Econ 424/Amath 540

Midterm Exam Solutions

This is a closed book and closed note exam. However, you are allowed one page of notes (8.5" by 11" double-sided) and the use of a calculator. Answer all questions and write all answers in a blue book or on separate sheets of paper. Time limit is 1 hours and 50 minutes. Total points = 104.

I. Return Calculations (20 pts, 4 points each)

Consider a 1-month investment in two assets: the Vanguard S&P 500 index (VFINX) and the Vanguard Emerging Markets Stock Index (VEIEX). Suppose you buy one share of the S&P 500 fund and one share of the emerging markets fund at the end of June, 2010 for $P_{vfinx,t-1} = 105.06$, $P_{veiex,t-1} = 28.64$, and then sell these shares at the end of July, 2010 for $P_{vfinx,t} = 109.04$, $P_{veiex,t} = 29.49$. (Note: these are actual adjusted closing prices taken from Yahoo!)

a. What are the simple 1-month returns for the two investments?

```
> p.vfinx.1 = 105.06
> p.vfinx.2 = 109.04
> p.veiex.1 = 28.64
> p.veiex.2 = 29.49

> r.vfinx = (p.vfinx.2 - p.vfinx.1)/p.vfinx.1
> r.veiex = (p.veiex.2 - p.veiex.1)/p.veiex.1
> r.vfinx
[1] 0.03788
> r.veiex
[1] 0.02968
```

b. What are the continuously compounded (cc) 1-month returns for the two investments?

```
> log(1 + r.vfinx)
[1] 0.03718
> log(1 + r.veiex)
[1] 0.02925
```

c. Assume you get the same monthly returns from part a. and b. every month for the next year. What are the annualized simple and cc returns?

```
> r.vfinx.a = (1 + r.vfinx)^12 - 1
> r.veiex.a = (1 + r.veiex)^12 - 1
> r.vfinx.a
[1] 0.5624
> r.veiex.a
[1] 0.4204
```

d. Continuing with c., how much will $10,000 invested in each fund be worth after 1 year?

```
> w0 = 10000
> w1.vfinx = w0*(1 + r.vfinx.a)
> w1.veiex = w0*(1 + r.veiex.a)
> w1.vfinx
[1] 15624
> w1.veiex
[1] 14204
```

e. At the end of June, 2011, suppose you have $10,000 to invest in VFINX and VEIEX over the next month. Suppose you purchase $2,000 worth of VFINX and the remainder in VEIEX. What are the portfolio weights in the two assets? Using the results from parts a. and b. compute the 1-month simple and cc portfolio returns.

```
> w0 = 10000
> x.vfinx = 2000/w0
> x.veiex = 1 - x.vfinx
> x.vfinx
[1] 0.2
> x.veiex
[1] 0.8

> r.p = x.vfinx*r.vfinx + x.veiex*r.veiex
> r.p
[1] 0.03132
> log(1 + r.p)
[1] 0.03084
```
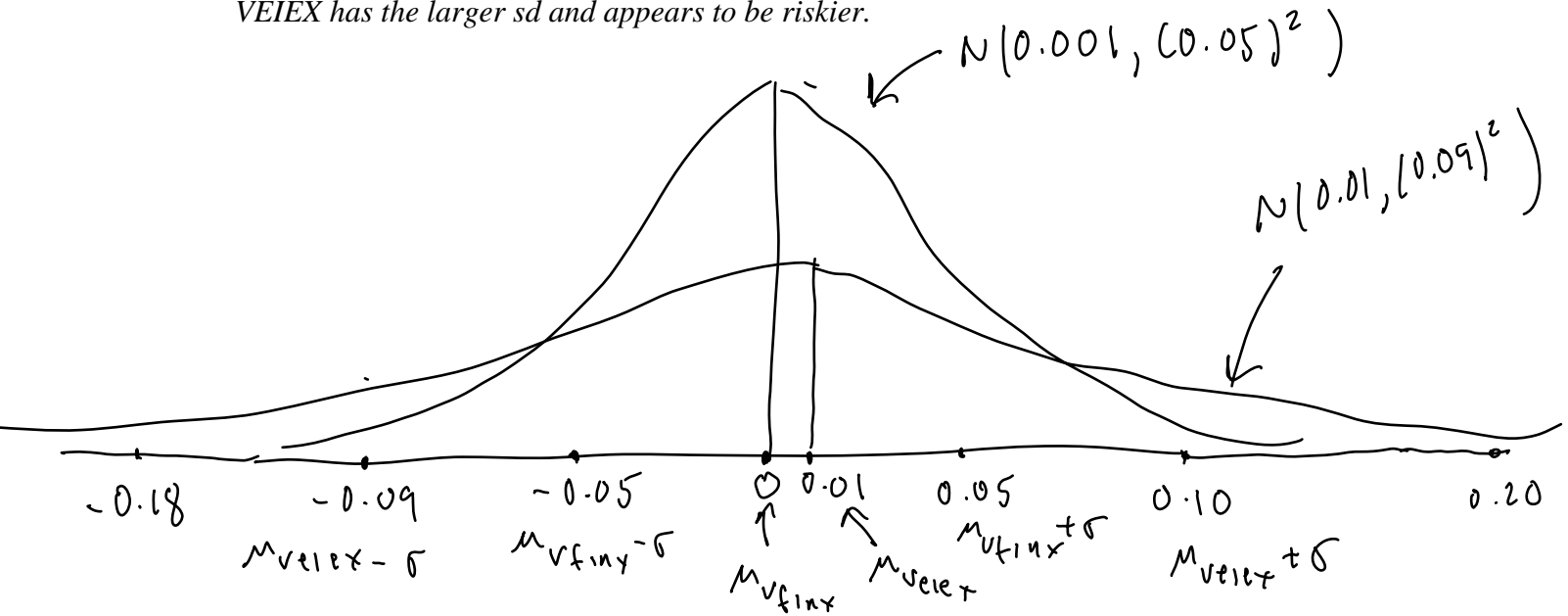
## II. Probability Theory (20 points, 4 points each)

