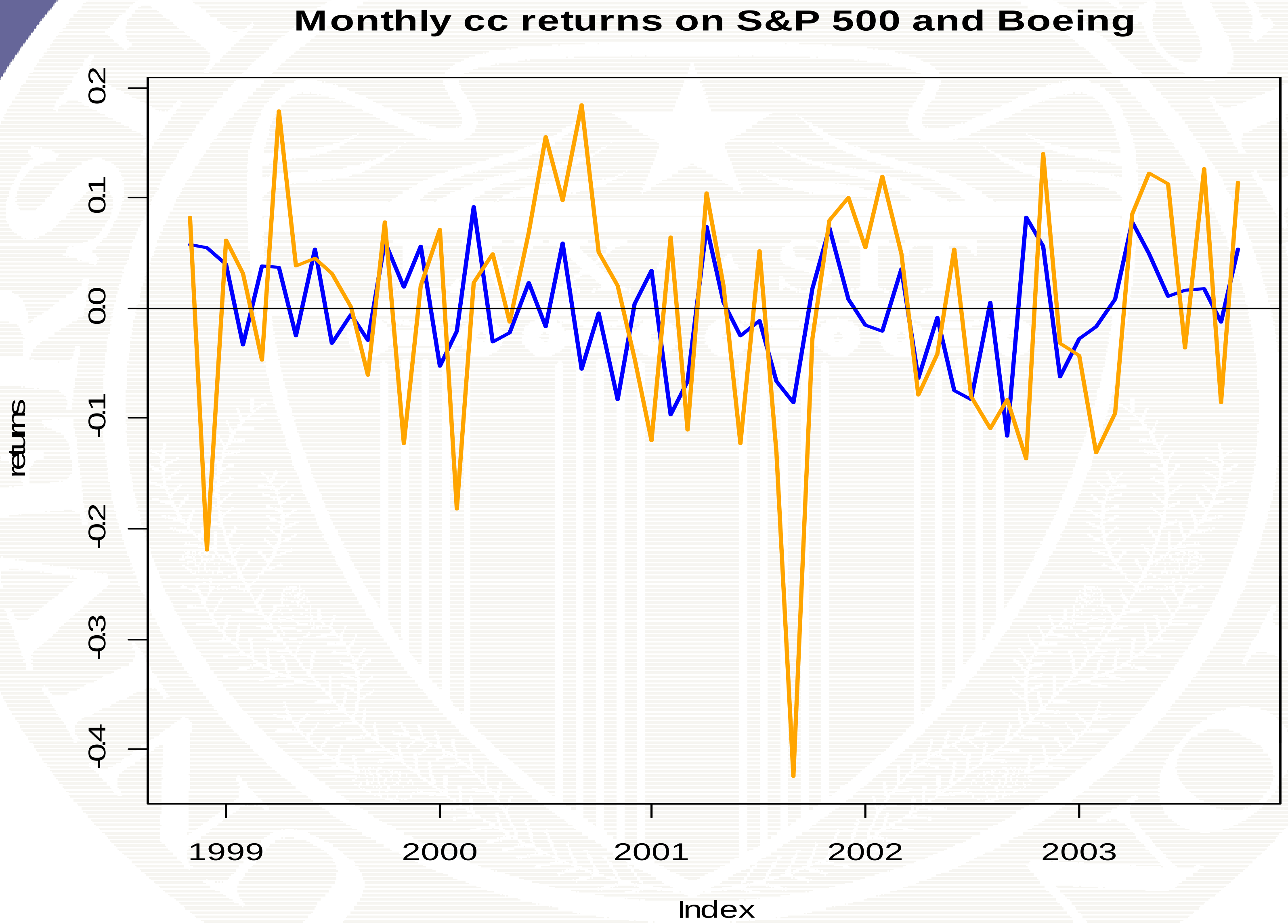
$$R_{nord} = \alpha_{nord} + \beta_{nord} R_M + \varepsilon_{nord}$$



β_{nord} = 1.5 = slope

R^2 = 0.328

Beta = 0.64



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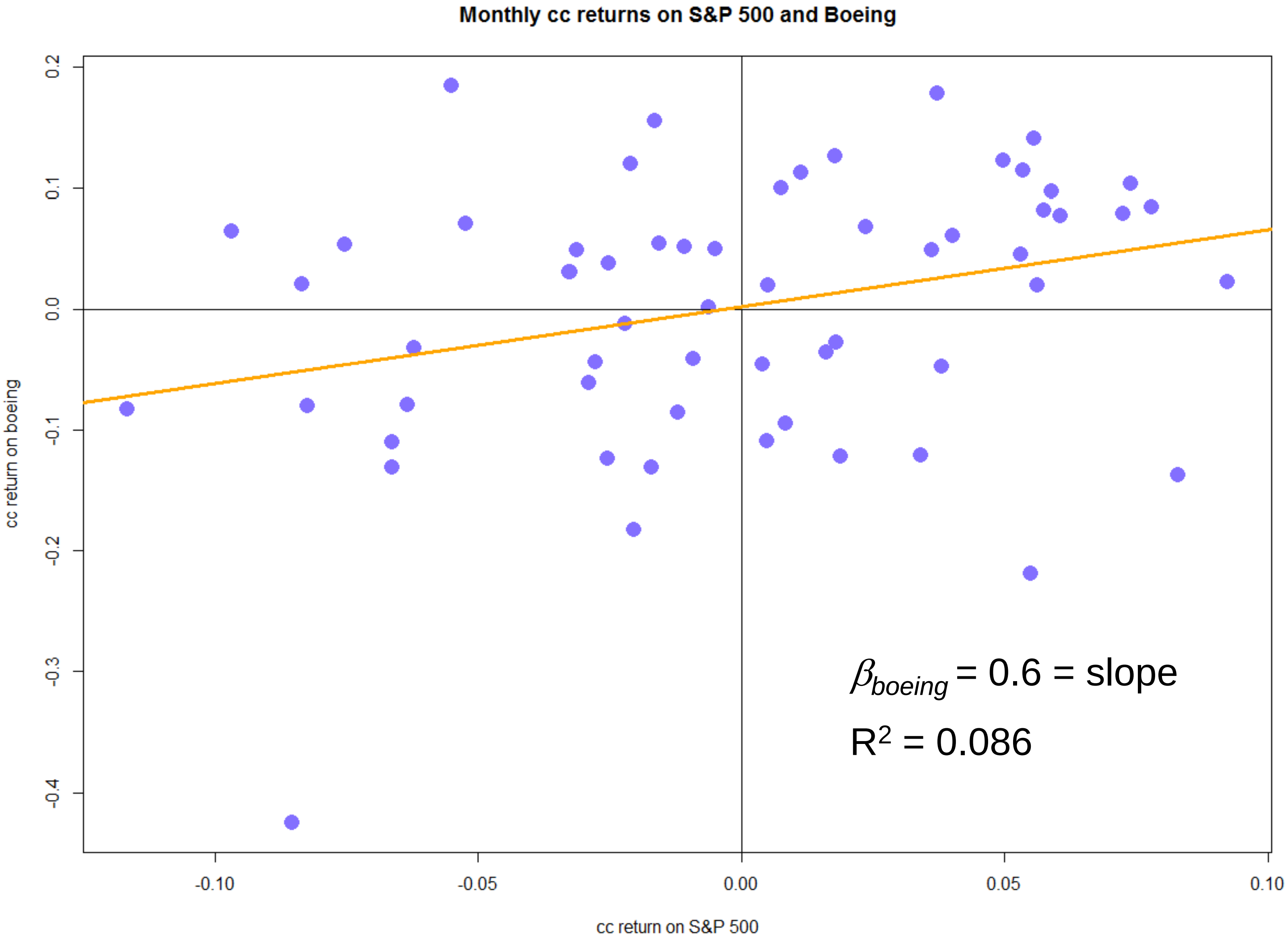
$$R_{boeing} = \alpha_{boeing} + \beta_{boeing} R_M + \varepsilon_{boeing}$$



