University of Washington
Department of Economics

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Econ 424/CFRM 462

Midterm Exam Solutions

I. Return Calculations (20 pts, 4 points each)

Consider a 60-month (5 year) investment in two assets: the Vanguard S&P 500 index (VFINX) and Amazon stock (AMZN). Suppose you buy one share of the S&P 500 fund and one share of Amazon stock at the end of January, 2009 for $P_{vfinx,t-60} = 68.18,\ P_{amzn,t-60} = 58.82$, and then sell these shares at the end of January, 2014 for $P_{vfinx,t} = 164.1,\ P_{amzn,t} = 403$. (Note: these are actual adjusted closing prices taken from Yahoo!). In this question, you will see how much money you could have made if you invested in these assets right after the financial crisis.

a. What are the simple 60-month (5-year) returns for the two investments?

```
> p.amzn.1 = 58.82
> p.amzn.2 = 403
> p.vfinx.1 = 68.18
> p.vfinx.2 = 164.1
> r.vfinx = (p.vfinx.2 - p.vfinx.1)/p.vfinx.1
> r.amzn = (p.amzn.2 - p.amzn.1)/p.amzn.1
> r.vfinx
[1] 1.407 = 140.7%
> r.amzn
[1] 5.851 = 585.1%
```

b. What are the continuously compounded (cc) 60-month (5-year) returns for the two investments?

```
> log(1 + r.vfinx)
[1] 0.8783 = 87.83%
> log(1 + r.amzn)
[1] 1.924 = 192.4%
```

c. Suppose you invested $1,000 in each asset at the end of January, 2009. How much would each investment be worth at the end of January, 2014?

```
> w0 = 1000
> w1.vfinx = w0*(1 + r.vfinx)
> w1.amzn = w0*(1 + r.amzn)
> w1.vfinx
[1] 2407
> w1.amzn
[1] 6851
```

d. What is the simple compound annual return on the two 5 year investments?

```
> r.vfinx.a = (1 + r.vfinx)^(1/5) - 1
> r.amzn.a = (1 + r.amzn)^(1/5) - 1
> r.vfinx.a
[1] 0.192 = 19.2% per year
> r.amzn.a
[1] 0.4695 = 46.95% per year (wow!!)
```

e. At the end of January, 2009, suppose you have $1,000 to invest in VFINX and AMZN over the next 60 months (5 years). Suppose you purchase $400 worth of VFINX and the remainder in AMZN. What are the portfolio weights in the two assets? Using the results from parts a. and b. compute the 5-year simple and cc portfolio returns.

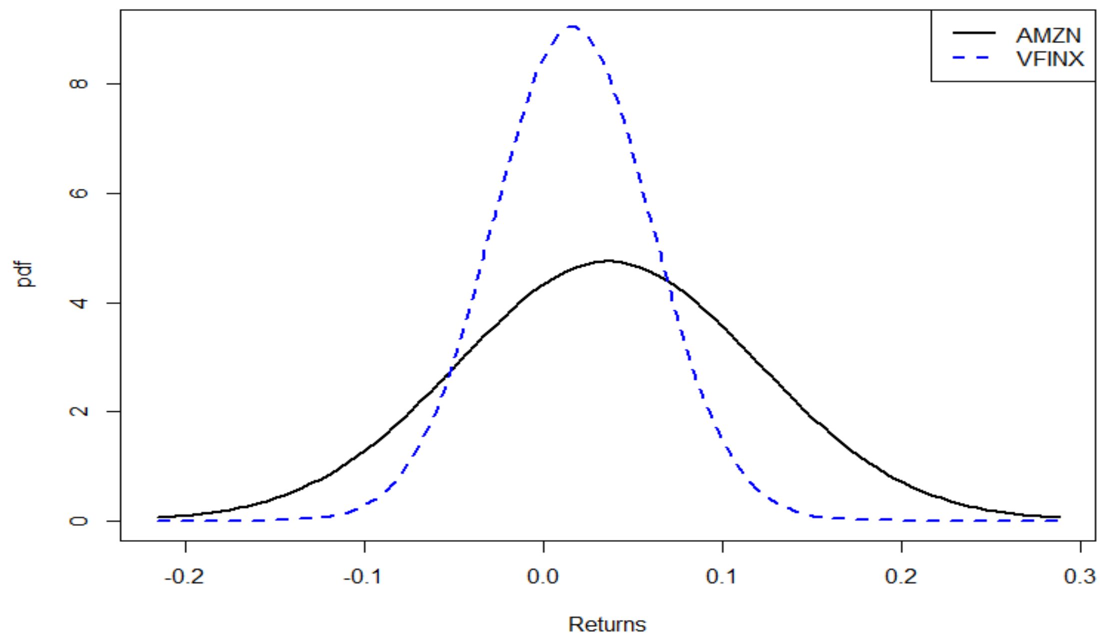
```
# portfolio weights
> w0 = 1000
> x.vfinx = 400/w0
> x.amzn = 1 - x.vfinx
> x.vfinx
[1] 0.4
> x.amzn
[1] 0.6

# portfolio returns
> r.p = x.vfinx*r.vfinx + x.amzn*r.amzn
> r.p
[1] 4.074 = 407.4%
> log(1 + r.p)
[1] 1.624 = 162.4%
```

II. Probability Theory (20 points, 4 points each)

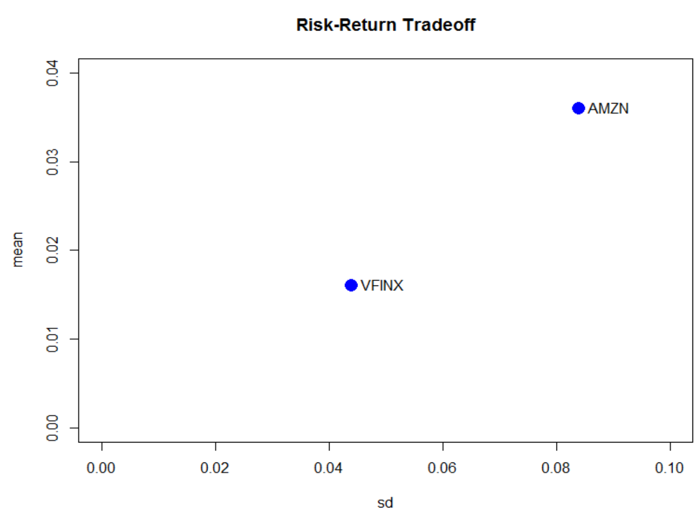
Let R_{vfinx} and R_{amzn} denote the monthly *simple* returns on VFINX and AMZN and suppose that $R_{vfinx} \sim iid\ N(0.016,(0.044)^2)$, $R_{amzn} \sim iid\ N(0.036,(0.084)^2)$.

a. Sketch the normal distributions for the two assets on the same graph. Show the mean values and the ranges mean ± 2 sd. Which asset appears to be the most risky?