Let r_{vfinx} and r_{veiex} denote the monthly *continuously compounded* returns on VFINX and VEIEX and suppose that $r_{vfinx} \sim iid\ N(0.001,(0.05)^2)$, $r_{veiex} \sim iid\ N(0.01,(0.09)^2)$.

a. Sketch the normal distributions for the two assets on the same graph. Show the mean values and the ranges mean ± 2 sd. Which asset appears to be the most risky?

VEIEX has the larger sd and appears to be riskier.

$N(0.001, (0.05)^2)$

$N(0.01, (0.09)^2)$

-0.18   -0.09   -0.05   0   0.01   0.05   0.10   0.20

$\mu_{veiex} - \sigma$ $\mu_{vfiny} - \sigma$ μ_{vfiny} μ_{veiex} $\mu_{vfinx} + \sigma$ $\mu_{veiex} + \sigma$

b. Plot the risk-return tradeoff for the two assets. That is, plot the mean values of each asset on the y-axis and plot the sd values on the x-axis. What relationship do you see?

μ

0.05

0.01

VFINY

VEIEX

0.05   0.10   σ
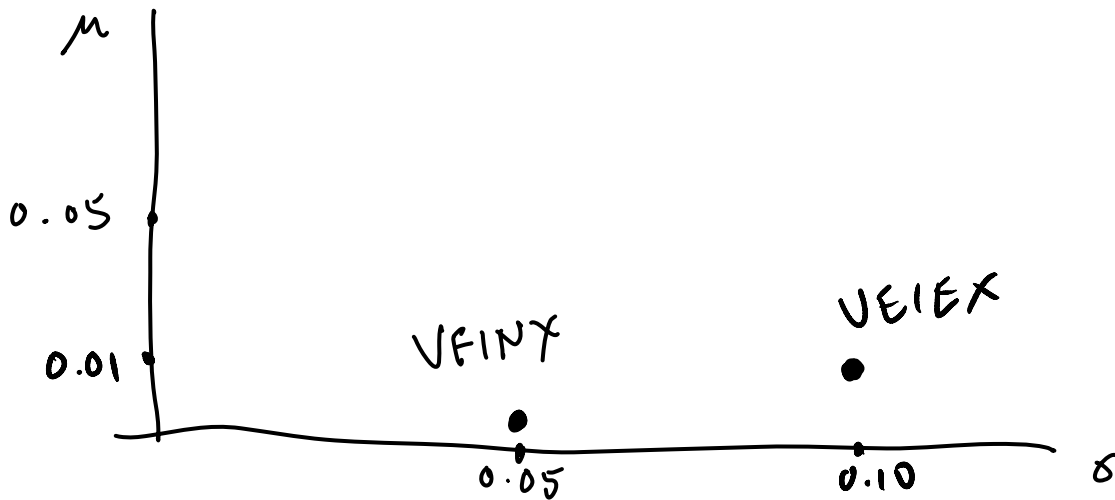
Here we see a positive relationship between risk and return.

c. Let W_0 = \$100,000 be the initial wealth invested in each asset. Compute the 5% monthly Value-at-Risk for each asset. (Hint: $q_{0.05}^Z = -1.645$)

```
> mu.vfinx = 0.001
> mu.veiex = 0.01
> sigma.vfinx = 0.05
> sigma.veiex = 0.09
> w0 = 100000
> q.vfinx = mu.vfinx + sigma.vfinx*(-1.645)
> q.veiex = mu.veiex + sigma.veiex*(-1.645)
> VaR.05.vfinx = w0*(exp(q.vfinx) - 1)
> VaR.05.veiex = w0*(exp(q.veiex) - 1)
> VaR.05.vfinx
[1] -7804
> VaR.05.veiex
[1] -12894
```

d. Give an expression for the 12-month (annual) continuously compounded return, $r_t(12)$, in terms of the monthly continuously compounded returns. Using this expression compute $E[r_t(12)]$, and $SD(r_t(12))$ for each asset.

$$r_t \sim iid \ N(\mu, \sigma^2)$$
$$r_t(12) = r_t + r_{t-1} + \cdots + r_{t-11}$$
$$E[r_t(12)] = E[r_t + r_{t-1} + \cdots + r_{t-11}] = E[r_t] + E[r_{t-1}] + \cdots + E[r_{t-11}] = 12\mu$$
$$\text{var}(r_t(12)) = \text{var}(r_t + r_{t-1} + \cdots + r_{t-11}) = \text{var}(r_t) + \text{var}(r_{t-1}) + \cdots + \text{var}(r_{t-11}) = 12\sigma^2$$
$$SD(r_t(12)) = \sqrt{12}\sigma$$

