University of Washington — Spring 2015
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Econ 424

Midterm Exam Solutions

This is a closed book and closed note exam. However, you are allowed one page of notes (8.5” by 11” or A4 double-sided) and the use of a calculator. Answer all questions and write all answers directly on the exam in the space provided. If you need more space, you may use extra sheets of paper. Time limit is 1 hour and 50 minutes. Total points = 116.

I. Return Calculations (20 pts, 4 points each)

Consider a 60-month (5 year) investment in two assets: the Vanguard S&P 500 index (VFINX) and Apple stock (AAPL). Suppose you buy one share of the S&P 500 fund and one share of Apple stock at the end of January, 2010 for $P_{vfinx,t-60} = 89.91$, $P_{aapl,t-60} = 25.88$, and then sell these shares at the end of January, 2015 for $P_{vfinx,t} = 184.2$, $P_{aapl,t} = 116.7$. (Note: these are actual adjusted closing prices taken from Yahoo!). In this question, you will see how much money you could have made if you invested in these assets right after the financial crisis.

a. What are the simple 60-month (5-year) returns for the two investments?

```
> r.vfinx = (p.vfinx.2 - p.vfinx.1)/p.vfinx.1
> r.AAPL = (p.AAPL.2 - p.AAPL.1)/p.AAPL.1
> r.vfinx
[1] 1.05
> r.AAPL
[1] 3.51
```

b. What are the continuously compounded (cc) 60-month (5-year) returns for the two investments?

```
> log(1 + r.vfinx)
[1] 0.717
> log(1 + r.AAPL)
[1] 1.51
```

c. Suppose you invested $1,000 in each asset at the end of January, 2010. How much would each investment be worth at the end of January, 2015?

```
> w0 = 1000
> w1.vfinx = w0*(1 + r.vfinx)
> w1.AAPL = w0*(1 + r.AAPL)
> w1.vfinx
[1] 2049
> w1.AAPL
[1] 4509
```

d. What is the compound annual return on the two 5 year investments?

```
> r.vfinx.a = (1 + r.vfinx)^(1/5) - 1
> r.AAPL.a = (1 + r.AAPL)^(1/5) - 1
> r.vfinx.a
[1] 0.154
> r.AAPL.a
[1] 0.352
```

e. At the end of January, 2010, suppose you have $1,000 to invest in VFINX and AAPL over the next 60 months (5 years). Suppose you purchase $400 worth of VFINX and the remainder in AAPL. What are the portfolio weights in the two assets? Using the results from parts a. and b. compute the 5-year simple and cc portfolio returns.

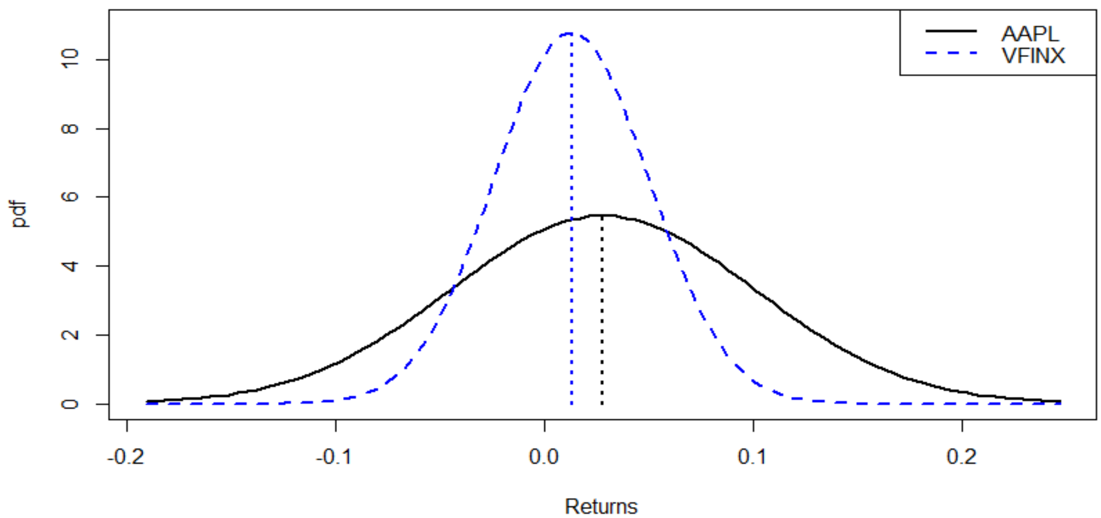
```
> w0 = 1000
> x.vfinx = 400/w0
> x.AAPL = 1 - x.vfinx
> x.vfinx
[1] 0.4
> x.AAPL
[1] 0.6

> r.p = x.vfinx*r.vfinx + x.AAPL*r.AAPL
> r.p
[1] 2.53
> log(1 + r.p)
[1] 1.26
```