β_{boeing} = 0.6 = slope

R^2 = 0.086

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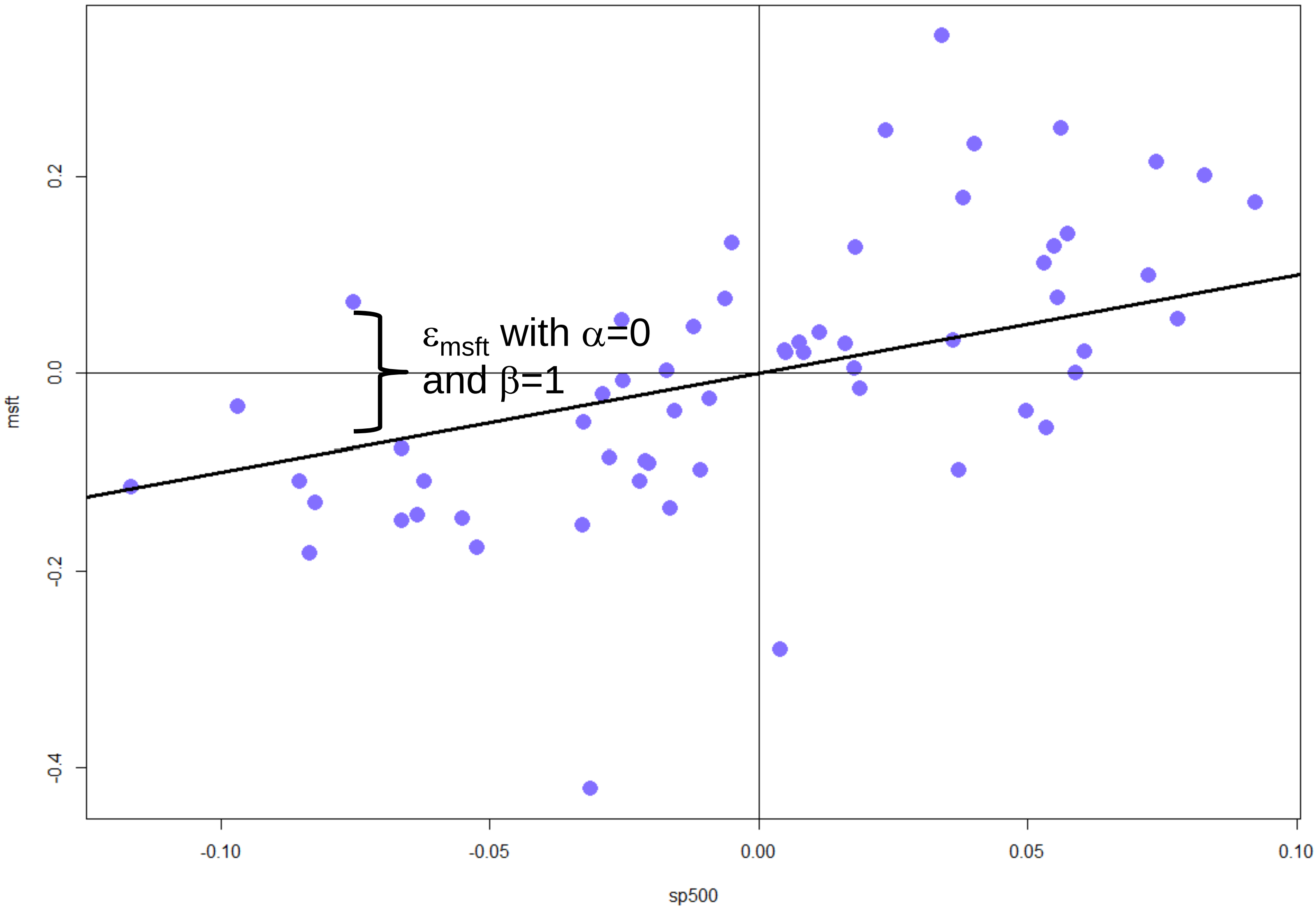
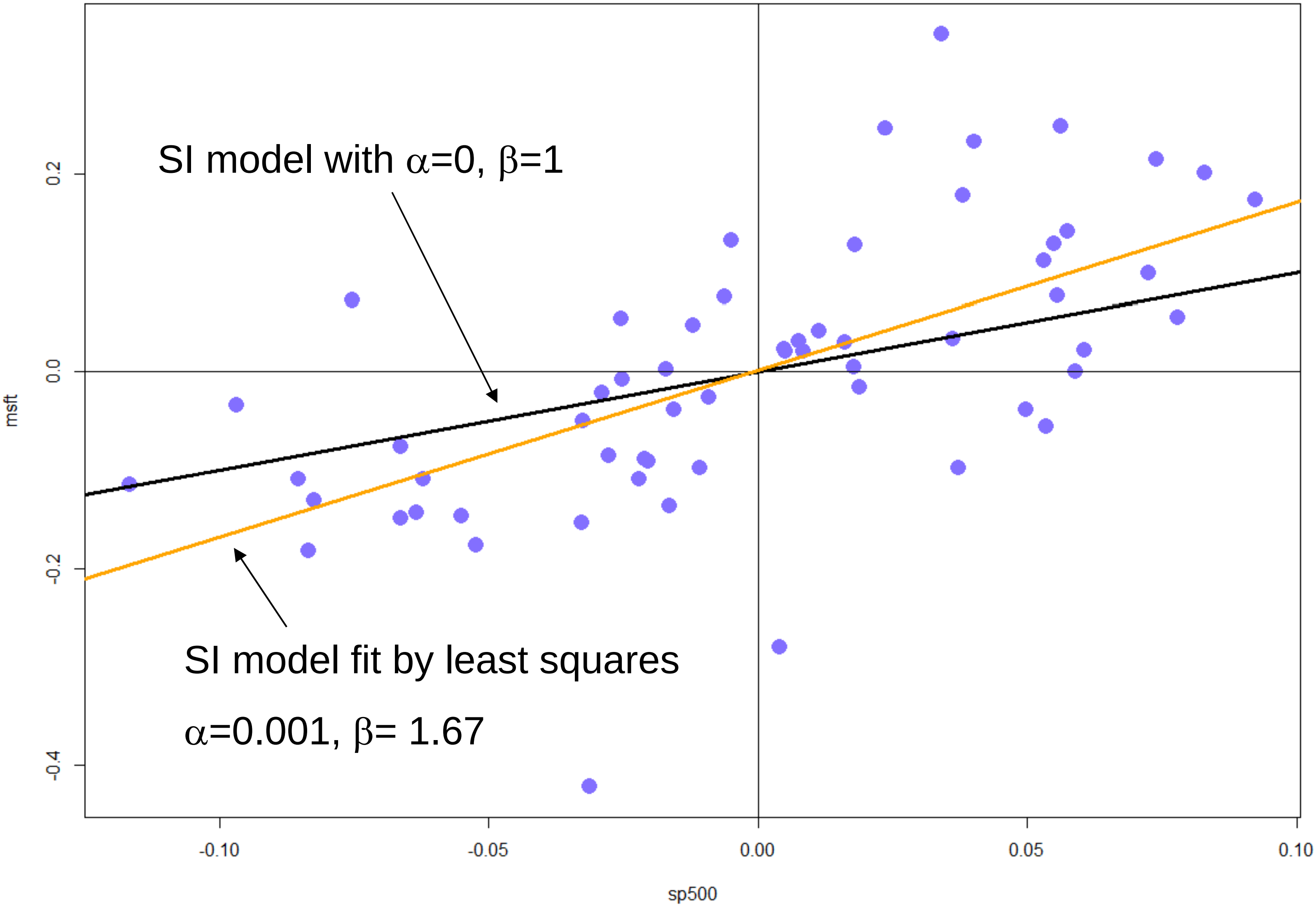
SI Model for MSFT with α=0 and β=1