Using the above formulas, we have

```
> mu.vfinx.a = 12*mu.vfinx
> sigma.vfinx.a = sqrt(12)*sigma.vfinx
> mu.vfinx.a
[1] 0.012
> sigma.vfinx.a
[1] 0.1732
```

```
> mu.veiex.a = 12*mu.veiex
> sigma.veiex.a = sqrt(12)*sigma.veiex
> mu.veiex.a
[1] 0.12
> sigma.veiex.a
[1] 0.3118
```

e. Continuing with d., let $W_0 = \$100{,}000$ be the initial wealth invested in each asset. Compute the 5% annual Value-at-Risk for each asset.

```
> w0 = 100000
> q.vfinx.a = mu.vfinx.a + sigma.vfinx.a*(-1.645)
> q.veiex.a = mu.veiex.a + sigma.veiex.a*(-1.645)
> VaR.05.vfinx.a = w0*(exp(q.vfinx.a) - 1)
> VaR.05.veiex.a = w0*(exp(q.veiex.a) - 1)

> VaR.05.vfinx.a
[1] -23885
> VaR.05.veiex.a
[1] -32488
```

III. Matrix Algebra (16 points, 4 points each)

Let R_i denote the simple return on asset i $(i = 1,2,\ldots,n)$ with $E[R_i] = \mu_i$, $\mathrm{var}(R_i) = \sigma_i^2$ and $\mathrm{cov}(R_i, R_j) = \sigma_{ij}$.

a. Define the $n \times 1$ vectors containing the returns and expected returns (call these R and μ), and define the $n \times n$ covariance matrix (call this Σ).

$$
\mathbf{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}, \quad
\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \quad
\Sigma = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{1,n} & \cdots & \sigma_n^2 \end{pmatrix}
$$

b. Let x_i and y_i $(i = 1,2,\ldots,n)$ denote the shares of wealth invested in asset i in two portfolios. Define the $n \times 1$ vectors of portfolio weights for these two portfolios. Denote these vectors x and y. Using matrix algebra, give expressions for the restriction that the weights in each portfolio must sum to one.

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad
\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad
\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad
\mathbf{x'1} = 1, \; \mathbf{y'1} = 1
$$

c. Using matrix algebra, give expressions for the returns, expected returns and variances for the two portfolios.

$$R_{p,x} = \mathbf{x}'\mathbf{R}, \; R_{p,y} = \mathbf{y}'\mathbf{R},$$

$$\mu_{p,x} = \mathbf{x}'\mu, \; \mu_{p,y} = \mathbf{y}'\mu,$$

$$\sigma^2_{p,x} = \mathbf{x}'\Sigma\mathbf{x}, \; \sigma^2_{p,y} = \mathbf{y}'\Sigma\mathbf{y}$$

d. Using matrix algebra, give an expression for the covariance between the returns on portfolio x and the returns on portfolio y.

$$\sigma_{x,y} = \mathbf{x}'\Sigma\mathbf{y}$$

IV. Time Series Concepts (8 points, 4 points each)

a. Let $\{Y_t\}$ represent a stochastic process. Under what conditions is $\{Y_t\}$ covariance stationary?

$$E[Y_t] = \mu$$
$$\mathrm{var}(Y_t) = \sigma^2$$
$$\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j \; \text{(depends on } j \text{ and not } t\text{)}$$

b. Realizations from four stochastic processes are given in Figure 1 below. Which processes appear to be covariance stationary and which processes appear to be non-stationary? Briefly justify your answers.
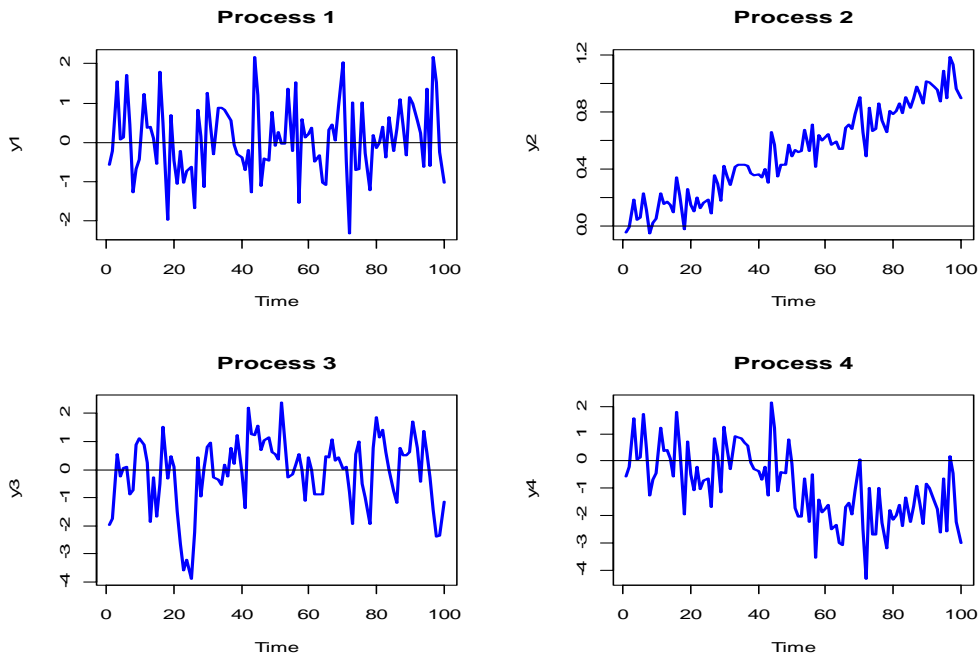
Figure 1: Realizations from four stochastic processes.

- *Processes 1 and 3 appear to be covariance stationary. The means and volatilities appear constant over time.*
- *Processes 2 and 4 appear to be non-stationary. Process 2 has a clear deterministic trend, so the mean is not constant over time. Process 4 appears to have two distinct means. The first mean is roughly zero for the first half of the sample, and the second mean is roughly -2 for the second half of the sample.*

Figure 2 below shows a realization of a stochastic process representing a monthly time series of overlapping 2-month continuously compounded returns, where the 1-month continuously compounded returns follow a Gaussian White noise process.

c. Based on the sample autocorrelations, which time series process is most appropriate for describing the series: MA(1) or AR(1)?