II. Probability Theory (20 points, 4 points each)

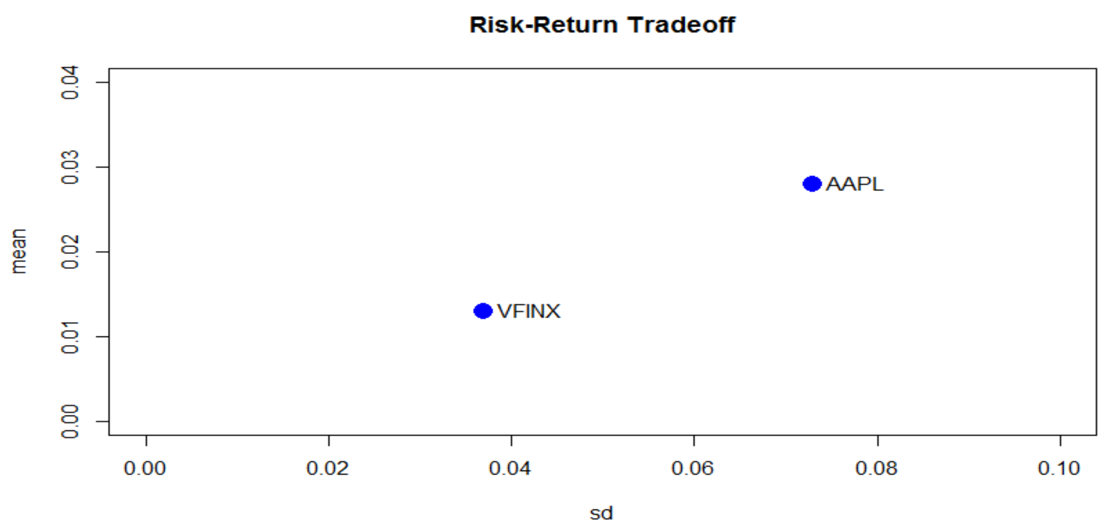
Let R_{vfinx} and R_{AAPL} denote the monthly *simple* returns on VFINX and AAPL and suppose that $R_{vfinx} \sim iid\ N(0.013, (0.037)^2)$, $R_{aapl} \sim iid\ N(0.028, (0.073)^2)$.

a. Sketch the normal distributions for the two assets on the same graph. Show the mean values and the ranges mean ± 2 sd. Which asset appears to be the most risky?



AAPL has a higher SD which means its normal pdf is more spread out than the normal pdf of VFINX. There is more uncertainity, hence more risk, in the return for AAPL.

b. Plot the risk-return tradeoff for the two assets. That is, plot the mean values of each asset on the y-axis and plot the sd values on the x-axis. What relationship do you see?



AAPL has both a higher expected return and standard deviation (risk) than VFINX. This is the stylized risk-return tradeoff.

c. Let W_0 = \$1,000 be the initial wealth invested in each asset. Compute the 1% monthly Value-at-Risk values for each asset. (Hint: $q_{0.01}^{Z} = -2.326$).

```
> w0 = 1000
> q.vfinx.01 = mu.vfinx + sigma.vfinx*(-2.326)
> q.AAPL.01 = mu.AAPL + sigma.AAPL*(-2.326)
> VaR.01.vfinx = w0*q.vfinx.01
> VaR.01.AAPL = w0*q.AAPL.01
> VaR.01.vfinx
[1] -73.1
> VaR.01.AAPL
[1] -142
```

d. Continuing with c., state in words what the 1% Value-at-Risk numbers represent (i.e., explain what 1% Value-at-Risk for a one month \$1,000 investment means)

With 1% probability (or one month in every 100 months), a one month \$1000 investment in VFINX will lose \$73.1 or more.

With 1% probability (or one month in every 100 months), a one month \$1000 investment in AAPL will lose \$142 or more.

e. The normal distribution can be used to characterize the probability distribution of monthly simple returns or monthly continuously compounded returns. What are two problems with using the normal distribution for simple returns? Given these two problems, why might it be better to use the normal distribution for continuously compounded returns?

Problem 1: Simple returns are bounded from below by -1. The normal distribution is defined over -∞ to +∞ and so it is possible returns to be less than -1 with positive probability if they are normally distributed

Problem 2: Multi-period simple returns are multiplicative, not additive. So if simple returns are normally distributed then multi-period returns are not normally distributed (because the product of two normal random variables is not normally distributed).

The normal distribution is more appropriate for continuously compounded returns because continuously compounded returns are defined over -∞ to +∞ and multi-period continuously compounded returns are additive.

III. Time Series Concepts (16 points, 4 points each)

a. Let $\{Y_t\}$ represent a stochastic process. Under what conditions is $\{Y_t\}$ covariance stationary?

$$E[Y_t] = \mu$$
$$\mathrm{var}(Y_t) = \sigma^2$$
$$\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j \text{ (depends on } j \text{ and not } t\text{)}$$

b. Realizations from four stochastic processes are given in Figure 1 below.