AMZN has a higher SD which means its normal pdf is more spread out than the normal pdf of VFINX. There is more uncertainity, hence more risk, in the return for AMZN.

b. Plot the risk-return tradeoff for the two assets. That is, plot the mean values of each asset on the y-axis and plot the sd values on the x-axis. What relationship do you see?



AMZN has both a higher expected return and standard deviation (risk) than VFINX. This is the stylized risk-return tradeoff.

c. Let W_0 = $1,000 be the initial wealth invested in each asset. Compute the 1% monthly Value-at-Risk values for each asset. (Hint: $q_{0.01}^{Z} = -2.326$).

```
> q.vfinx.01 = mu.vfinx + sigma.vfinx*(-2.326)
> q.amzn.01 = mu.amzn + sigma.amzn*(-2.326)
> VaR.01.vfinx = w0*q.vfinx.01
> VaR.01.amzn = w0*q.amzn.01
> VaR.01.vfinx
[1] -86.34
> VaR.01.amzn
[1] -159.4
```

d. Continuing with c., state in words what the 1% Value-at-Risk numbers represent (i.e., explain what 1% Value-at-Risk for a one month $1,000 investment means)

With 1% probability (or one month in every 100 months), a one month $1000 investment in VFINX will lose $86.34 or more.

With 1% probability (or one month in every 100 months), a one month $1000 investment in AMZN will lose $159.4 or more.

e. The normal distribution can be used to characterize the probability distribution of monthly simple returns or monthly continuously compounded returns. What are two problems with using the normal distribution for simple returns? Given these two problems, why might it be better to use the normal distribution for continuously compounded returns?

Problem 1: Simple returns are bounded from below by -1. The normal distribution is defined over -∞ to +∞ and so it is possible returns to be less than -1 with positive probability if they are normally distributed

Problem 2: Multi-period simple returns are multiplicative, not additive. So if simple returns are normally distributed then multi-period returns are not normally distributed (because the product of two normal random variables is not normally distributed).

The normal distribution is more appropriate for continuously compounded returns because continuously compounded returns are defined over -∞ to +∞ and multi-period continuously compounded returns are additive.

III. Time Series Concepts (16 points, 4 points each)

a. Let $\{Y_t\}$ represent a stochastic process. Under what conditions is $\{Y_t\}$ covariance stationary?

$$E[Y_t] = \mu$$
$$\mathrm{var}(Y_t) = \sigma^2$$
$$\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j \text{ (depends on } j \text{ and not } t\text{)}$$

b. Realizations from four stochastic processes are given in Figure 1 below.

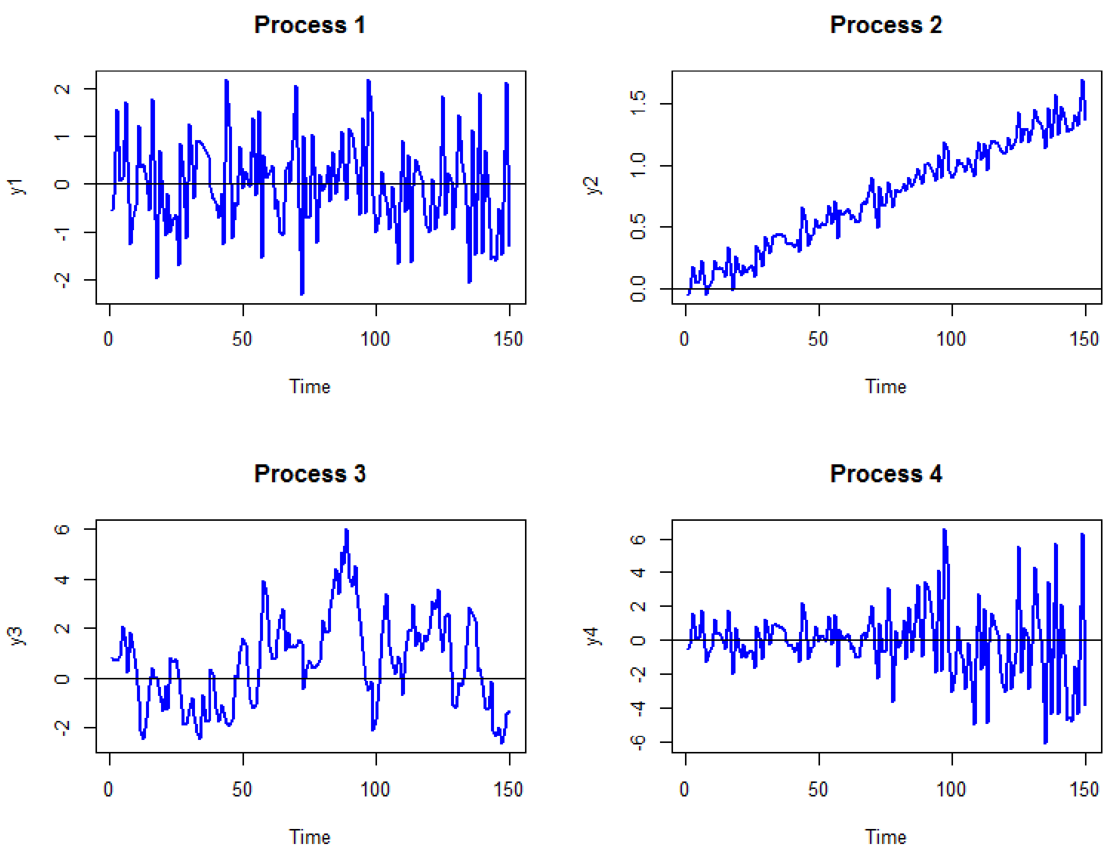


Figure 1: Realizations from four stochastic processes.