Based on the sample autocovariance function (SACF), an MA(1) model looks more appropriate than an AR(1) model. In an MA(1) model, $\rho_1 = \dfrac{\theta}{1+\theta^2}$ and $\rho_j = 0$ for $j > 1$ and in an AR(1) model $\rho_j = \phi^j$. The SACF shows $\hat{\rho}_1 \approx 0.42$ and $\hat{\rho}_j \approx 0$ for $j > 1$ which is consistent with an MA(1) model.

Overlapping 2-month cc return
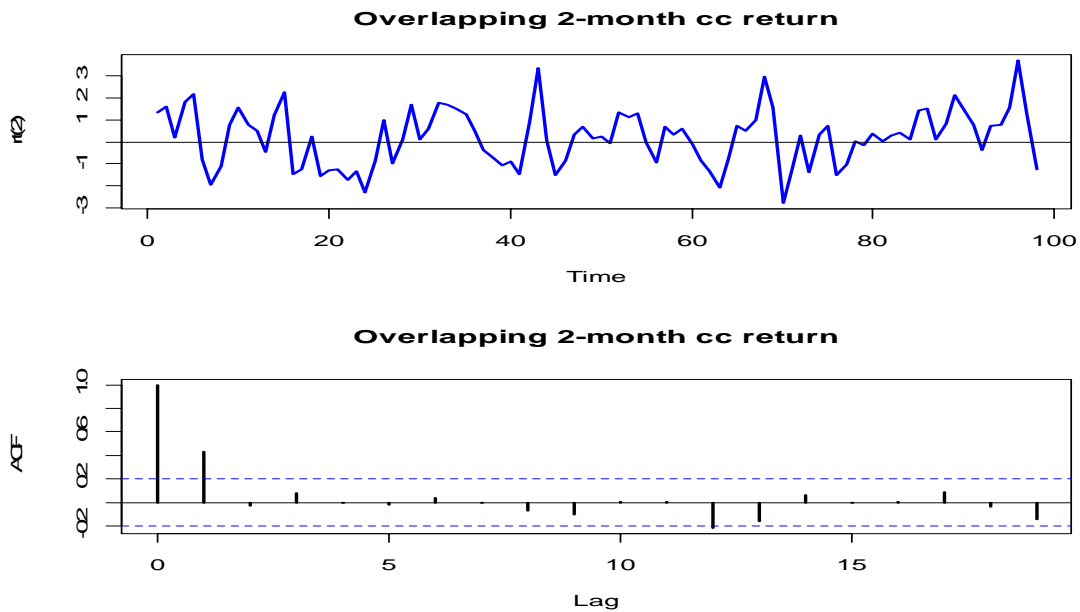
d. If you think the process is an AR(1) process, what do you think is the value of the autoregressive parameter ϕ? If you think the process is a MA(1) process, what do you think is the value of the moving average parameter θ?

If the processes was an AR(1), then $\phi \approx \hat{\rho}_1 \approx 0.42$. If the process was an MA(1) then

$$\frac{\theta}{1+\theta^2} = \rho_1 \approx 0.42 \Rightarrow \theta = 0.42 \times (1+\theta^2) = 0.42 + 0.42 \times \theta^2 \Rightarrow 0.42 \times \theta^2 - \theta + 0.42 = 0$$

This is a quadratic equation in θ. Using the quadratic formula, there are two solutions

$$\theta = \frac{1 \pm \sqrt{1 - 4(0.42)^2}}{2(0.42)} = 1.836 \text{ and } 0.554.$$

V. Descriptive Statistics (24 points, 4 points each)

Figure 3 shows monthly continuously compounded returns on the Vanguard S&P 500 index (VFINX) and the Vanguard Emerging Markets index (VEIEX) over the 5-year period September 2005, through September 2010. For this period there are T=60 monthly observations.
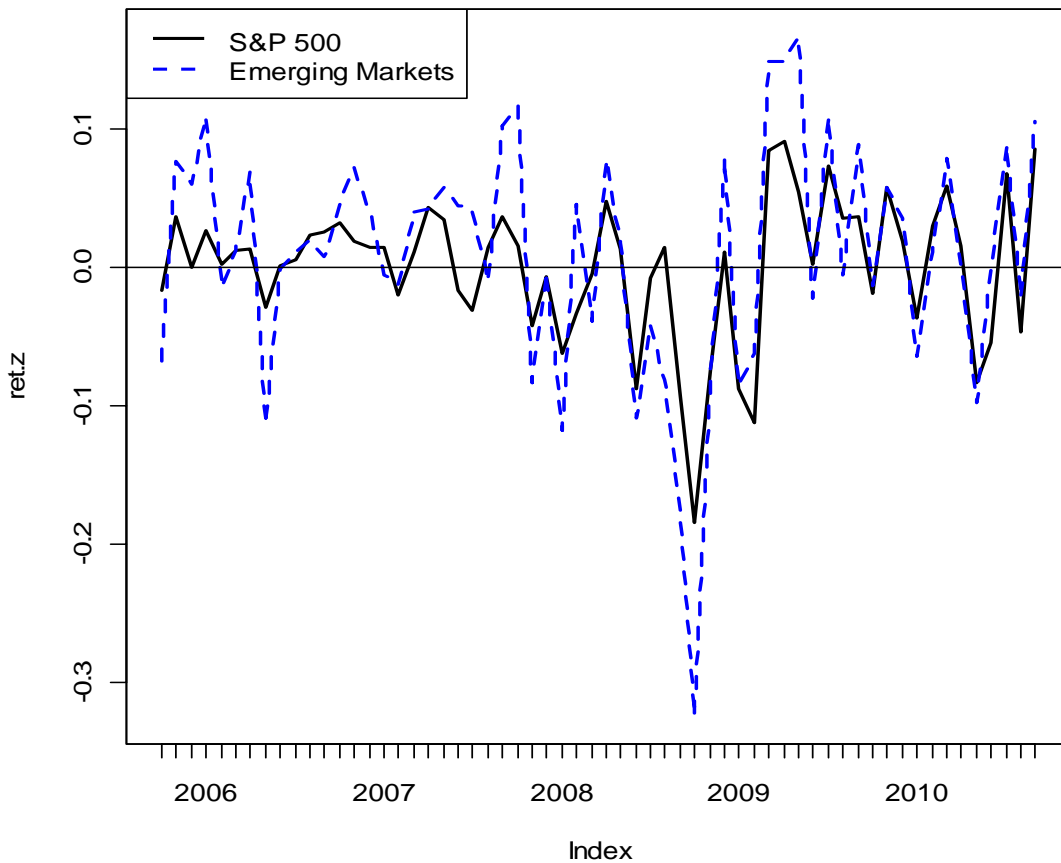
Figure 2: Monthly continuously compounded returns on two Vanguard Funds.