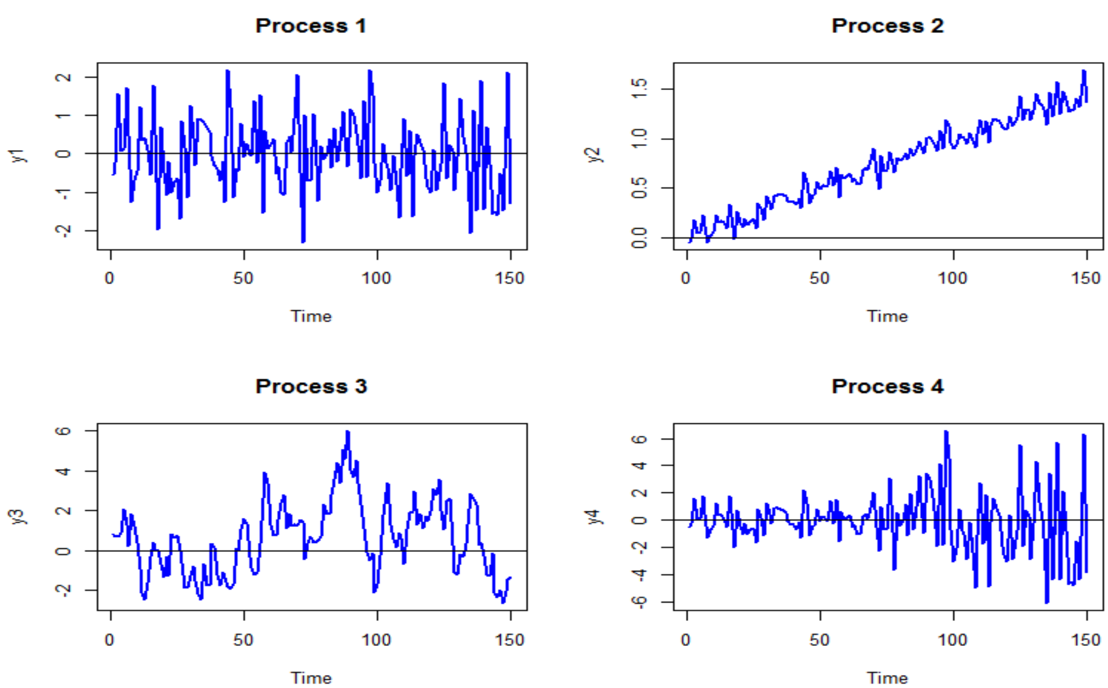


Figure 1: Realizations from four stochastic processes.

Which processes appear to be covariance stationary and which processes appear to be non-stationary? Briefly justify your answers.

- *Processes 1 and 3 appear to be covariance stationary. The means and volatilities appear constant over time and the series exhibit mean reversion (when the series gets above or below the mean it reverts back to the mean)*
- *Processes 2 and 4 appear to be non-stationary. Process 2 has a clear deterministic trend, so the mean is not constant over time. Process 4 appears to have two distinct volatilities (low in the first half and high in the second half).*

c. The CER model for cc returns

$$r_t = \mu + \varepsilon_t,\ \varepsilon_t \sim iid\ N(0,\sigma^2)\,,$$

implies that the log price follows a random walk with drift

$$\ln P_t = \ln P_{t-1} + r_t = \ln P_0 + \mu t + \sum_{s=1}^{t} \varepsilon_s\ .$$

Show that $E[\ln P_t]$ and $\mathrm{var}(\ln P_t)$ depend on *t* so that $\ln P_t$ is non-stationary.

$E\left[\ln P_t\right] = E[\ln P_0] + E[\mu t] + \sum_{s=1}^{t} E\left[\varepsilon_s\right] = \ln P_0 + \mu t$ which depends on *t*.

$\mathrm{var}(\ln P_t) = \mathrm{var}\left(\ln P_0 + \mu t + \sum_{s=1}^{t} \varepsilon_s\right) = \mathrm{var}\left(\sum_{s=1}^{t} \varepsilon_s\right) = \sum_{s=1}^{t} \mathrm{var}(\varepsilon_s) = t\sigma^2$ which depends on *t*.

d. Suppose the time series $\{X_t\}$ is independent white noise. That is,

$$X_t \sim iid\ (0,\sigma^2)$$

Define two new time series $\{Y_t\}$ and $\{Z_t\}$ where $Y_t = X_t^2$ and $Z_t = |Y_t|$. Are $\{Y_t\}$ and $\{Z_t\}$ also independent white noise processes? Why or why not?

$\{Y_t\}$ and $\{Z_t\}$ are independent processes because any function of independent random variables are also independent. The process are covariance stationary because any function of covariance stationary processes are covariance stationary (provided the mean and variance and autocovariances are finite). Hence, the processes will have constant variance. The processes will not have mean zero. So they will behave like a white noise process with a non-zero mean.

IV. Matrix Algebra (16 points, 4 points each)

Let R_i denote the simple return on asset i ($i = 1, \ldots, N$) with $E[R_i] = \mu_i$, var(R_i) = σ_i^2 and cov(R_i, R_j) = σ_{ij}. Define the ($N \times 1$) vectors $\boldsymbol{R} = (R_1, \ldots, R_N)'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{x} = (x_1, \ldots, x_N)'$, $\boldsymbol{y} = (y_1, \ldots, y_N)'$, and $\mathbf{1} = (1, \ldots, 1)'$ and the ($N \times N$) covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ contain portfolio weights (investment shares) that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolios defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ give the matrix algebra expression for the portfolio returns, ($R_{p,x}$ and $R_{p,y}$) and the portfolio expected returns ($\mu_{p,x}$ and $\mu_{p,y}$).

$$R_{p,x} = \mathbf{x}'\mathbf{R},\ R_{p,y} = \mathbf{y}'\mathbf{R}$$
$$E[R_{p,x}] = \mu_{p,x} = \mathbf{x}'\boldsymbol{\mu},\ E[R_{p,y}] = \mu_{p,y} = \mathbf{y}'\boldsymbol{\mu}$$

b. For the portfolios defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ give the matrix algebra expression for the constraint that the portfolio weights sum to one.

$\mathbf{x}'\mathbf{1} = 1$ and $\mathbf{y}'\mathbf{1} = 1$

c. For the portfolios defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ give the matrix algebra expression for the portfolio variances ($\sigma_{p,x}^2$ and $\sigma_{p,y}^2$), and the covariance between $R_{p,x}$ and $R_{p,y}$ (σ_{xy}).

$$\sigma_{p,x}^2 = \mathrm{var}(R_{p,x}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x},\ \sigma_{p,y}^2 = \mathrm{var}(R_{p,y}) = \mathbf{y}'\boldsymbol{\Sigma}\mathbf{y}$$
$$\sigma_{xy} = \mathrm{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}$$

d. In the expression for the portfolio variance $\sigma_{p,x}^2$, how many variance terms are there? How many covariance terms are there?

There are N variance terms and N(N-1) total covariance terms (N(N-1)/2 unique covariance terms).

V. Descriptive Statistics (32 points, 4 points each)

Figure 3 shows monthly simple returns on the Vanguard S&P 500 index (VFINX) and Apple stock (AAPL) over the 5-year period January 2010, through January 2015. For this period there are T=60 monthly returns.

