University of Washington   Fall 2009
Department of Economics   Eric Zivot

Economics 424

Midterm Exam

This is a closed book and closed note exam. However, you are allowed one page of notes
(double-sided). Answer all questions and write all answers in a blue book or on separate
sheets of paper. Time limit is 1 hours and 50 minutes. Total points = 84.

I. Return Calculations (16 pts, 4 points each)

Consider a one year investment in two assets: the Vanguard S&P 500 index (VFINX) and
the Vanguard Short Term Bond mutual fund (VBISX). Suppose you buy one share of the
S&P 500 fund and one share of the bond fund at the end of September, 2008 for
$P_{sp500,t-1} = 104.61$, $P_{bond,t-1} = 9.70$, and then sell these shares at the end of September,
2009 for $P_{sp500,t} = 97.45$, $P_{bond,t} = 10.46$. (Note: these are actual closing prices taken from
Yahoo!)

a. What are the simple annual returns for the two investments?

```
> p.sp500.1 = 104.61
> p.sp500.2 = 97.45
> p.bond.1 = 9.70
> p.bond.2 = 10.46

# a) simple returns on sp500 and bond
> r.sp500 = (p.sp500.2 - p.sp500.1)/p.sp500.1
> r.bond = (p.bond.2 - p.bond.1)/p.bond.1
> r.sp500
[1] -0.06844
> r.bond
[1] 0.07835
```

b. What are the continuously compounded annual returns for the two investments?

```
# b) cc returns on Amazon and sp500
> log(1 + r.sp500)
[1] -0.0709
> log(1 + r.bond)
[1] 0.07543
```

c. Assume you get the same annual returns from part a. every year for the next 10 years. How much will $10,000 invested in each fund be worth after 10 years?

```
> w0 = 10000
> w1.sp500 = w0*(1 + r.sp500)^10
> w1.bond = w0*(1 + r.bond)^10
> w1.sp500
[1] 4921
> w1.bond
[1] 21262
```

d. The annual inflation rate between September 2008 and September 2009 was about -1% (yes, we actually had *deflation*!). Using this information, determine the simple and continuously compounded real annual returns on S&P 500 and the bond fund.

```
> inflat = -0.01
# simple real returns
> r.sp500.real = (1+r.sp500)/(1+inflat) - 1
> r.bond.real = (1+r.bond)/(1+inflat) - 1
> r.sp500.real
[1] -0.05904
> r.bond.real
[1] 0.08924
# cc real returns
> log(1+r.sp500.real)
[1] -0.06085
> log(1+r.bond.real)
[1] 0.08548
```

II. Probability Theory and Matrix Algebra (16 points, 4 points each)

Let r_t denote the monthly continuously compounded return on an asset and suppose that $r_t \sim iid\ N(0.01,(0.05)^2)$.

a. What is the relationship between the continuously compounded return r_t and the simple return R_t? Given this relationship, what is the probability distribution of $1 + R_t$?

$$r_t = \ln(1+R_t) \Rightarrow R_t = e^{r_t} - 1 \Rightarrow 1 + R_t = e^{r_t}$$

Because $r_t \sim iid\ N(0.01,(0.05)^2)$, it follows that $1+R_t \sim$ lognormal$(0.01,(0.05)^2)$.

b. Give an expression for the 6-month continuously compounded return, $r_t(6)$, in terms of the monthly continuously compounded returns. Using this expression compute $E[r_t(6)]$, $\text{var}(r_t(6))$, and $SD(r_t(6))$. What is the probability distribution of $r_t(6)$?