Which processes appear to be covariance stationary and which processes appear to be non-stationary? Briefly justify your answers.

- *Processes 1 and 3 appear to be covariance stationary. The means and volatilities appear constant over time and the series exhibit mean reversion (when the series gets above or below the mean it reverts back to the mean)*
- *Processes 2 and 4 appear to be non-stationary. Process 2 has a clear deterministic trend, so the mean is not constant over time. Process 4 appears to have two distinct volatilities (low in the first half and high in the second half).*

Figure 2 below shows a realization of a stochastic process representing a monthly time series of overlapping 2-month continuously compounded returns $r_t(2)$, where the 1-month continuously compounded returns follow a Gaussian White noise process.

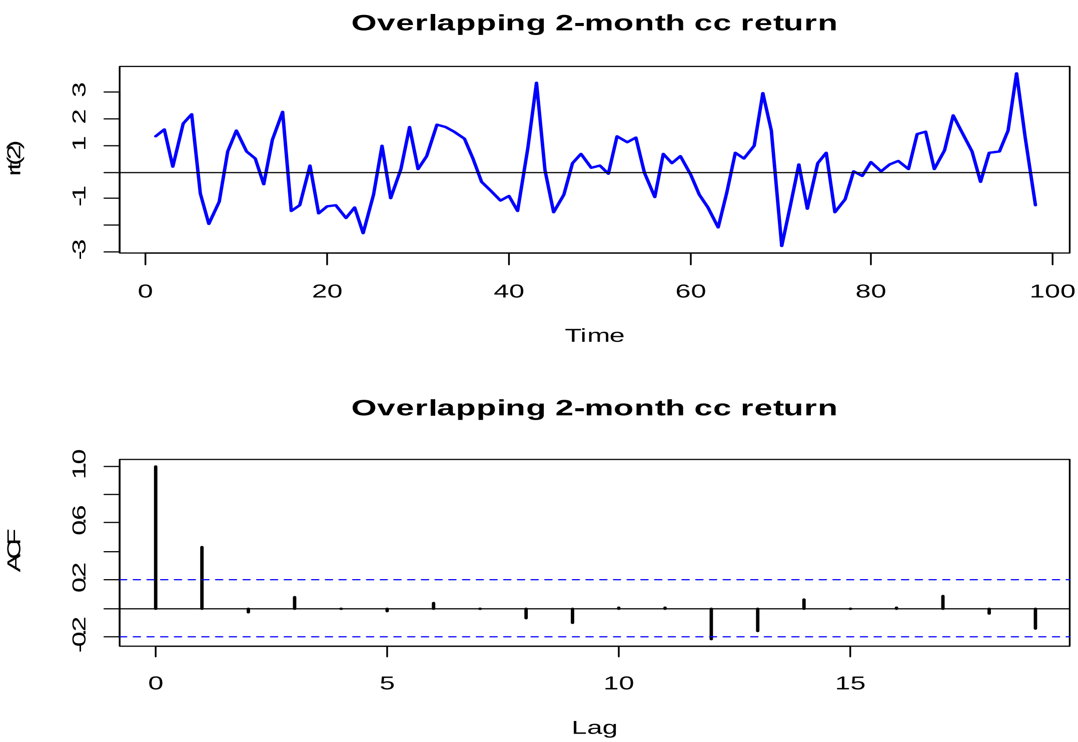


Figure 2 Overlapping 2-month returns

c. Based on the sample autocorrelations, which time series process is most appropriate for describing the series: MA(1) or AR(1)? Justify your answer.

Based on the sample autocovariance function (SACF), an MA(1) model looks more appropriate than an AR(1) model. In an MA(1) model, $\rho_1 = \dfrac{\theta}{1+\theta^2}$ *and* $\rho_j = 0 \text{ for } j > 1$

and in an AR(1) model $\rho_j = \phi^j$ *. The SACF shows* $\hat{\rho}_1 \approx 0.42$ and $\hat{\rho}_j \approx 0$ for $j > 1$ *which is consistent with an MA(1) model.*

d. If you think the process is an AR(1) process, what do you think is the value of the autoregressive parameter ϕ? If you think the process is a MA(1) process, what do you think is the value of the moving average parameter θ?

If the processes was an AR(1), then $\phi \approx \hat{\rho}_1 \approx 0.42$. *If the process was an MA(1) then*

$$\frac{\theta}{1+\theta^2} = \rho_1 \approx 0.42 \Rightarrow \theta = 0.42 \times (1+\theta^2) = 0.42 + 0.42 \times \theta^2 \Rightarrow 0.42 \times \theta^2 - \theta + 0.42 = 0$$

This is a quadratic equation in θ. *Using the quadratic formula, there are two solutions*

$$\theta = \frac{1 \pm \sqrt{1-4(0.42)^2}}{2(0.42)} = 1.836 \text{ and } 0.554.$$

IV. Matrix Algebra (16 points, 4 points each)

Let R_i denote the simple return on asset i (i = 1, …, N) with $E[R_i] = \mu_i$, var(R_i) = σ_i^2 and cov(R_i, R_j) = σ_{ij}. Define the ($N \times 1$) vectors $\mathbf{R} = (R_1, \ldots, R_N)'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{x} = (x_1, \ldots, x_N)'$, $\boldsymbol{y} = (y_1, \ldots, y_N)'$, and $\mathbf{1} = (1, \ldots, 1)'$ and the ($N \times N$) covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ contain portfolio weights (investment shares) that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolios defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ give the matrix algebra expression for the portfolio returns, ($R_{p,x}$ and $R_{p,y}$) and the portfolio expected returns ($\mu_{p,x}$ and $\mu_{p,y}$).

$$R_{p,x} = \mathbf{x}'\mathbf{R},\ R_{p,y} = \mathbf{y}'\mathbf{R}$$
$$E[R_{p,x}] = \mu_{p,x} = \mathbf{x}'\boldsymbol{\mu},\ E[R_{p,y}] = \mu_{p,y} = \mathbf{y}'\boldsymbol{\mu}$$

b. For the portfolios defined by the vectors ***x*** and ***y*** give the matrix algebra expression for the constraint that the portfolio weights sum to one.