ε_{msft} with α=0 and β=1

SI Model Fit by Least Squares



SI model with α=0, β=1

SI model fit by least squares

α=0.001, β= 1.67

Example Data

Monthly continuously compounded returns on S&P 500, Starbucks, Microsoft, Nordstrom and Boeing

```
> head(si.df)
     sp500     sbux     msft     nord   boeing
1  0.05744  0.06079  0.14155  0.31280  0.08202
2  0.05484  0.19625  0.12835 -0.07138 -0.21923
3  0.04019 -0.07471  0.23258  0.18243  0.06103
4 -0.03282  0.01524 -0.15346 -0.03172  0.03069
5  0.03806  0.05947  0.17738  0.01545 -0.04702
6  0.03724  0.27495 -0.09734 -0.14975  0.17825

> colnames(si.df)
[1] "sp500"  "sbux"   "msft"   "nord"   "boeing"
```

Least Squares in R

```
> msft.fit = lm(msft~sp500,data=si.df)
> class(msft.fit)
[1] "lm“

> names(msft.fit)
 [1] "coefficients"  "residuals"     "effects“  "rank"
 [5] "fitted.values" "assign"        "qr"
"df.residual"
 [9] "xlevels"       "call"          "terms”    "model"

> msft.fit$coef
 (Intercept)       sp500
   0.001199    1.697067
```

$$\hat{\alpha} \qquad\qquad \hat{\beta}$$

LM Print Method

```
> msft.fit
Call:
lm(formula = msft ~ sp500, data = si.df)

Coefficients:
(Intercept)          sp500
     0.0012         1.6971
```

$\hat{\alpha}$ $\hat{\beta}$

LM Summary Method

```
> summary(msft.fit)
Call:
lm(formula = msft ~ sp500, data = si.df)

Residuals:
    Min      1Q  Median      3Q     Max
-0.3690 -0.0540  0.0050  0.0469  0.2828

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.0012     0.0140    0.09     0.93
sp500         1.6971     0.2808    6.04  1.2e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’

Residual standard error: 0.109 on 58 degrees of freedom
Multiple R-squared: 0.386, Adjusted R-squared: 0.376
F-statistic: 36.5 on 1 and 58 DF,  p-value: 1.16e-07
```

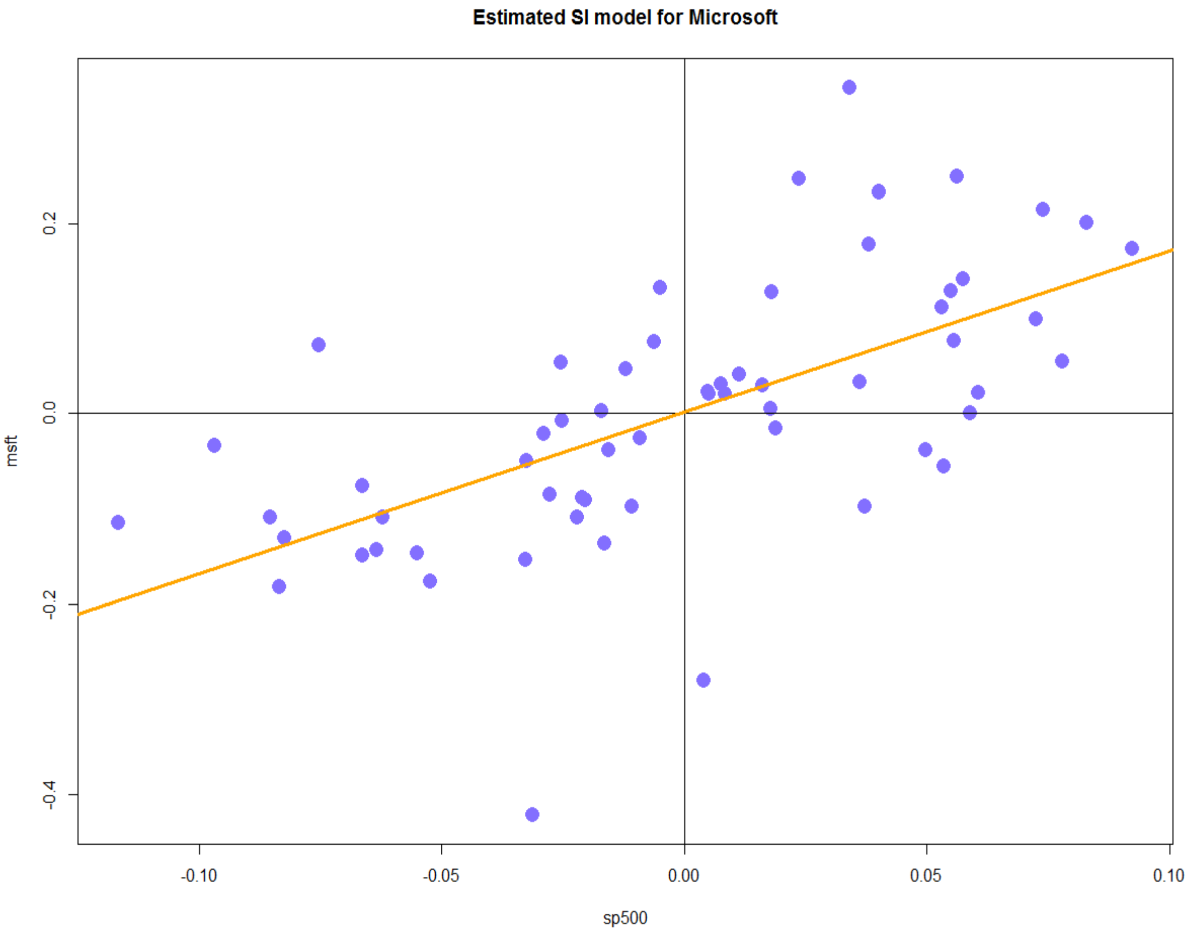
$SE(\hat{\alpha})$

$SE(\hat{\beta})$

$\hat{\sigma}_{\varepsilon}$

R^2

Scatterplot with Regression Line



```
> plot(si.df$sp500, si.df$msft, pch=16, lwd=2, col="blue")
> abline(msft.fit, col="orange", lwd=2)
> abline(h=0, v=0)
```

95% Confidence Intervals

$$\hat{\beta} \pm 2 \times SE(\hat{\beta})$$

$$1.697 \pm 2 \times (0.2808)$$

$$= [1.135,\ 2.259]$$

Note: 95% confidence interval is pretty big!