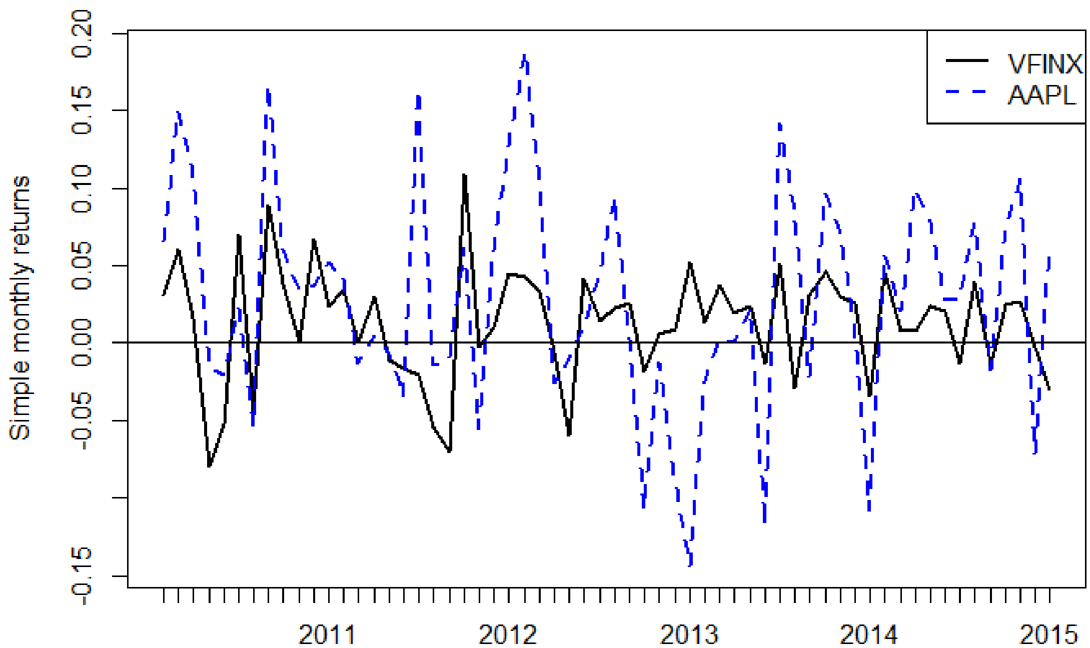


Figure 2: Monthly simple returns on two assets.

a. Do the monthly returns from the two assets look like realizations from a covariance stationary stochastic process? Why or why not?

Recall, covariance stationarity implies that the mean, variance and autocovariances are constant over time. Visually it looks like both mean values are constant over time and both series exhibit mean reversion (both series fluctuate up and down about the mean value). The volatilities of VFINX and AAPL look fairly constant over time. Overall, both series look like they could be realizations from a covariance stationary process.

b. Compare and contrast the return characteristics of the two assets. In addition, comment on any common features, if any, of the two return series.

AAPL looks to have a slightly larger mean than VFINX and clearly a larger volatility. AAPL also has many more large returns (positive and negative) than VFINX. The two returns do not look like they move very closely together (e.g. when VFINX goes up (down) it is not always the case that AAPL goes up (down)). Hence, it looks like the correlation between VFINX and AAPL is a small positive number.

c. Figure 4 below gives the cumulative simple returns (equity curve) of each fund which represents the growth of $1 invested in each fund over the sample period. Which fund performed better over the sample?

A $1 investment in AAPL clearly did better than a $1 investment in VFINX. For AAPL $1 grew to about $5, and for VFINX $1 grew to only about $2.

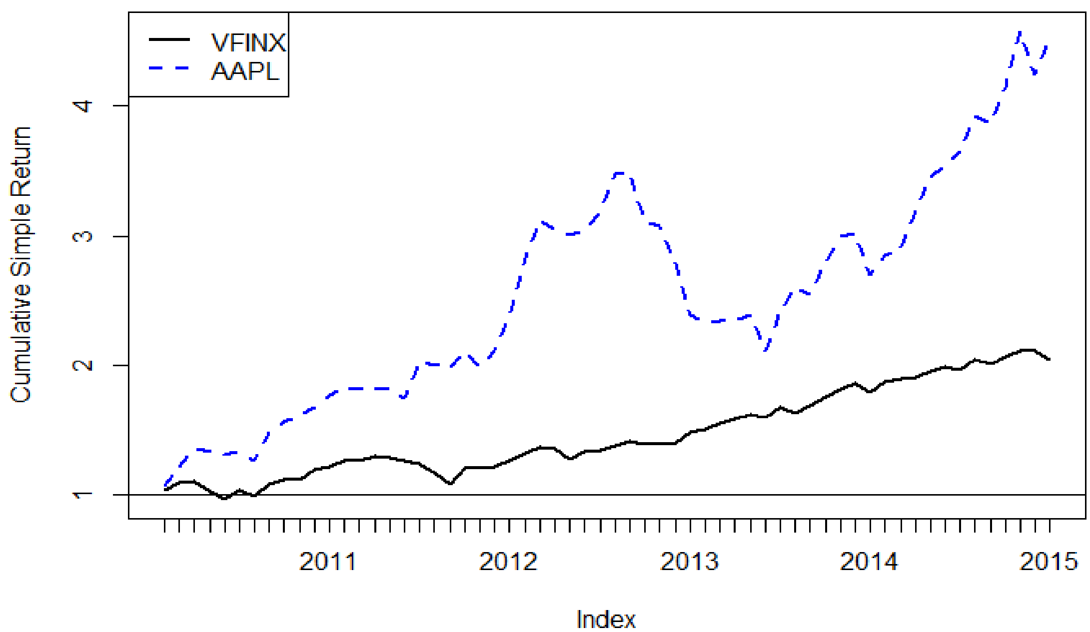
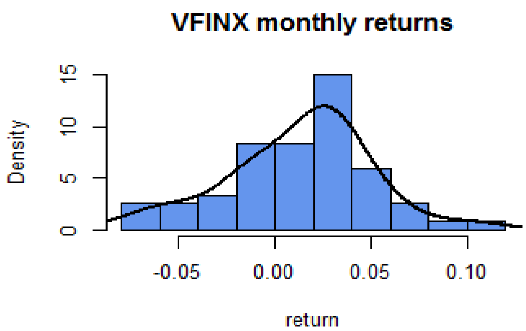
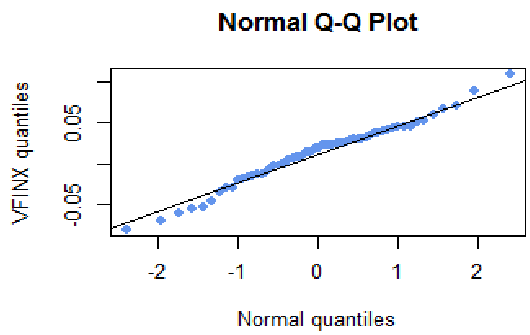
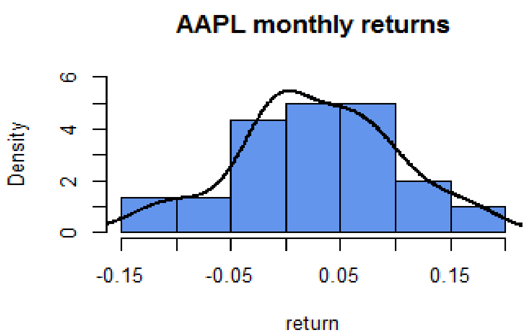