$\mathbf{x}'\mathbf{1} = 1$ and $\mathbf{y}'\mathbf{1} = 1$

c. For the portfolios defined by the vectors ***x*** and ***y*** give the matrix algebra expression for the portfolio variances ($\sigma^2_{p,x}$ and $\sigma^2_{p,y}$), and the covariance between $R_{p,x}$ and $R_{p,y}$ (σ_{xy}).

$$\sigma^2_{p,x} = \text{var}(R_{p,x}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x},\ \ \sigma^2_{p,y} = \text{var}(R_{p,y}) = \mathbf{y}'\boldsymbol{\Sigma}\mathbf{y}$$
$$\sigma_{xy} = \text{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}$$

d. In the expression for the portfolio variance $\sigma^2_{p,x}$, how many variance terms are there? How many covariance terms are there?

There are N variance terms and N(N-1) total covariance terms (N(N-1)/2 unique covariance terms).

V. Descriptive Statistics (32 points, 4 points each)

Figure 3 shows monthly simple returns on the Vanguard S&P 500 index (VFINX) and Amazon stock (AMZN) over the 5-year period January 2009, through January 2014. For this period there are T=60 monthly returns.

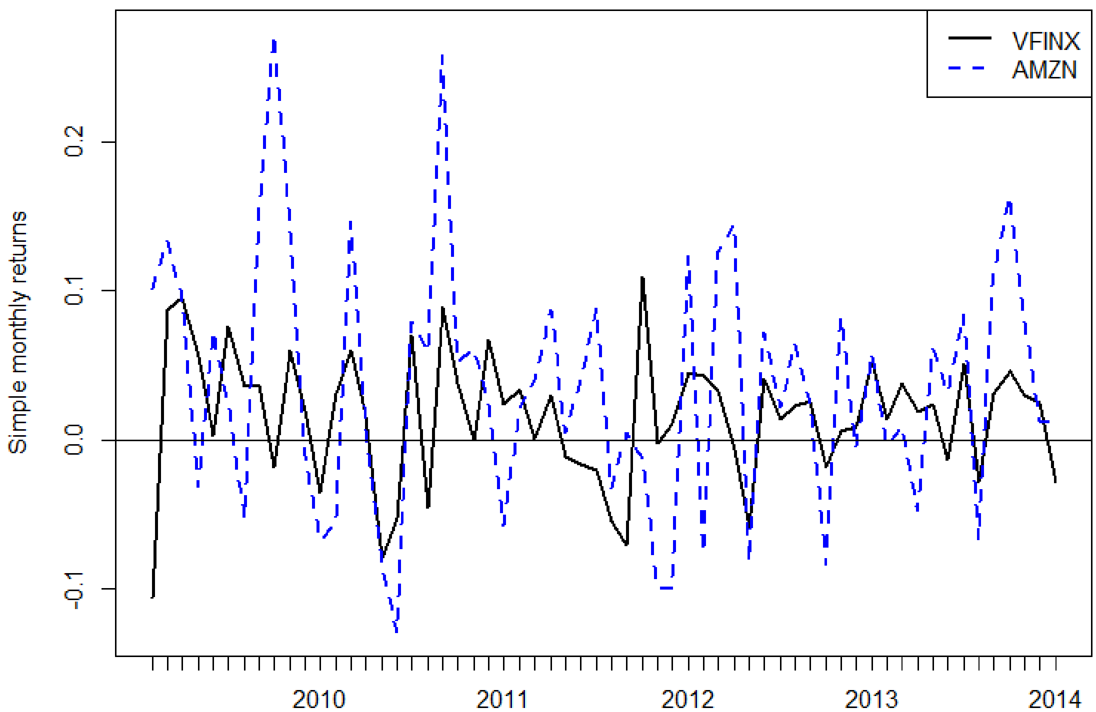


Figure 3: Monthly simple returns on two assets.

a. Do the monthly returns from the two assets look like realizations from a covariance stationary stochastic process? Why or why not?

Recall, covariance stationarity implies that the mean, variance and autocovariances are constant over time. Visually it looks like both mean values are constant over time and both series exhibit mean reversion. The volatility of VFINX looks fairly constant over time whereas the volatility of AMZN look a bit higher in the first half of the sample but it is not obvious. Overall, both series look like they could be realizations from a covariance stationary process.

b. Compare and contrast the return characteristics of the two assets. In addition, comment on any common features, if any, of the two return series.

AMZN looks to have a slightly larger mean than VFINX and clearly a larger volatility. AMZN also has many more large returns than VFINX. The two returns do not look like they move very closely together (e.g. when VFINX goes up (down) it is not always the case that AMZN goes up (down)). Hence, it looks like the correlation between VFINX and AMZN is a small positive number.

c. Figure 4 below gives the cumulative simple returns (equity curve) of each fund which represents the growth of $1 invested in each fund over the sample period. Which fund performed better over the sample?

A $1 investment in AMZN clearly did better than a $1 investment in VFINX. For AMZN $1 grew to about $7, and for VFINX $1 grew to only about $2.