a. Do the monthly continuously compounded returns from the two funds look like realizations from a covariance stationary stochastic process? Why or why not?

Recall, covariance stationarity implies that the mean, variance and autocovariances are constant over time. Both processes appear to have changing means and variances. The correlation between the processes may also be changing over time. Clearly the volatilities are higher and the means are lower during the crisis period. The series seem to move together more closely during the crisis period as well. Given these observations, it does not appear that these two series are covariance stationary.

b. Comment on any common features, if any, of the two return series. Recall, the financial crisis reached its peak toward the end of 2008.

Both series tend to move together, indicating a positive correlation. The positive correlation appears to be greater during and after the crisis period. Also, the volatilities of both series seem to increase around the same time.
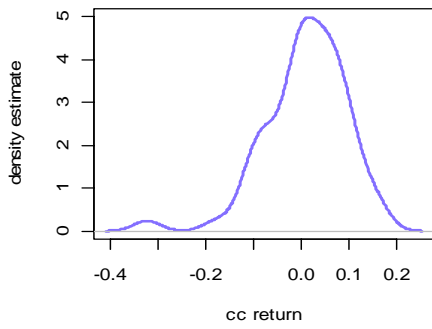
The figures below gives some graphical diagnostics of the return distributions for the S&P 500 index and the emerging markets fund.
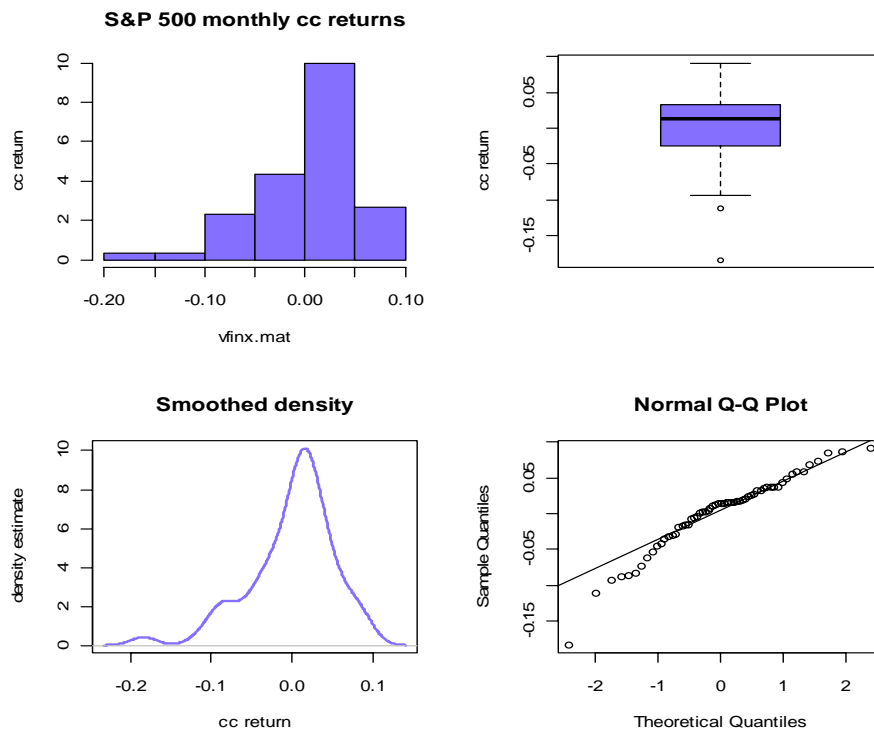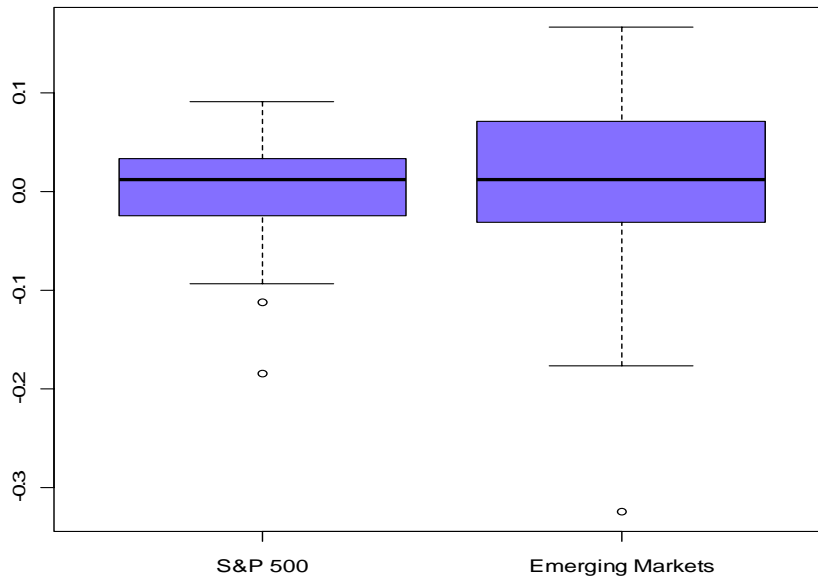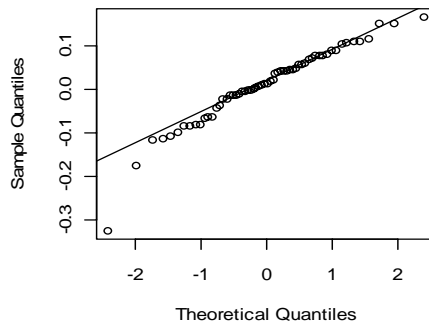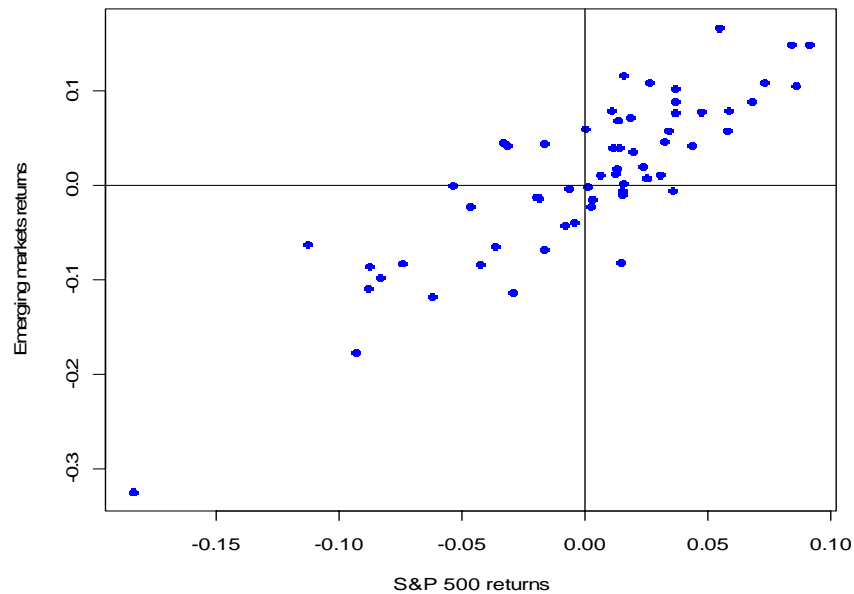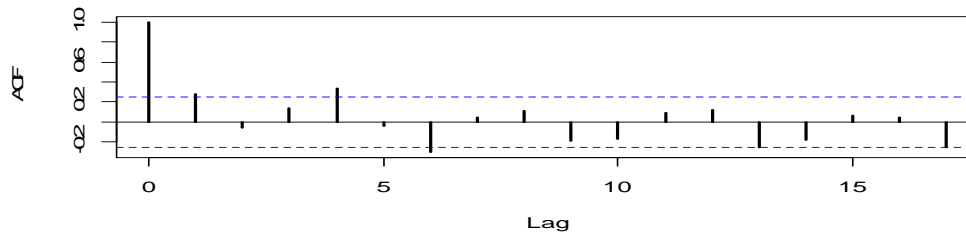
Emerging markets fund monthly cc return
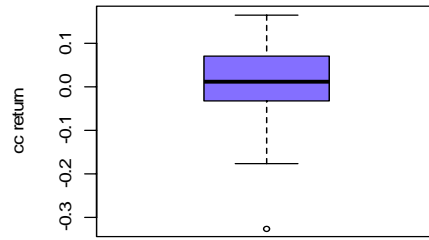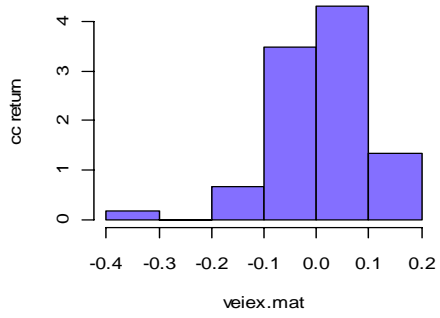
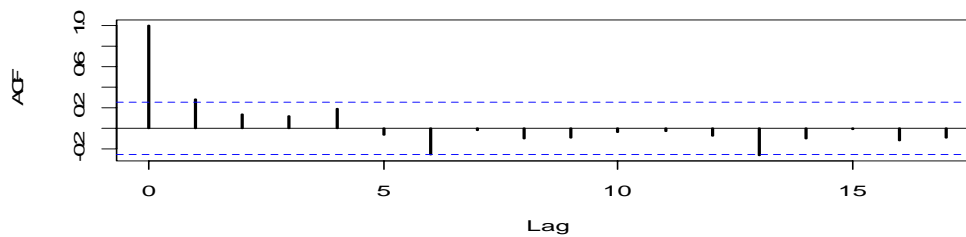**Smoothed density**

Normal Q-Q Plot

Vanguard S&P 500 Index

Vanguard Emerging Markets Index

c. Do the returns on the S&P500 index and the emerging markets fund look normally distributed? Briefly justify your answer.