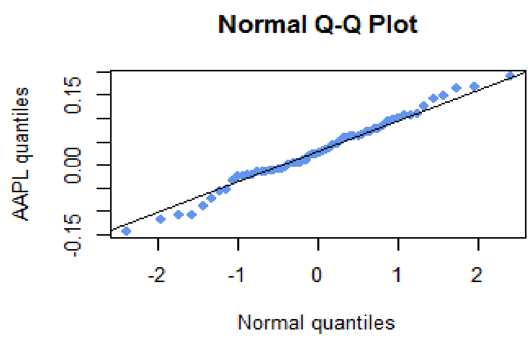
Figure 3 Cumulative simple returns on two assets.

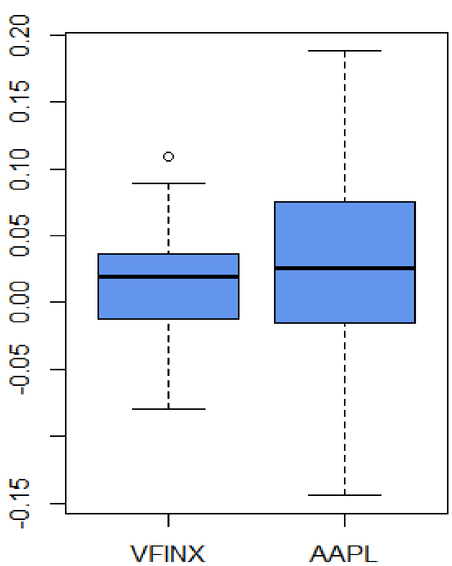
The figures below give some graphical diagnostics of the return distributions for VFINX and AAPL.