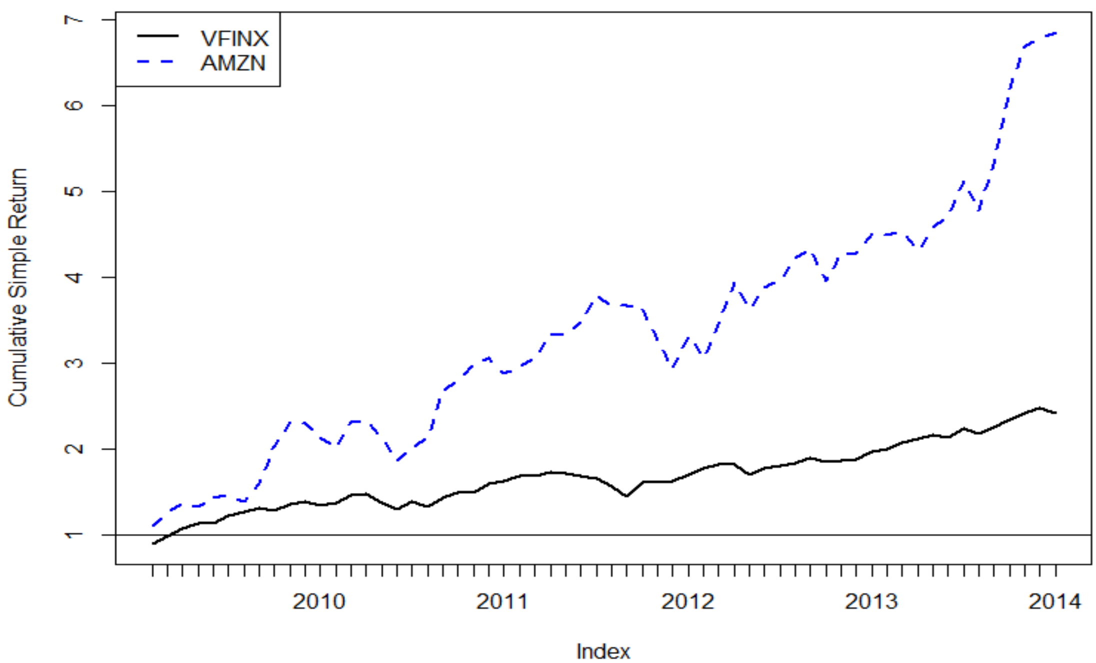
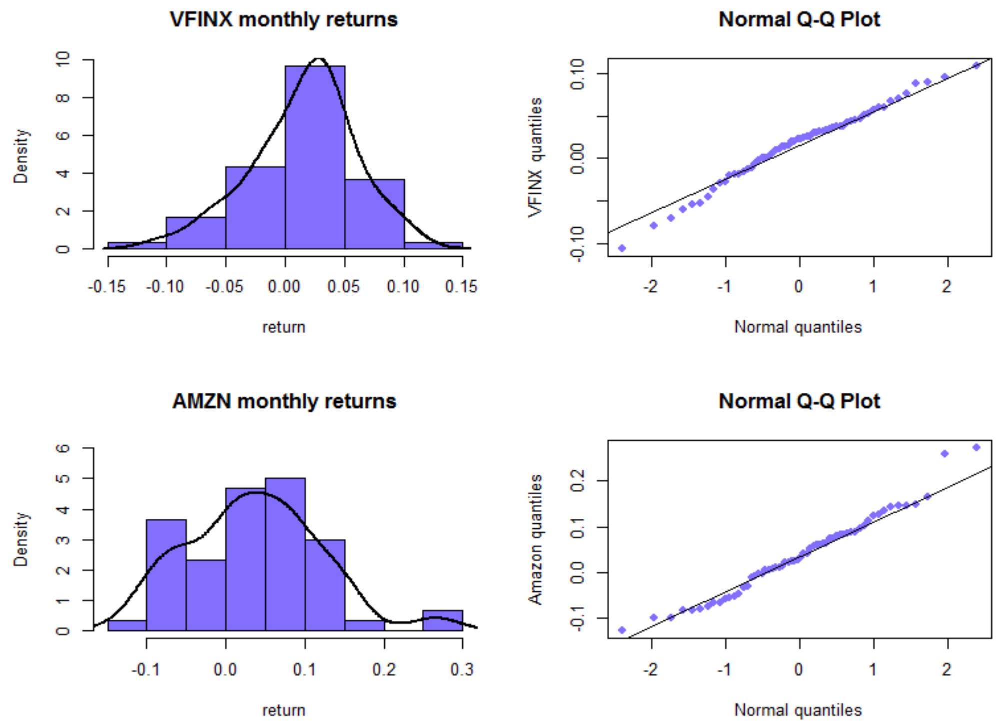
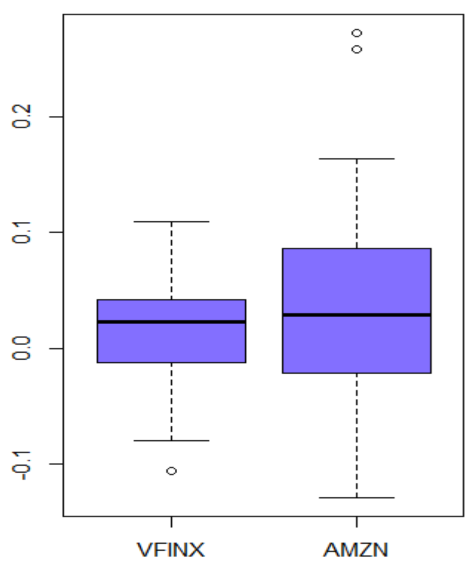
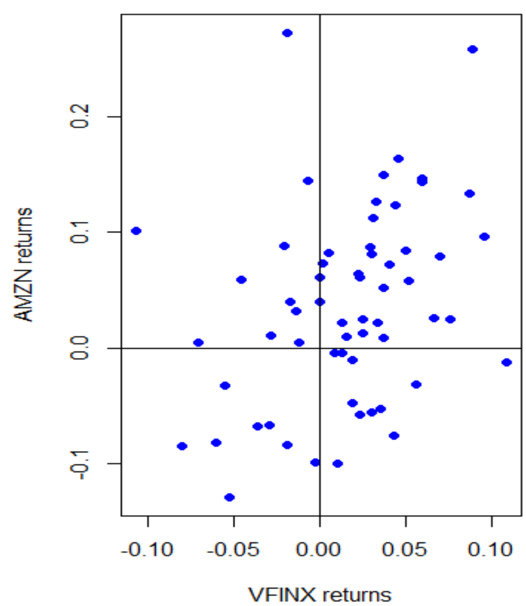