=> β is not very precisely estimated for individual stocks

Confidence Intervals In R

```
> confint(msft.fit, level=0.95)
                2.5 %  97.5 %
(Intercept) -0.02688 0.02928
sp500        1.13489 2.25925
```

LM Extractor Functions

```
> coef(msft.fit)
(Intercept)       sp500
   0.001199    1.697067
```

$$\hat{\varepsilon}_t = R_t - \hat{\alpha} - \hat{\beta} R_{Mt}$$

```
> residuals(msft.fit)[1:5]
1998-02-01 1998-03-01 1998-04-01 1998-05-01 1998-06-01
   0.04286    0.03408    0.16318   -0.09897    0.11159
```

$$\hat{R}_t = \hat{\alpha} + \hat{\beta} R_{Mt}$$

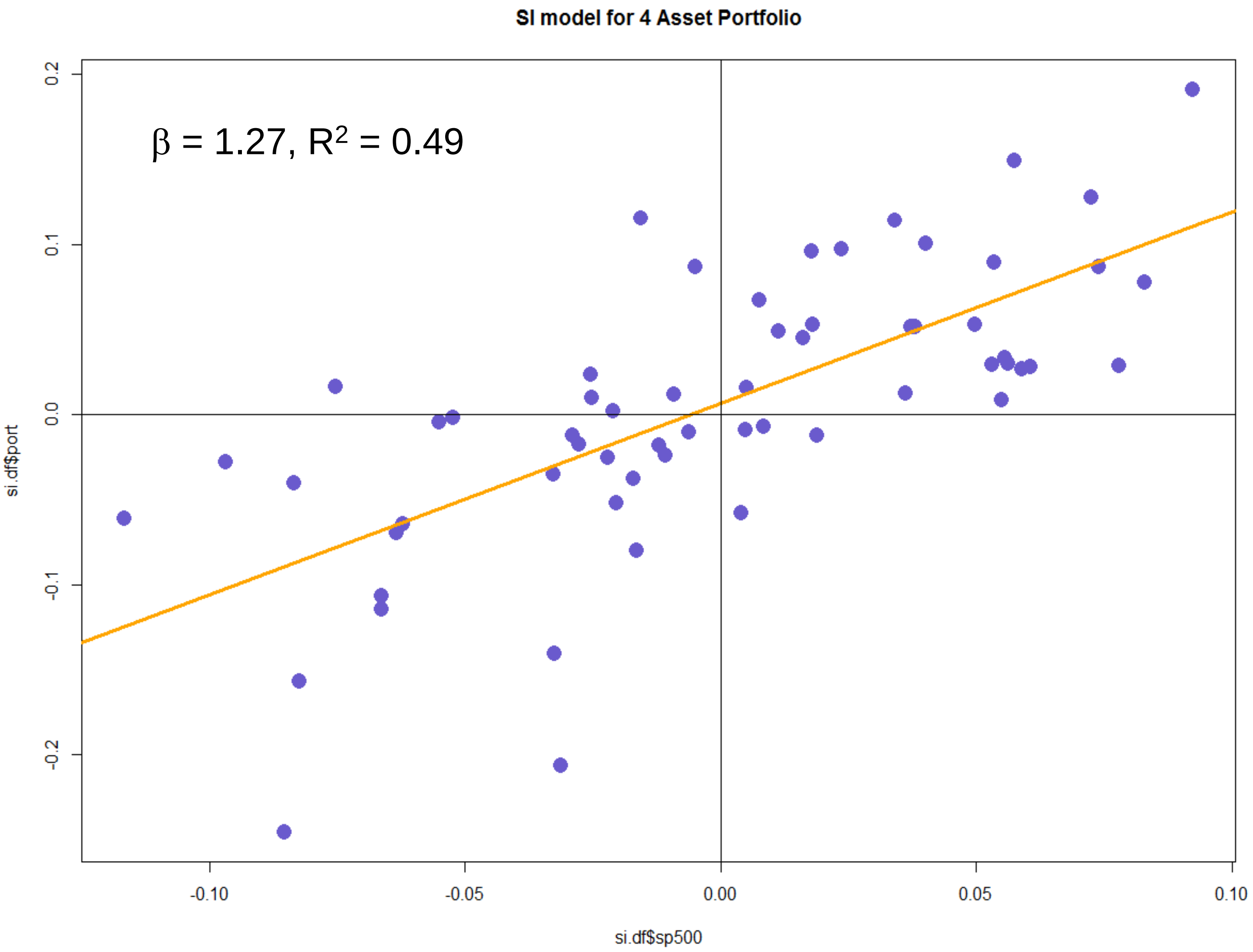
```
> fitted(msft.fit)[1:5]
1998-02-01 1998-03-01 1998-04-01 1998-05-01 1998-06-01
   0.09869    0.09427    0.06941   -0.05449    0.06579
```

SI Model for 4 Asset Portfolio

```
> port = (si.df$sbux + si.df$msft + si.df$nord +
+          si.df$boeing)/4
> new.data = data.frame(si.df,port)
> port.fit = lm(port~sp500,data=new.data)
```

```
> summary(port.fit)
```

```
Call: lm(formula = port ~ sp500, data = new.data)
Residuals:
     Min        1Q    Median      3Q    Max
 -0.1776 -0.03609 -0.002005 0.04635 0.1264

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept) 0.0065 0.0075     0.8616  0.3924
      sp500 1.1276 0.1510     7.4668  0.0000

Residual standard error: 0.05842 on 58 degrees of freedom
Multiple R-Squared: 0.4901
```



$\beta = 1.27, R^2 = 0.49$

Portfolio Beta

```
# show beta of portfolio = weighted avg of individual betas

> beta.sbux = coef(lm(sbux~sp500,data=si.df))[2]
> beta.msft = coef(lm(msft~sp500,data=si.df))[2]
> beta.nord = coef(lm(nord~sp500,data=si.df))[2]
> beta.boeing = coef(lm(boeing~sp500,data=si.df))[2]

> (beta.sbux + beta.msft + beta.nord + beta.boeing)/4
    sp500
 1.127567

> coef(port.fit)[2]
    sp500
 1.127567
```

Single Index Model Fit

Asset	β	SE(β)	σ_ε	R^2
Boeing	0.638	0.273	0.106	0.09
Msft	1.697	0.281	0.109	0.39
Nord	1.508	0.283	0.110	0.33
Sbux	0.667	0.342	0.132	0.06
port	**1.128**	**0.151**	**0.058**	**0.49**

Single stocks vs. portfolio

- Portfolio β is closer to 1
- Portfolio β is estimated more precisely (SE is smaller)
- Portfolio σ_ε is smaller (diversification effect)
- Portfolio R^2 is higher (diversification effect)

Estimating Covariances

```
> beta.vec
  SBUX   MSFT   NORD BOEING
0.6666 1.6971 1.5080 0.6386

> sig2.sp500
[1] 0.002537

> cov.market = sig2.sp500*(beta.vec%*%t(beta.vec))
> cov.market
         SBUX     MSFT     NORD   BOEING
[1,] 0.001127 0.002870 0.002550 0.001080
[2,] 0.002870 0.007307 0.006493 0.002750
[3,] 0.002550 0.006493 0.005769 0.002443
[4,] 0.001080 0.002750 0.002443 0.001035
```

Estimating Covariances

```
> D.mat =
diag(c(sig2e.sbux,sig2e.msft,sig2e.nord,sig2e.boeing))
> D.mat
        [,1]    [,2]    [,3]    [,4]
[1,] 0.01719 0.00000 0.00000 0.00000
[2,] 0.00000 0.01161 0.00000 0.00000
[3,] 0.00000 0.00000 0.01179 0.00000
[4,] 0.00000 0.00000 0.00000 0.01101

> cov.si = cov.market + D.mat
> cov.si
        SBUX     MSFT     NORD   BOEING
[1,] 0.01832 0.002870 0.002550 0.001080
[2,] 0.00287 0.018913 0.006493 0.002750
[3,] 0.00255 0.006493 0.017564 0.002443
[4,] 0.00108 0.002750 0.002443 0.012045
```

Estimating Covariances

```
# compare with sample covariance matrix
> print(cov.hat,digits=4)
            sbux      msft     nord    boeing
  sbux 0.0183176 0.0055003 0.002735 0.0001221
  msft 0.0055003 0.0189132 0.006987 0.0001189
  nord 0.0027354 0.0069870 0.017564 0.0037662
boeing 0.0001221 0.0001189 0.003766 0.0120446

> print(cov.si,digits=4)
        SBUX     MSFT     NORD   BOEING
[1,] 0.01832 0.002870 0.002550 0.001080
[2,] 0.00287 0.018913 0.006493 0.002750
[3,] 0.00255 0.006493 0.017564 0.002443
[4,] 0.00108 0.002750 0.002443 0.012045
```