$$r_t(6) = r_t + r_{t-1} + \cdots + r_{t-5} = \sum_{j=0}^{5} r_{t-j}$$

$$E[r_t(6)] = \sum_{j=0}^{5} E[r_{t-j}] = \sum_{j=0}^{5} 0.01 = 6 \times (0.01) = 0.06$$

$$\text{var}(r_t(6)) = \text{var}\left(\sum_{j=0}^{5} r_{t-j}\right) = \sum_{j=0}^{5} \text{var}(r_{t-j}) = 6 \times (0.05)^2 = 0.015$$

$$SD(r_t(6)) = \sqrt{6}(0.05) = \sqrt{0.015} = 0.1224$$

Therefore, $r_t(6) \sim N(0.06,(0.1224)^2)$

Let R_i denote the continuously compounded return on asset i ($i = 1,2,3$) with $E[R_i] = \mu_i$, $\text{var}(R_i) = \sigma_i^2$ and $\text{cov}(R_i, R_j) = \sigma_{ij}$. Define the 3×1 vectors

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}, \mathbf{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and the 3×3 covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ represent portfolio weights (i.e., shares of wealth invested in the three assets).

c. Using matrix algebra, give expressions for the returns, expected returns and variances for the two portfolios.

$$R_{p,x} = \mathbf{x}'\mathbf{R}, R_{p,y} = \mathbf{y}'\mathbf{R},$$

$$\mu_{p,x} = \mathbf{x}'\mathbf{\mu}, \mu_{p,x} = \mathbf{y}'\mathbf{\mu},$$

$$\sigma_{p,x}^2 = \mathbf{x}'\Sigma\mathbf{x}, \sigma_{p,x}^2 = \mathbf{y}'\Sigma\mathbf{y}$$

d. Using matrix algebra, give expressions for the restriction that the weights in each portfolio must sum to one, and give an expression for the covariance between the returns on portfolio $\mathbf{x}$ and the returns on portfolio $\mathbf{y}$.

$$\sum_{i=1}^{3} x_i = \mathbf{x}'\mathbf{1} = 1, \ \sum_{i=1}^{3} y_i = \mathbf{y}'\mathbf{1} = 1$$

$$\operatorname{cov}(R_{p,x}, R_{p,y}) = \operatorname{cov}(\mathbf{x}'\mathbf{R}, \mathbf{y}'\mathbf{R}) = \mathbf{x}'\Sigma\mathbf{y}$$

III. Time Series Concepts (16 points, 4 points each)

a. Let $\{Y_t\}$ represent a stochastic process. Under what conditions is $\{Y_t\}$ covariance stationary?

$$E[Y_t] = \mu$$
$$\operatorname{var}(Y_t) = \sigma^2$$
$$\operatorname{cov}(Y_t, Y_{t-j}) = \gamma_j \ \text{(depends on } j \text{ and not } t\text{)}$$

b. Realizations from four stochastic processes are given in Figure 1 below. Which processes appear to be covariance stationary and which processes appear to be non-stationary? Briefly justify your answers.
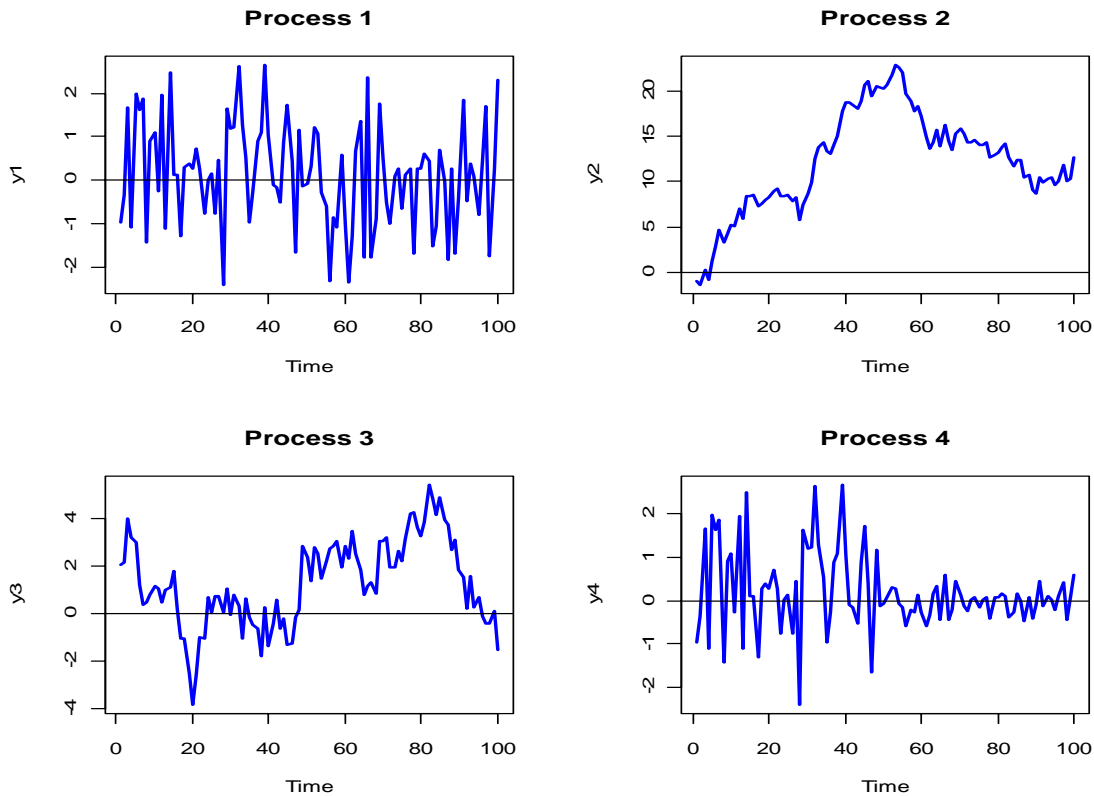
Figure 1: Realizations from four stochastic processes.