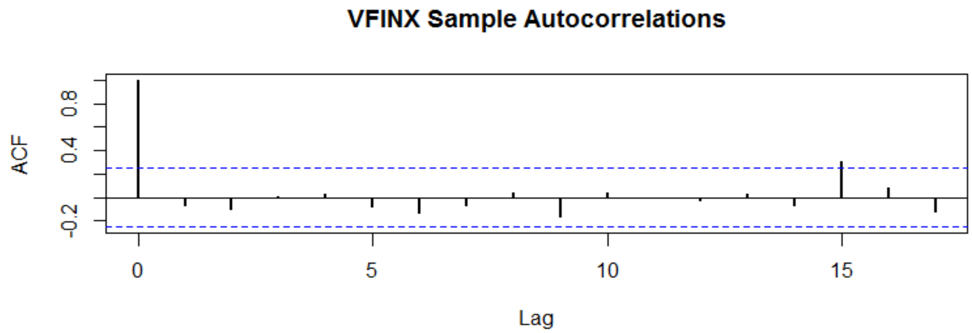
Figure 4 Cumulative simple returns on two assets.

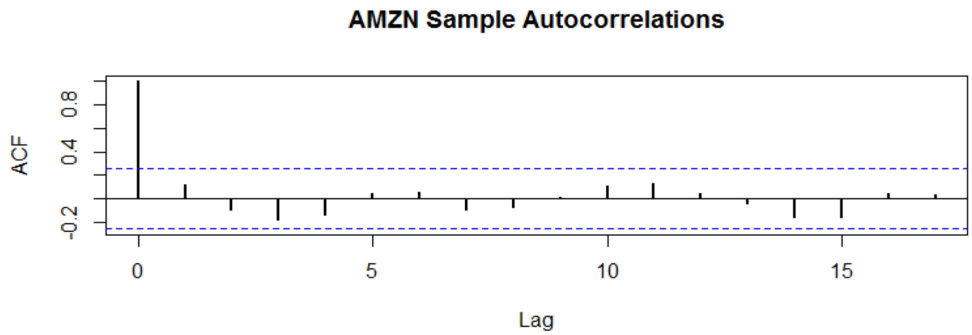
The figures below give some graphical diagnostics of the return distributions for VFINX and AMZN.


VFINX monthly returns
Density
return
Normal Q-Q Plot
VFINX quantiles
Normal quantiles
AMZN monthly returns
Density
return
Normal Q-Q Plot
Amazon quantiles
Normal quantiles


VFINX
AMZN


AMZN returns
VFINX returns





The following table summarizes some sample descriptive statistics of the monthly returns

Statistic	VFINX	AMZN
Mean ($\hat{\mu}$)	0.01571	0.03597
Standard Deviation ($\hat{\sigma}$)	0.04415	0.08489
Skewness	-0.4224	0.3811
Excess Kurtosis	0.1587	0.1550
1% empirical quantile ($\hat{q}_{0.01}$)	-0.09083	-0.11182
Lag 1 autocorrelation ($\hat{\rho}_1$)	-0.073	0.122
Lag 2 autocorrelation ($\hat{\rho}_2$)	-0.096	-0.099
Lag 3 autocorrelation ($\hat{\rho}_3$)	0.010	-0.185

The sample correlation between VFINX and AMZN returns is 0.3435.

d. Do the returns on VFINX and AMZN look normally distributed? Briefly justify your answer.

Both the returns on VFINX and AMZN look pretty normally distributed. The histograms are roughly bell shaped (VFINX has a slight negative skewness and AMZN has a slight positive skewness). Also, the normal qq-plots for both assets are mostly linear. The qq-plot for AMZN shows a slightly fatter right tail than implied by the normal distribution and there are two large outliers in the boxplot. Finally, the excess kurtosis values for both series are close to zero (about 0.15) which is consistent with the normal distribution.

e. Which asset appears to be riskier? Briefly justify your answer.

AMZN has a larger SD than VFINX (0.08 vs. 0.04) so there is more uncertainty in the return for AMZN. This means there is more risk with AMZN. However, VFINX has a negative skewness whereas AMZN has a positive skewness so it appears that VFINX has a larger tail risk than AMZN.

f. Does there appear to be any contemporaneous linear dependence between the returns on VFINX and AMZN? Briefly justify your answer.

The scatterplot of the AMZN and VFINX returns looks pretty much like a shotgun blast with a slight rightward tilt indicating very weak positive linear dependence. This is confirmed by the sample correlation value of 0.34.

g. Do the monthly returns show any evidence of (linear) time dependence? Briefly justify your answer.

For both series the SACF plots show no significant autocorrelations for lags less than 15 (no bars are above the dotted lines). All of the sample autocorrelations are small (close to zero) and show no distinct pattern over time. Hence, there does not appear to be any linear time dependence in the returns.

h. Let W_0 = $1,000 be the initial wealth invested in each asset. Compute the 1% monthly historical Value-at-Risk for each asset.