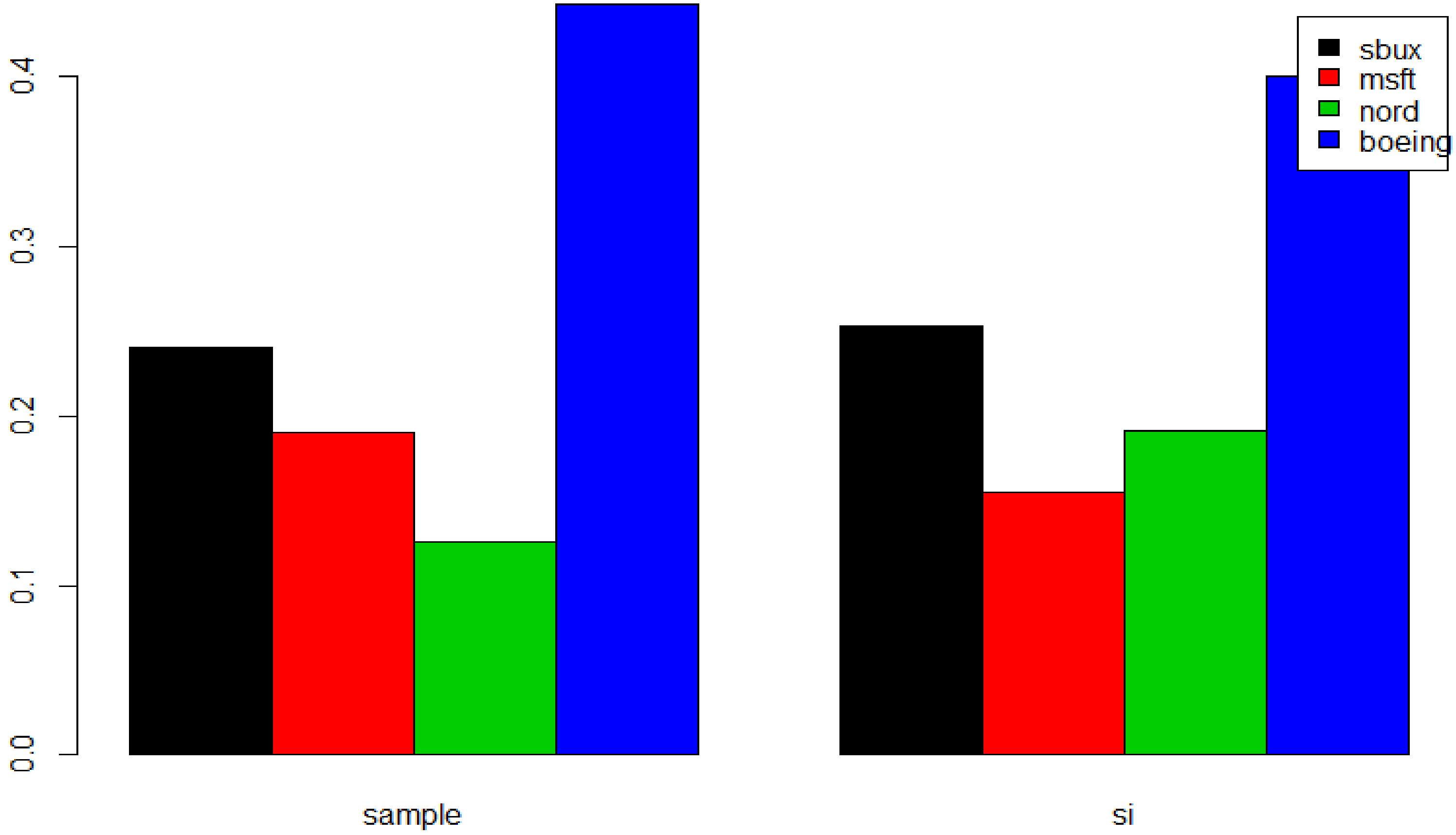
Estimating Correlations

```
> print(cor.hat,digits=4)
           sbux     msft   nord   boeing
  sbux 1.000000 0.295506 0.1525 0.008218
  msft 0.295506 1.000000 0.3833 0.007876
  nord 0.152500 0.383348 1.0000 0.258940
boeing 0.008218 0.007876 0.2589 1.000000

> print(cor.si,digits=4)
          sbux   msft   nord  boeing
  sbux 1.00000 0.1542 0.1422 0.07271
  msft 0.15419 1.0000 0.3562 0.18218
  nord 0.14218 0.3562 1.0000 0.16798
boeing 0.07271 0.1822 0.1680 1.00000
```

Minimum Variance Portfolios

```
# use sample covariance matrix
> mu.hat = colMeans(si.df[, 2:5])
> gmin.port.sample = globalMin.portfolio(muhat, cov.hat)
> gmin.port.sample
Call:
globalMin.portfolio(er = muhat, cov.mat = cov.hat)

Portfolio expected return:     0.00543
Portfolio standard deviation:  0.07655
Portfolio weights:
  sbux   msft   nord boeing
0.2410 0.1907 0.1252 0.4430

# use single index model covariance matrix
> gmin.port.si = globalMin.portfolio(muhat, cov.si)
> gmin.port.si
Call:
globalMin.portfolio(er = muhat, cov.mat = cov.si)

Portfolio expected return:     0.005784
Portfolio standard deviation:  0.07743
Portfolio weights:
  sbux   msft   nord boeing
0.2528 0.1545 0.1918 0.4009
```

Minimum Variance Portfolio Weights



t-Values

In the R summary output, the t values are t-statistics for testing the hypothesis that the true coefficient is equal to zero

$$t_{\alpha=0} = \frac{\hat{\alpha}}{SE(\hat{\alpha})} = \frac{.0012}{.0140} = .0855$$