The processes that appear to be covariance stationary are Process 1 and Process 3. Both processes are not trending, so the mean is not changing over time, and both processes appear to have constant volatility over time. In fact, Process 1 is Gaussian White Noise, and Process 2 is an AR(1) process with $\phi = 0.9$.

Processes 2 and 4 appear to be non-stationary. Process 2 appears to wander away from its initial starting point and exhibits an upward drift. This is non-stationary behavior. In fact, Process 2 is a random walk. Process 4 shows higher volatility in the first half of the sample than in the second half. Hence, the volatility of this process is not constant over time which violates the stationarity conditions. In fact, Process 3 is a Gaussian White Noise process with $\sigma = 1$ in the first half of the sample and $\sigma = 0.25$ in the second half of the sample.

Figure 2 below shows a realization of a stochastic process representing a monthly time series of overlapping 2-month continuously compounded returns, where the 1-month continuously compounded returns follow a Gaussian White noise process.

c. Based on the sample autocorrelations, which time series process is most appropriate for describing the series: MA(1) or AR(1)?

Based on the sample autocovariance function (SACF), an MA(1) model looks more appropriate than an AR(1) model. In an MA(1) model, $\rho_1 = \dfrac{\theta}{1+\theta^2}$ and $\rho_j = 0$ for $j > 1$ and in an AR(1) model $\rho_j = \phi^j$. The SACF shows $\hat{\rho}_1 \approx 0.42$ and $\hat{\rho}_j \approx 0$ for $j > 1$ which is consistent with an MA(1) model.

d. If you think the process is an AR(1) process, what do you think is the value of the autoregressive parameter ϕ? If you think the process is a MA(1) process, what do you think is the value of the moving average parameter θ?

If the processes was an AR(1), then $\phi \approx \hat{\rho}_1 \approx 0.42$. If the process was an MA(1) then

$$\frac{\theta}{1+\theta^2} = \rho_1 \approx 0.42 \Rightarrow \theta = 0.42 \times (1+\theta^2) = 0.42 + 0.42 \times \theta^2 \Rightarrow 0.42 \times \theta^2 - \theta + 0.42 = 0$$

This is a quadratic equation in θ. Using the quadratic formula, there are two solutions

$$\theta = \frac{1 \pm \sqrt{1 - 4(0.42)^2}}{2(0.42)} = 1.836 \text{ and } 0.554.$$