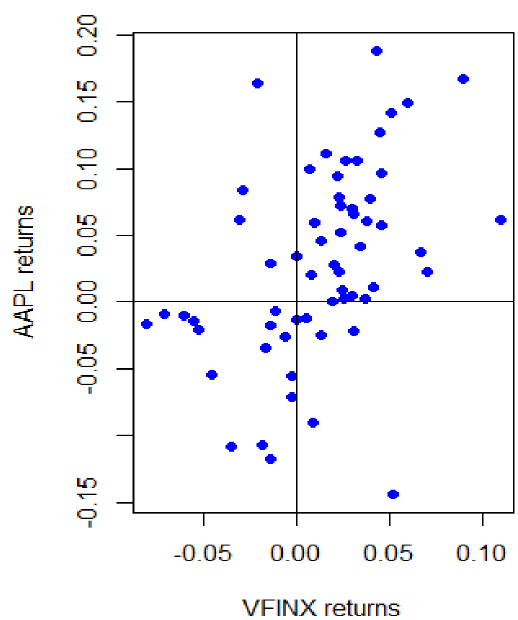


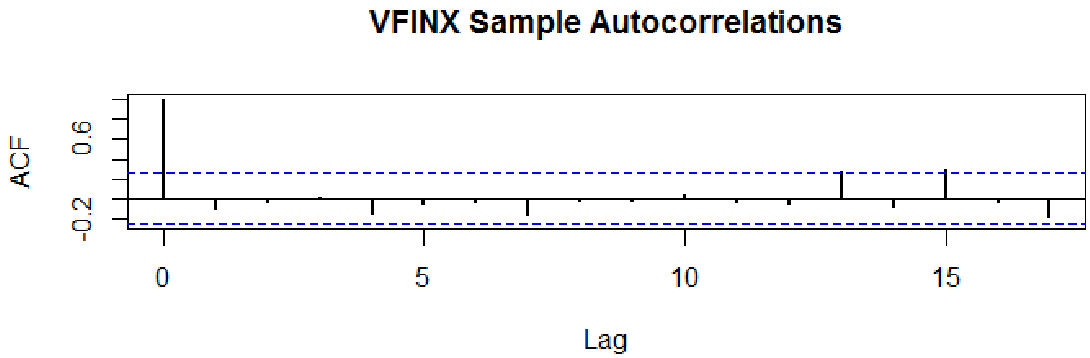


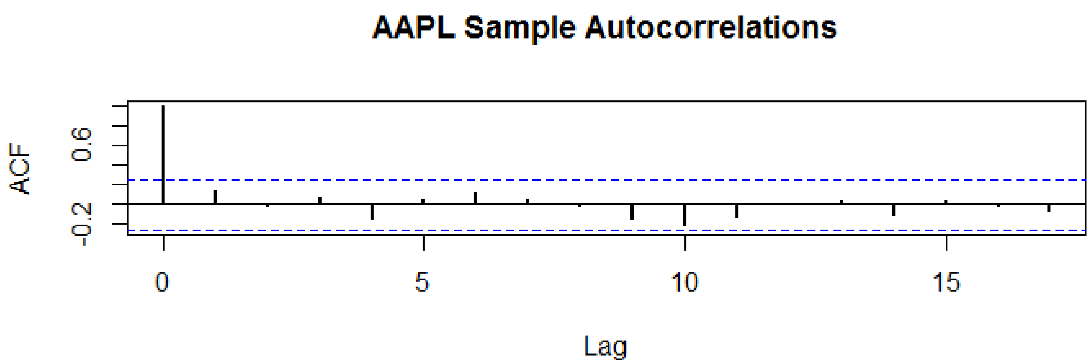












The following table summarizes some sample descriptive statistics of the T=60 monthly returns.

Table 1 Descriptive Statistics and CER model estimates

Statistic	VFINX	AAPL
Mean ($\hat{\mu}$)	0.0127	0.0280
Standard Deviation ($\hat{\sigma}$)	0.0374	0.0725
Skewness	-0.2634	-0.0738
Excess Kurtosis	0.286	-0.165
1% empirical quantile ($\hat{q}_{0.01}$)	-0.0744	-0.1289
Lag 1 autocorrelation ($\hat{\rho}_1$)	-0.110	0.136
Lag 2 autocorrelation ($\hat{\rho}_2$)	-0.031	-0.015
Lag 3 autocorrelation ($\hat{\rho}_3$)	0.011	0.070
Correlation between VFINX and AAPL ($\hat{\rho}_{VFINX,AMZN}$)	0.435	

d. Do the returns on VFINX and AAPL look normally distributed? Briefly justify your answer.

Both the returns on VFINX and AAPL look pretty normally distributed. The histograms are roughly bell shaped (VFINX has a slight negative skewness but not too large). Also, the normal qq-plots for both assets are mostly linear and the boxplots only show one moderate outlier for VFINX. Finally, the excess kurtosis values for both series are close to zero (about 0.286 for VFINX and -0.165 for AAPL) which is consistent with the normal distribution.

e. Which asset appears to be riskier? Briefly justify your answer.

AAPL has a larger SD than VFINX (0.0725 vs. 0.0374) so there is more uncertainty in the return for AAPL. Also, the 1% quantile for for AAPL (-0.1289) is more negative than the 1% quantile for VFINX (-0.0744) This means there is more tail risk with AAPL.

f. Does there appear to be any contemporaneous linear dependence between the returns on VFINX and AAPL? Briefly justify your answer.

The scatterplot of the AMZN and VFINX returns looks pretty much like a shotgun blast with a moderate rightward tilt indicating weak to moderate positive linear dependence. This is confirmed by the sample correlation value of 0.435.

g. Do the monthly returns show any evidence of (linear) time dependence? Briefly justify your answer.

For both series the SACF plots show no significant autocorrelations for lags less than 15 (no bars are above the dotted lines). All of the sample autocorrelations are small (close to zero) and show no distinct pattern over time. Hence, there does not appear to be any linear time dependence in the returns.

h. Let W_0 = $1,000 be the initial wealth invested in each asset. Compute the 1% monthly historical Value-at-Risk for each asset. (Recall, the data are simple returns here)

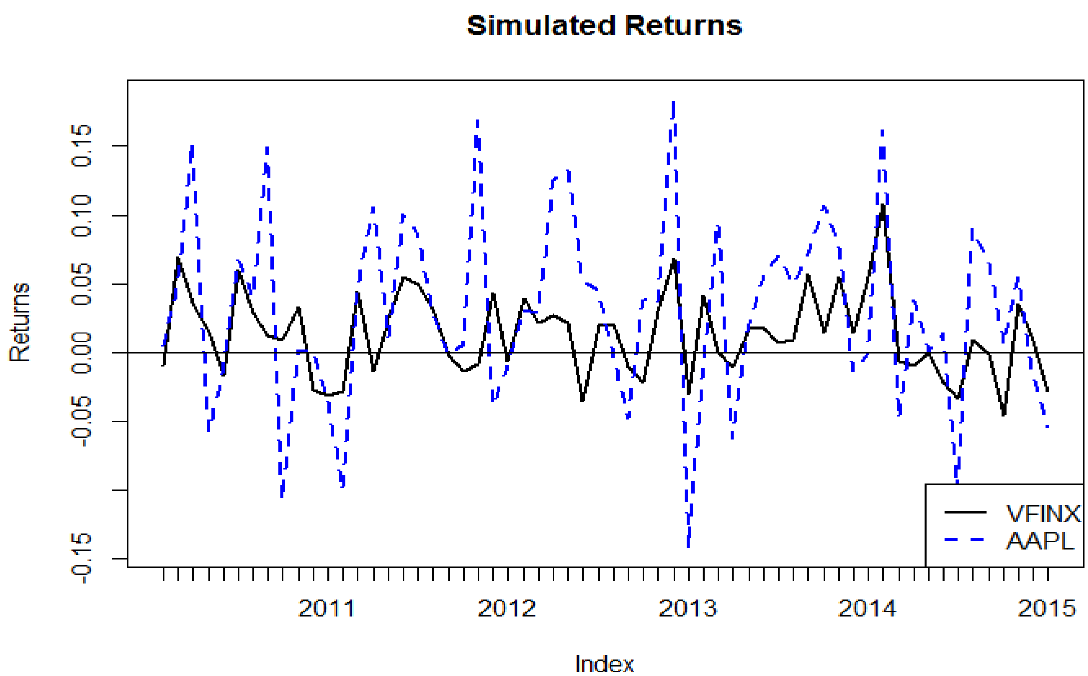
```
> W0 = 1000
> q.01 = apply(ret, 2, quantile, probs=0.01)
> VaR.vfinx.01 = W0*q.01["VFINX"]
> VaR.AAPL.01 = W0*q.01["AAPL"]
> VaR.vfinx.01
VFINX
-74.4
> VaR.AAPL.01
AAPL
-129
```