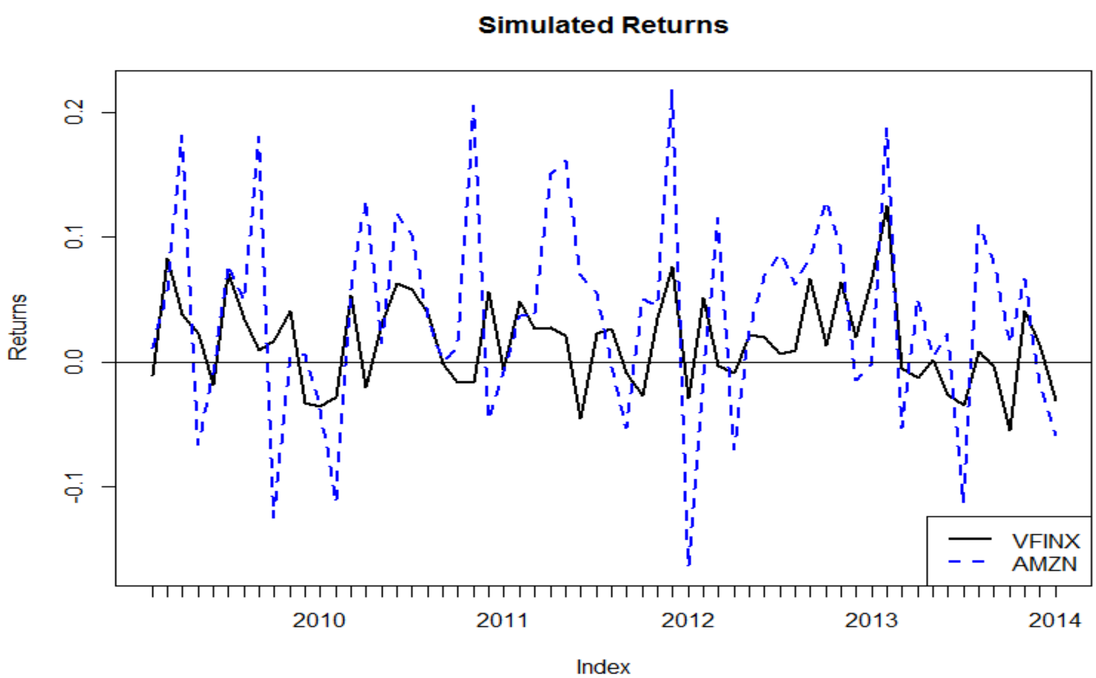
```
> W0 = 1000
> q.01 = apply(ret, 2, quantile, probs=0.01)
> VaR.vfinx.01 = W0*q.01["VFINX"]
> VaR.amzn.01 = W0*q.01["AMZN"]
> VaR.vfinx.01
 VFINX
-90.83
> VaR.amzn.01
  AMZN
-111.8
```

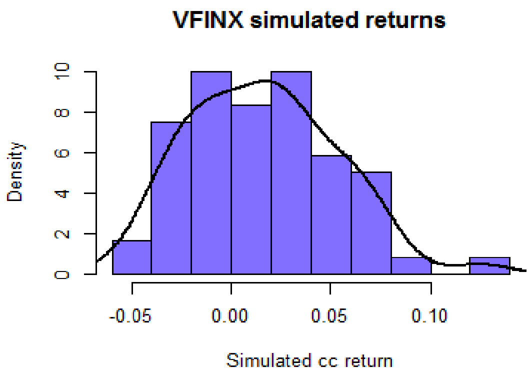
VI. Constant Expected Return Model (8 points)

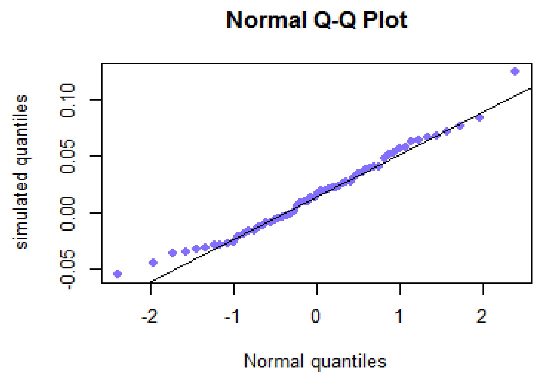
Consider the constant expected return (CER) model

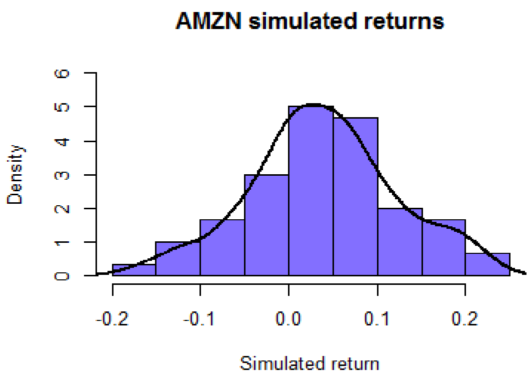
$$r_{it} = \mu_i + \varepsilon_{it},\ \ \varepsilon_{it} \sim iid\ N(0, \sigma_i^2)$$
$$\mathrm{cov}(r_{it}, r_{jt}) = \sigma_{ij},\ \ cor(r_{it}, r_{jt}) = \rho_{ij}$$

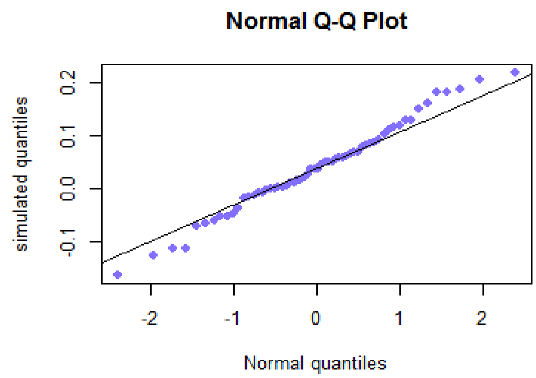
for the monthly simple returns on the Vanguard S&P500 index (VFINX) and Amazon stock (AMZN) presented in part V above. Below are simulated returns and some graphical descriptive statistics for VFINX and AMZN from the CER model calibrated using the sample estimates of the CER model parameters for the two assets (these are the sample statistics from the table of the previous question).