The returns on both the S&P 500 index and the emerging markets index exhibit non-normal behavior. The histograms show a long left tail (negative skewness) and the boxplot indicates a some large negative outliers (fat tails). The normal QQ-plot is not linear in the left tail. All of this is strong evidence against the normal distribution.

d. Which asset appears to be riskier? Briefly justify your answer.

The volatility of the emerging markets index is much larger than the volatility of the S&P 500 and appears to be riskier.

e. Based on the scatterplot of returns, does there appear to be any linear dependence between the returns on the S&P500 index and the emerging markets fund? Briefly justify your answer.

There is a strong positive linear association in the scatterplot. When the S&P 500 returns increase, the emerging markets returns also increase and the scatter lies quite close to a straight line.

f. Based on the sample autocorrelation functions (SACFs), do the returns show any strong temporal dependence?

Both sample autocorrelation plots show a positive first lag autocorrelation that is marginally statistically different from zero (the estimate of ρ_1 is just above the dotted line). The remaining sample autocorrelations are not very large. Hence there is weak evidence of temporal dependence that lasts at most one period.

VI. Constant Expected Return Model (8 points)

Consider the constant expected return model

$$r_{it} = \mu_i + \varepsilon_{it}, \ \varepsilon_{it} \sim iid \ N(0, \sigma_i^2)$$
$$\text{cov}(r_{it}, r_{jt}) = \sigma_{ij}, \ cor(r_{it}, r_{jt}) = \rho_{ij}$$
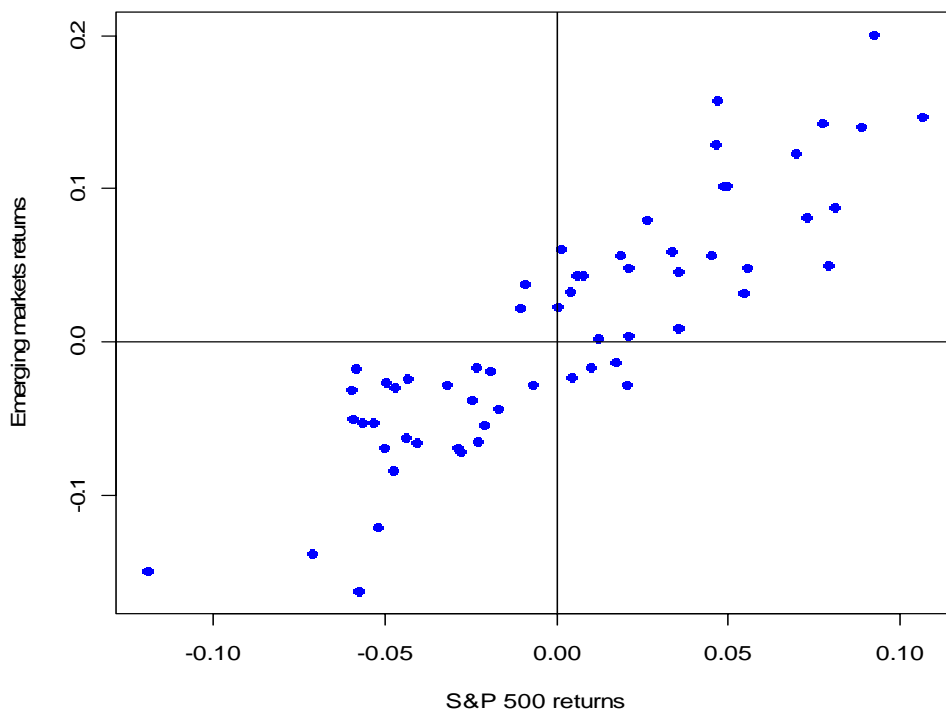
for the monthly continuously compounded returns on the Vanguard S&P500 index and the Vanguard emerging markets index presented in part V.