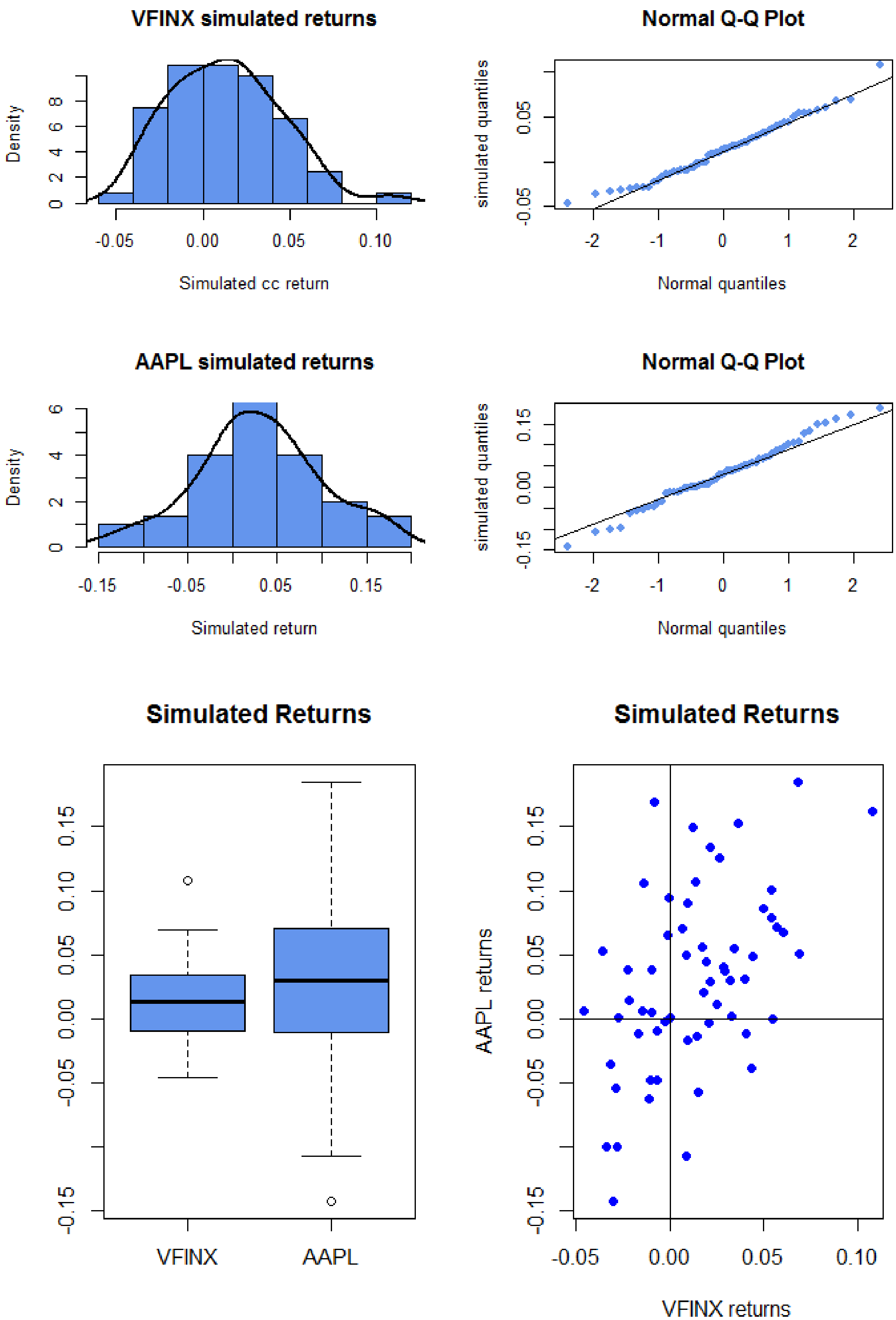
VI. Constant Expected Return Model (16 points)