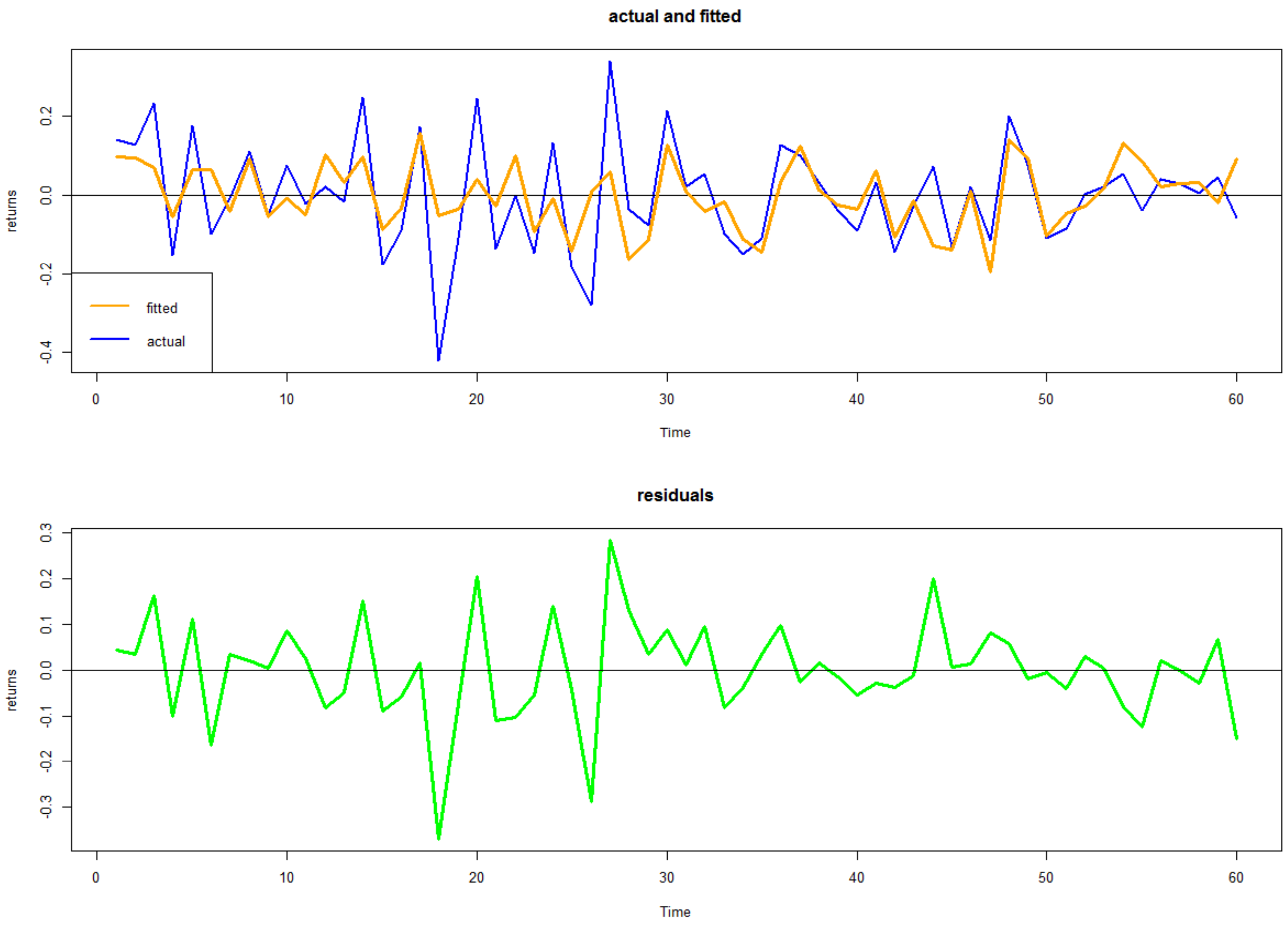
$$t_{\beta=0} = \frac{\hat{\beta}}{SE(\hat{\beta})} = \frac{1.6971}{.2808} = 6.0426$$

T-Stats for Specific Values

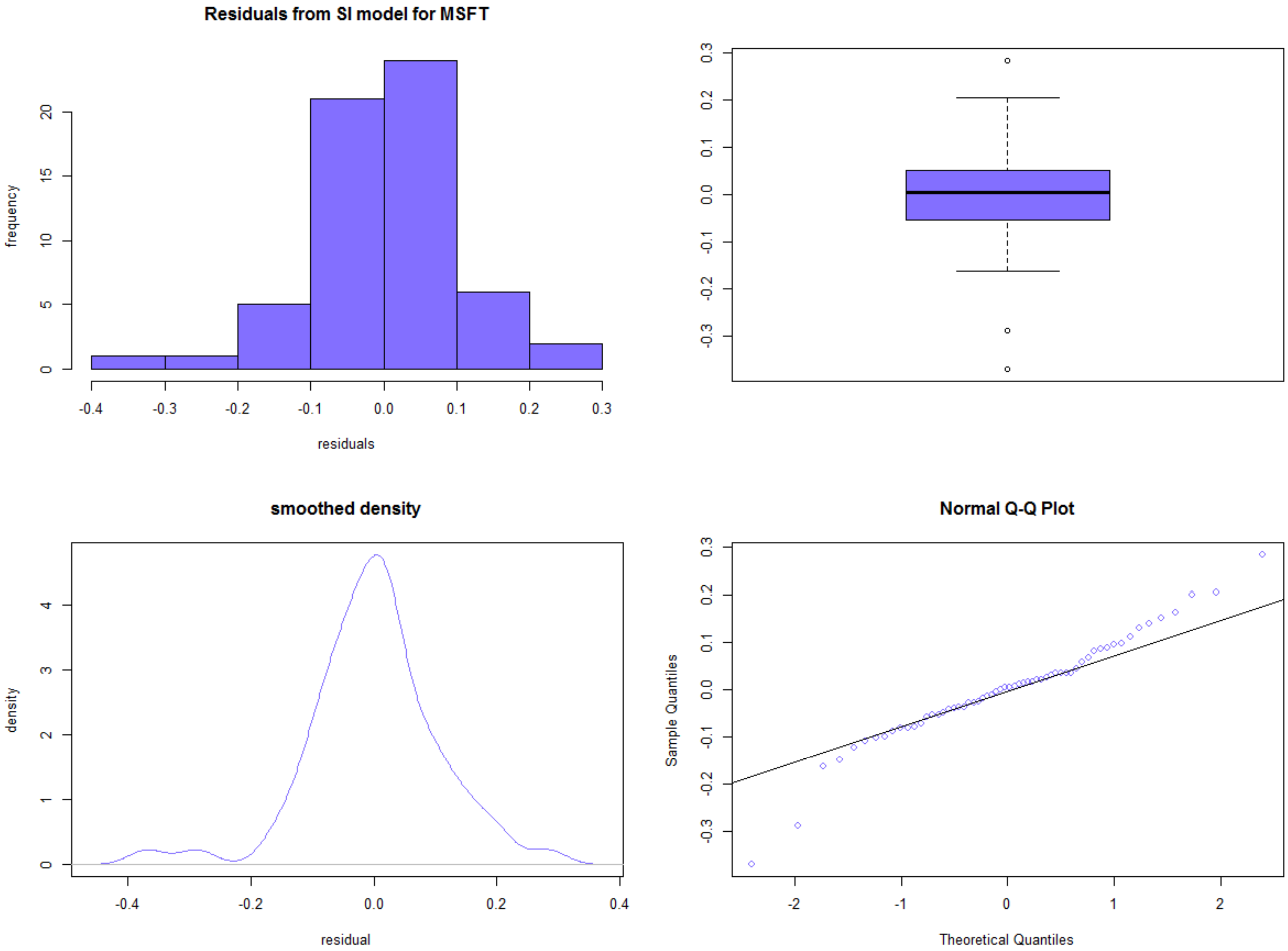
$$H_0 : \beta_i = 1 \text{ (asset has same risk as market)}$$

$$H_1 : \beta_i \neq 1 \text{ (asset risk is different than market)}$$

```
> msft.summary = summary(msft.fit)
> names(msft.summary)
 [1] "call"          "terms"         "residuals"     "coefficients"
 [5] "aliased"       "sigma"         "df"            "r.squared"
 [9] "adj.r.squared" "fstatistic"    "cov.unscaled"

> msft.summary$coef
            Estimate Std. Error t value  Pr(>|t|)
(Intercept) 0.001199    0.01403 0.08545 9.322e-01
sp500       1.697067    0.28085 6.04264 1.159e-07
> colnames(msft.summary$coef)
[1] "Estimate"   "Std. Error" "t value"    "Pr(>|t|)"
> se.beta = msft.summary$coef[2, "Std. Error"]

# t-stat for beta = 1
> t.stat = (coef(msft.fit)[2] - 1)/se.beta
> t.stat
sp500
2.482
```