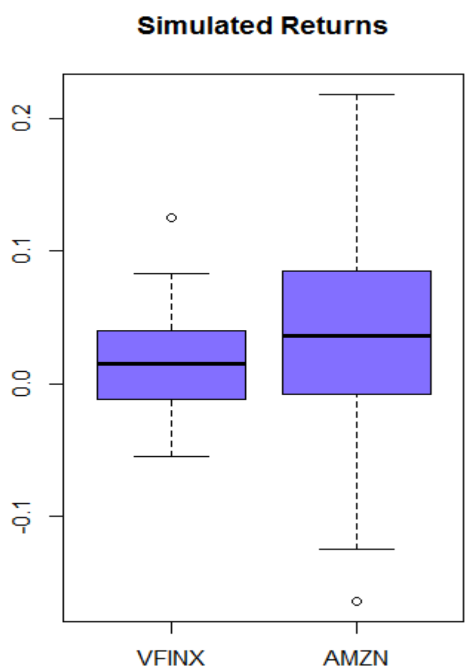


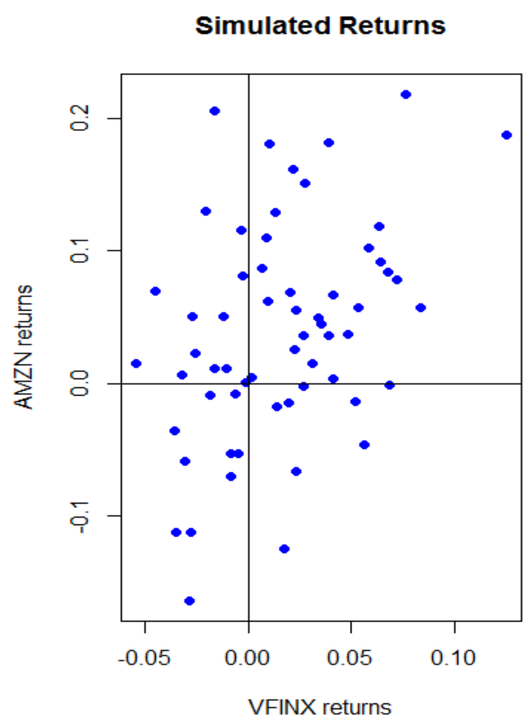

VFINX simulated returns
Density
0
2
4
6
8
10
-0.05
0.00
0.05
0.10
Simulated cc return

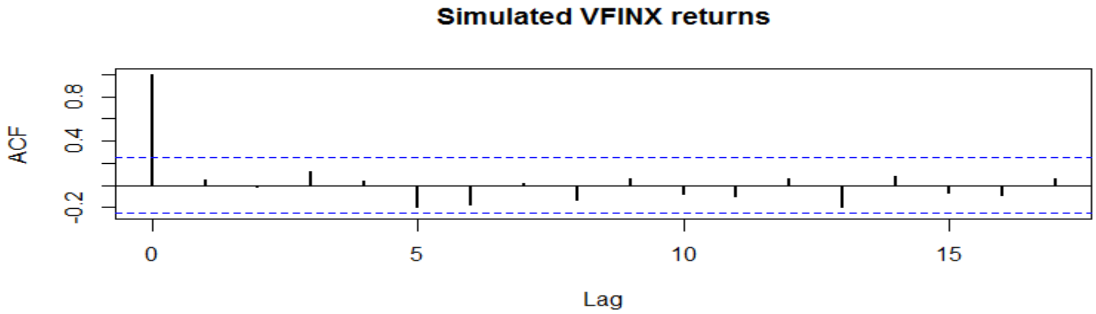

Normal Q-Q Plot
simulated quantiles
-0.05
0.00
0.05
0.10
-2
-1
0
1
2
Normal quantiles

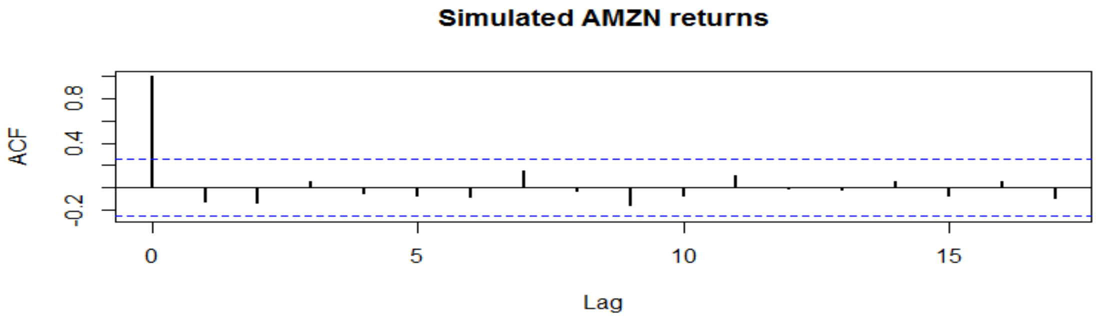

AMZN simulated returns
Density
0
1
2
3
4
5
6
-0.2
-0.1
0.0
0.1
0.2
Simulated return


Normal Q-Q Plot
simulated quantiles
-0.1
0.0
0.1
0.2
-2
-1
0
1
2
Normal quantiles


Simulated Returns
-0.1
0.0
0.1
0.2
VFINX
AMZN


Simulated Returns
AMZN returns
-0.1
0.0
0.1
0.2
-0.05
0.00
0.05
0.10
VFINX returns





a. Which features of the actual returns shown in part V are captured by the simulated CER model returns and which features are not?

Almost all features of the VFINX and AMZN returns are captured by the simulated CER model returns. There are only a few notable differences

- *Time plot of simulated returns looks very close to actual returns (AMZN has higher mean and volatility than VFINX; returns do not move very closely together)*
- *Histograms are roughly bell shaped and qq-plots are mostly linear. The negative skewness of actual VFINX returns and the positive skewness of the AMZN returns are not shown in the simulated returns*
- *Boxplots of simulated returns match closely the boxplots of the actual returns. However, the boxplot of the simulated returns do not show the positive outliers for AMZN*
- *Scatterplot of simulated returns looks like scatterplot of actual returns*
- *SACF of simulated returns look like SACF of actual returns.*

b. Does the CER model appear to be a good model for VFINX and AMZN returns? Why or why not?

Yes, the CER model appears to be a good model for VFINX and AMZN returns. Most of the important features of the actual data are captured by simulated data from the CER model.