Consider the constant expected return (CER) model

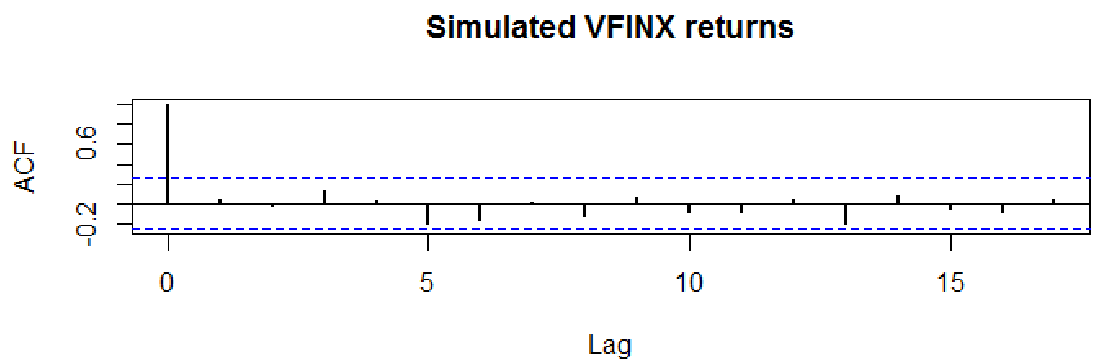
$$r_{it} = \mu_i + \varepsilon_{it},\ \varepsilon_{it} \sim iid\ N(0,\sigma_i^2)$$
$$\text{cov}(r_{it}, r_{jt}) = \sigma_{ij},\ cor(r_{it}, r_{jt}) = \rho_{ij}$$

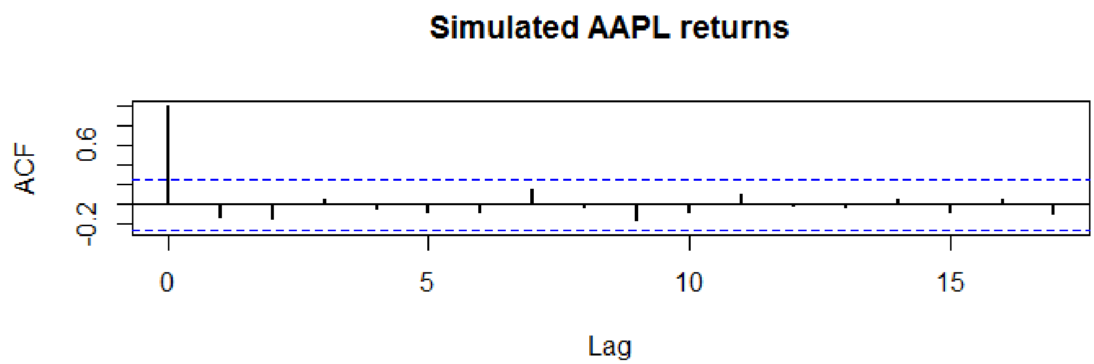
for the monthly simple returns on the Vanguard S&P500 index (VFINX) and Apple stock (AAPL) presented in part V above. Below are simulated returns and some graphical descriptive statistics for VFINX and AAPL from the CER model calibrated using the sample estimates of the CER model parameters for the two assets (these are the sample statistics from the table of the previous question).





VFINX simulated returns
Density
Simulated cc return
Normal Q-Q Plot
simulated quantiles
Normal quantiles
AAPL simulated returns
Density
Simulated return
Normal Q-Q Plot
simulated quantiles
Normal quantiles
Simulated Returns
VFINX
AAPL
Simulated Returns
AAPL returns
VFINX returns





a. Briefly describe how to create the simulated returns on VFINX and AAPL from the CER model. That is, tell me the algorithm for creating the simulated data. You don't have to show me R code to answer this question. Just tell me the steps required (e.g. pseudo-code) to simulate the data.

Here we want to do a multivariate simulation from the CER model in matrix form

$$\mathbf{R}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t,\ \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$$

To do this we need values for the 2 x 1 mean vector $\boldsymbol{\mu}$ *and 2 x 2 covariance matrix* $\boldsymbol{\Sigma}$. *We can get these values from the sample statistics in Table 1*

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} \mu_{vfinx} \\ \mu_{aapl} \end{pmatrix} = \begin{pmatrix} 0.0127 \\ 0.0280 \end{pmatrix} \text{ and } \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \sigma^2_{vfinx} & \sigma_{vfinx,aapl} \\ \sigma_{vfinx,aapl} & \sigma^2_{aapl} \end{pmatrix} = \begin{pmatrix} 0.0014 & 0.00118 \\ 0.00118 & 0.00526 \end{pmatrix}$$

where $0.0014 = \sigma^2_{vfinx} = (0.0374)^2$, $0.00526 = \sigma^2_{aapl} = (0.0725)^2$ *and* $0.00118 = \sigma_{vfinx,aapl} = \rho_{vfinx,aapl}\sigma_{vfinx}\sigma_{aapl}$. *Then we can simulate T observation on* $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ *(using the R function* `mvtnorm`*, for example) giving* $\tilde{\boldsymbol{\varepsilon}}_t$ *and create the simulated returns using* $\tilde{\mathbf{r}}_t = \hat{\boldsymbol{\mu}} + \tilde{\boldsymbol{\varepsilon}}_t$.

b. Which features of the actual returns shown in part V are captured by the simulated CER model returns and which features are not?

Almost all features of the VFINX and AAPL returns are captured by the simulated CER model returns. There are only a few notable differences

- *Time plot of simulated returns looks very close to actual returns (AAPL has higher mean and volatility than VFINX; returns do not move very closely together)*
- *Histograms are roughly bell shaped and qq-plots are mostly linear. The negative skewness of actual VFINX returns are not shown in the simulated returns. This histogram of the simulated VFINX returns shows positive skewness.*
- *Boxplots of simulated returns match closely the boxplots of the actual returns.*
- *Scatterplot of simulated returns looks like scatterplot of actual returns*
- *SACF of simulated returns look like SACF of actual returns.*

c. Does the CER model appear to be a good model for VFINX and AAPL returns? Why or why not?

Yes, the CER model appears to be a good model for VFINX and AMZN returns. Most of the important features of the actual data are captured by simulated data from the CER model.