In the Single Index model the errors should behave like White Noise

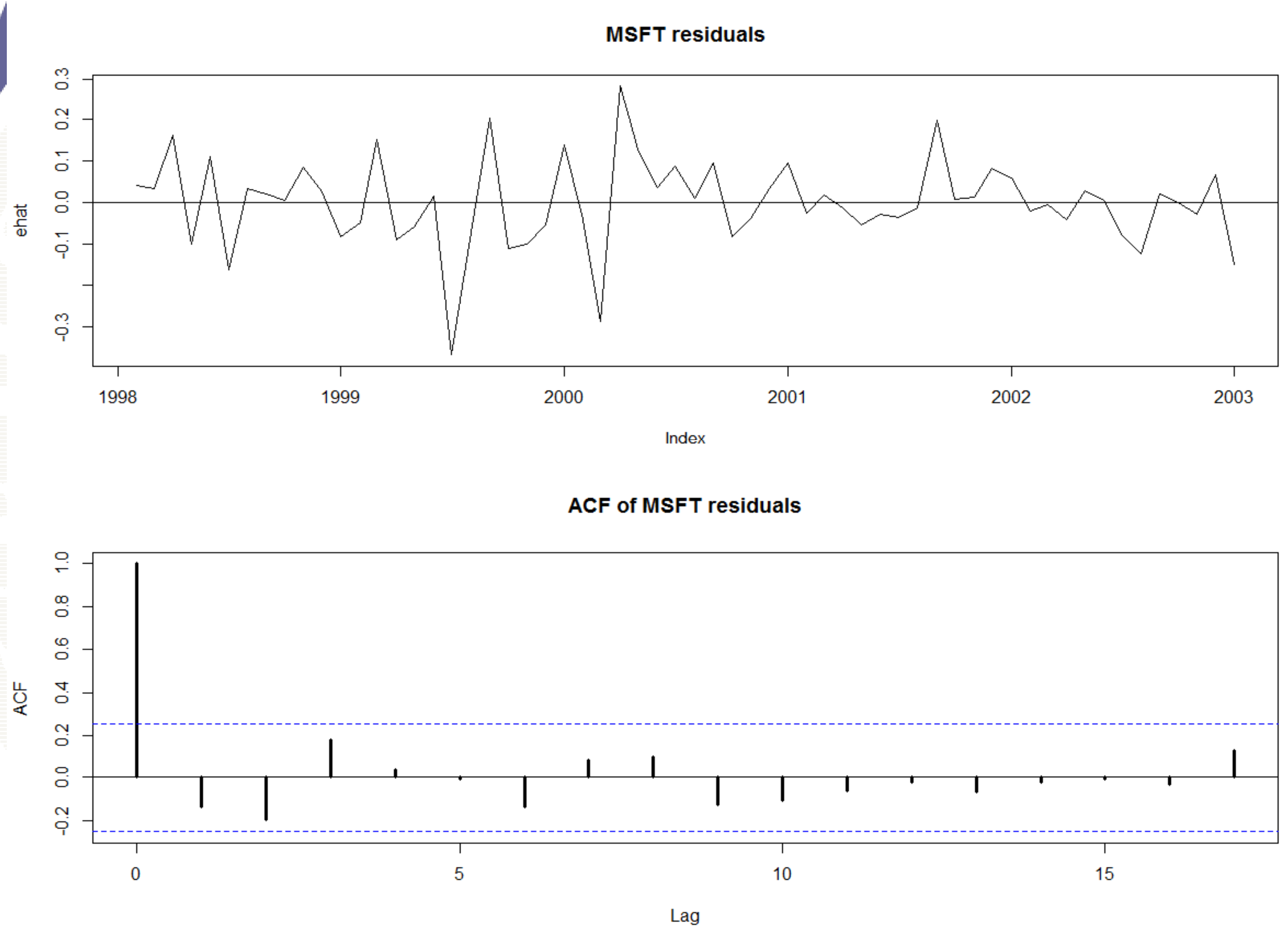


Residuals from SI model for MSFT

smoothed density

Normal Q-Q Plot

In the Single Index model the errors should be Normally Distributed



MSFT residuals

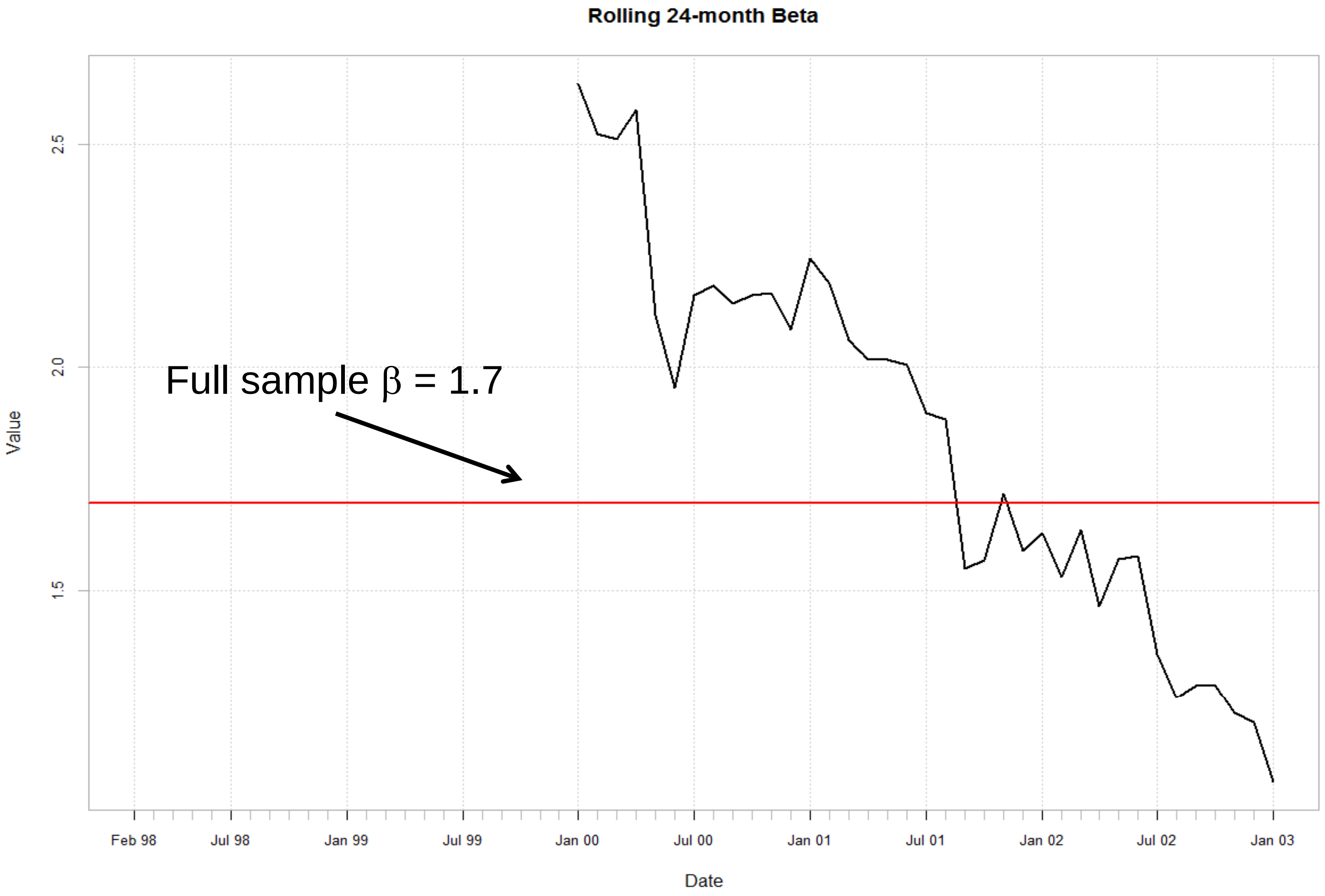
ACF of MSFT residuals

In the Single Index model the errors should behave like White Noise

Rolling Regression

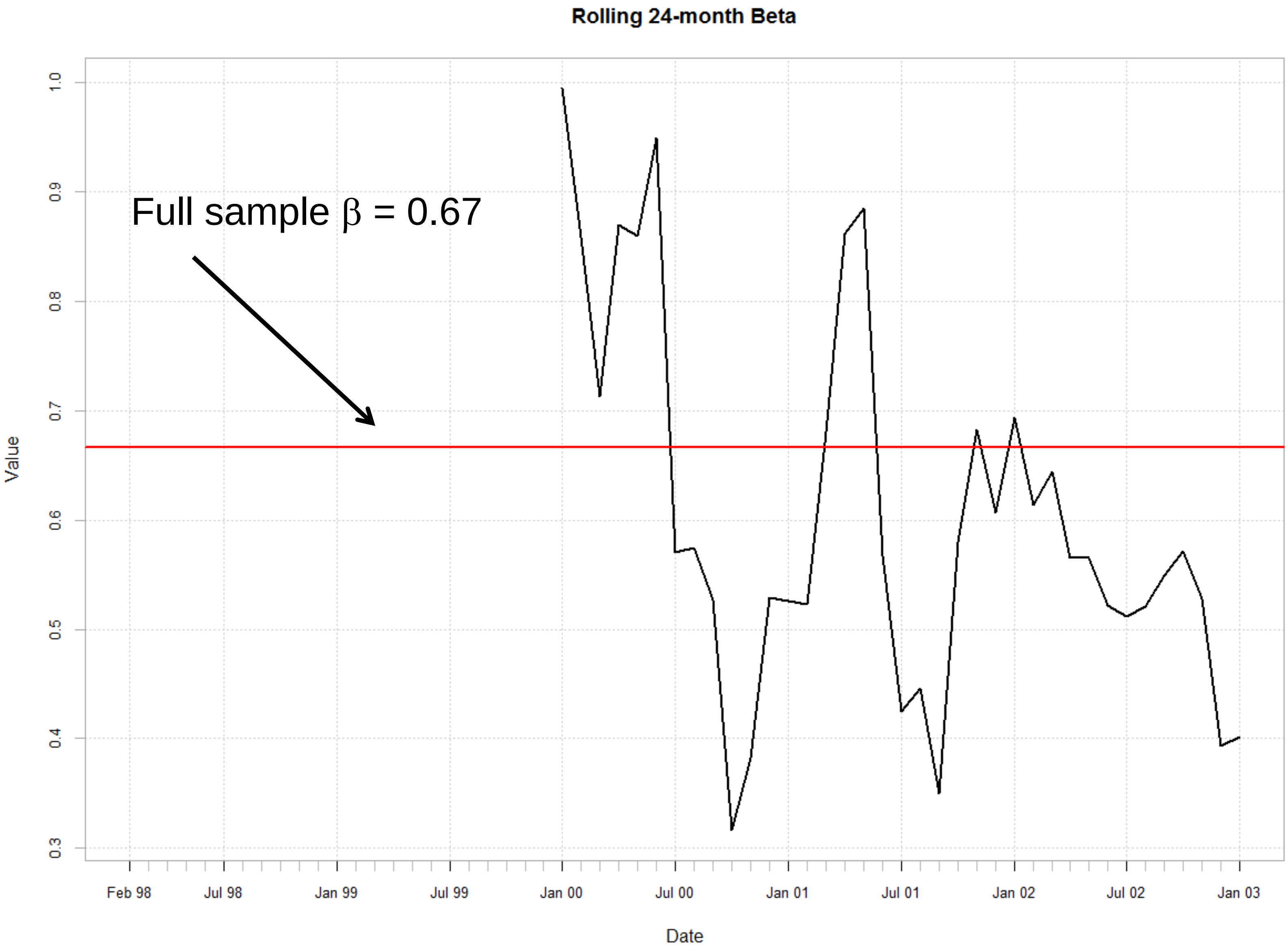
```