Below are simulated returns for the S&P 500 index and the Emerging Markets index from the CER model calibrated using the sample estimates of the CER model parameters for the two assets.
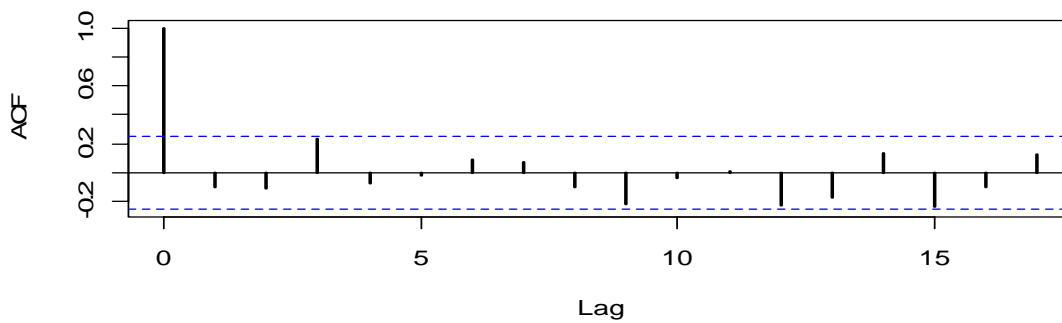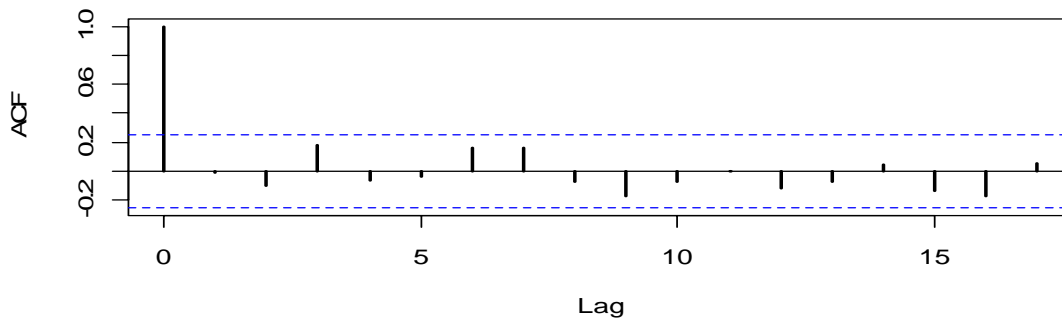
Simulated Returns
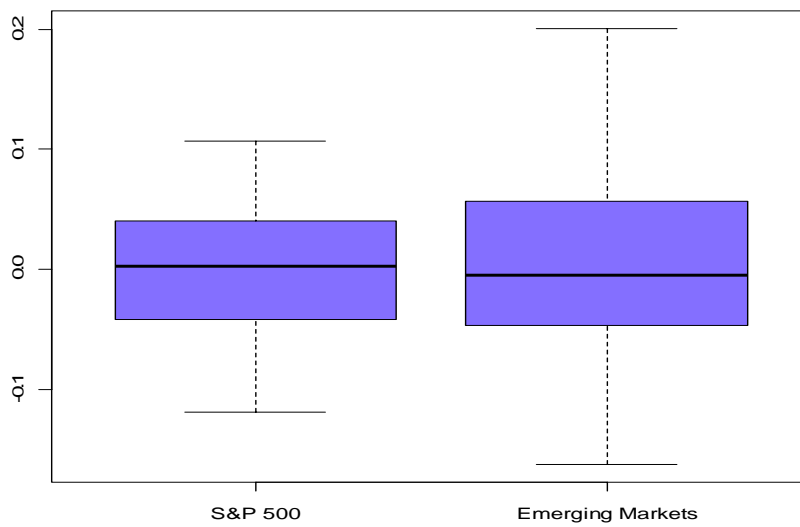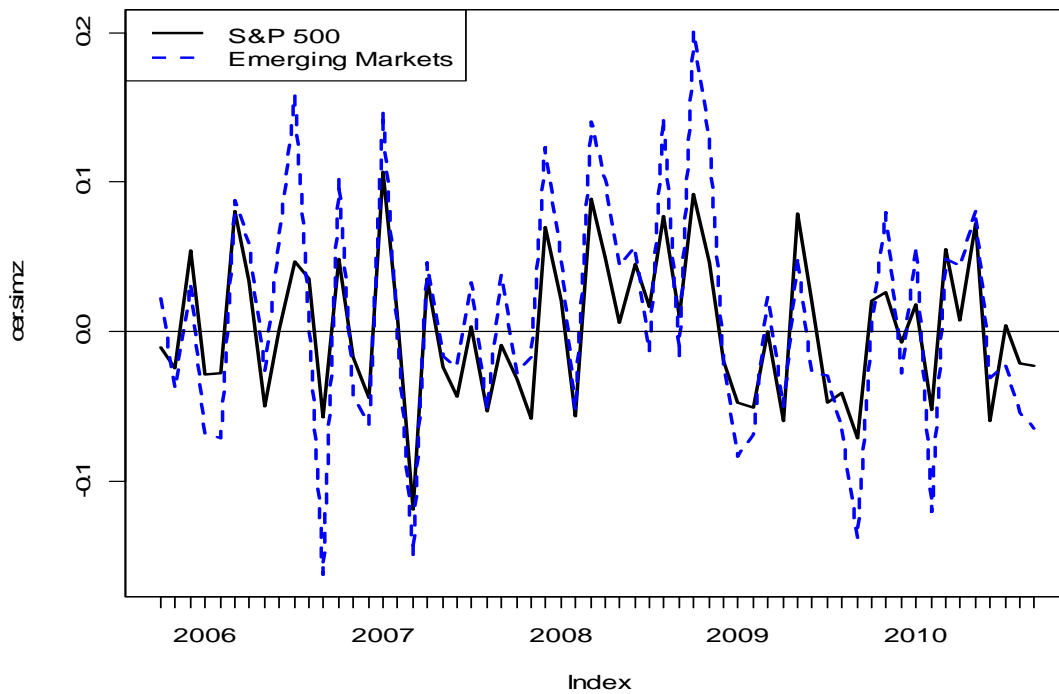
Simulated Returns

Simulated Vanguard S&P 500 Index

Simulated Vanguard Emerging Markets Index

a. Which features of the actual returns shown in part V are captured by the simulated returns and which features are not?

The features of the actual returns captured by the simulated returns are:

- *The mean of VEIEX is slightly higher than VFINX*
- *VEIEX has higher volatility than VFINX*
- *Strong positive correlation between VEIEX and VFINX*
- *The lack of time dependence (autocorrelation) in the returns*

The features not captured by the simulated returns:

- *The non-normality of the observed data. Both series have negative skewness and some large outliers.*
- *The changing volatility and correlation during and after the crisis period.*