# Use PerformanceAnalytics function
# chart.RollingRegression for 24-month rolling regression

> chart.RollingRegression(si.z[,"msft",drop=F],
+                         si.z[,"sp500",drop=F], width=24)

# Add horizonal line at full sample regression estimate
> abline(h=beta.vals["msft"], lwd=2, col="red")
```

24 month rolling β’s for MSFT



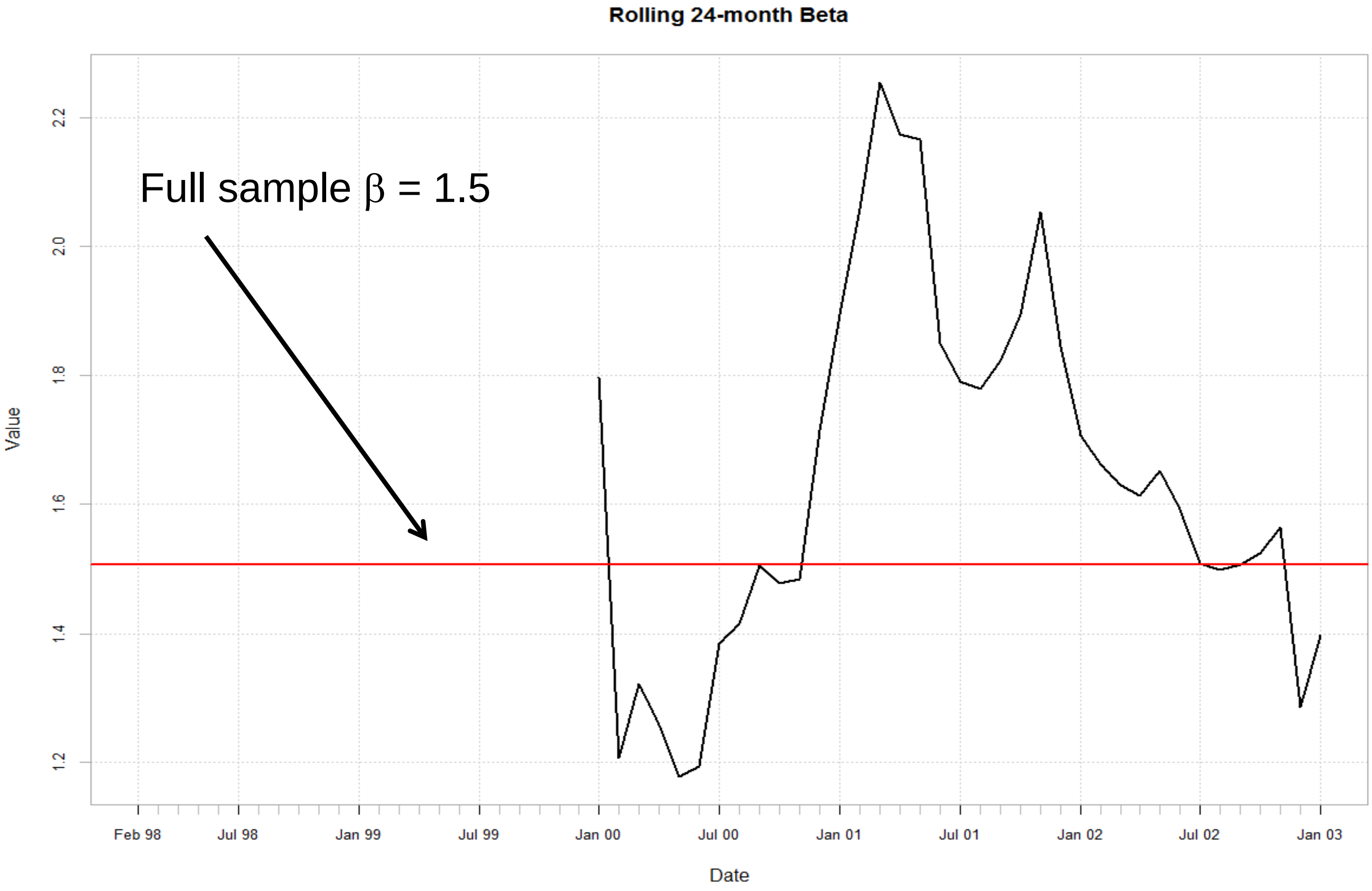
Full sample β = 1.7

24 month rolling β's for SBUX



Full sample β = 0.67

24 month rolling β’s for NORD



Full sample β = 1.5

24 month rolling β's for Boeing