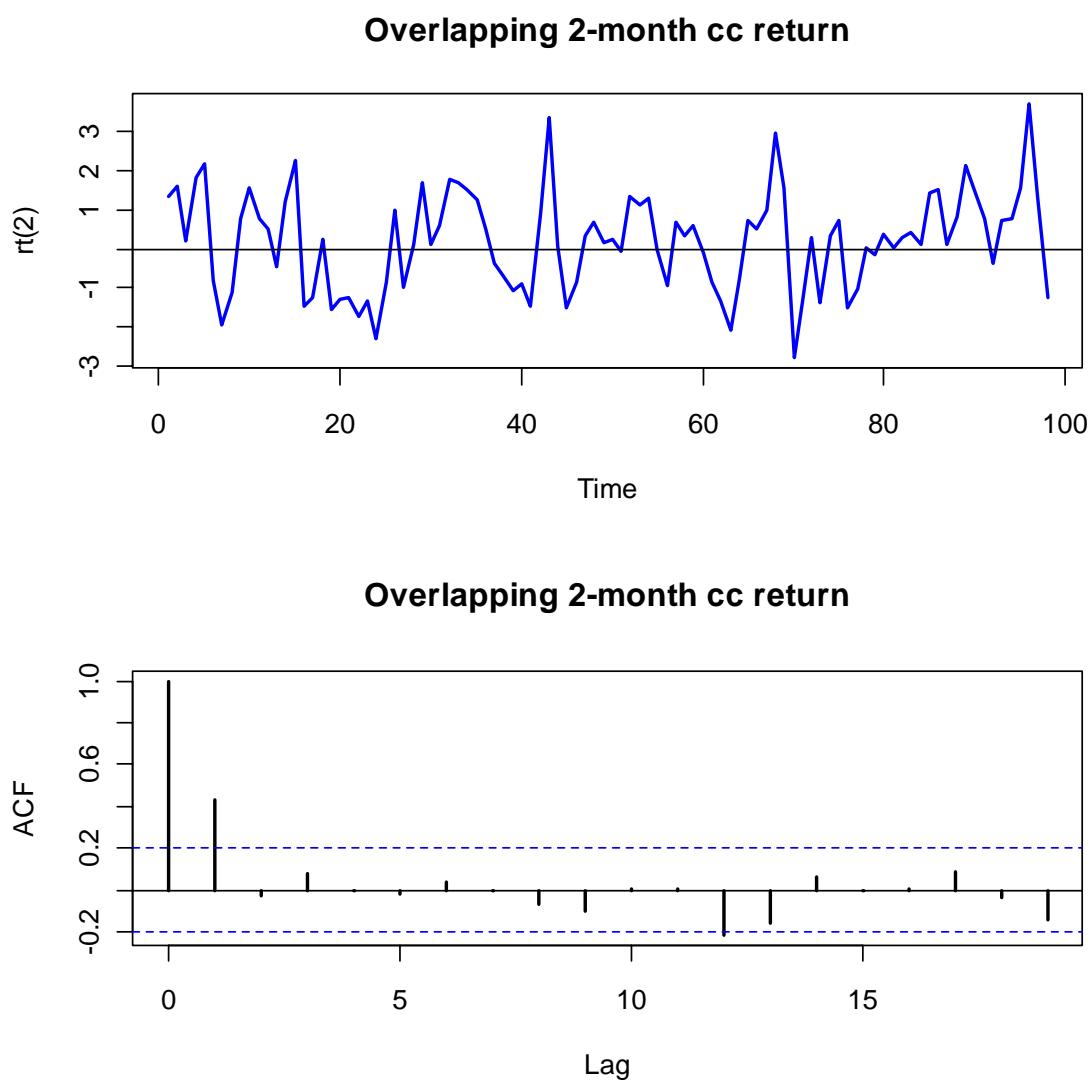## Overlapping 2-month cc return

Overlapping 2-month cc return

Figure 2: Stochastic process representing overlapping 2-month continuously compounded returns.

IV. Descriptive Statistics (16 points, 4 points each)

Figure 3 shows monthly continuously compounded returns on the Vanguard S&P 500 index (VFINX) and the Vanguard Short Term Bond mutual fund (VBISX) over the 5-year period October 2004, through October 2009. For this period there are $T=60$ monthly observations.
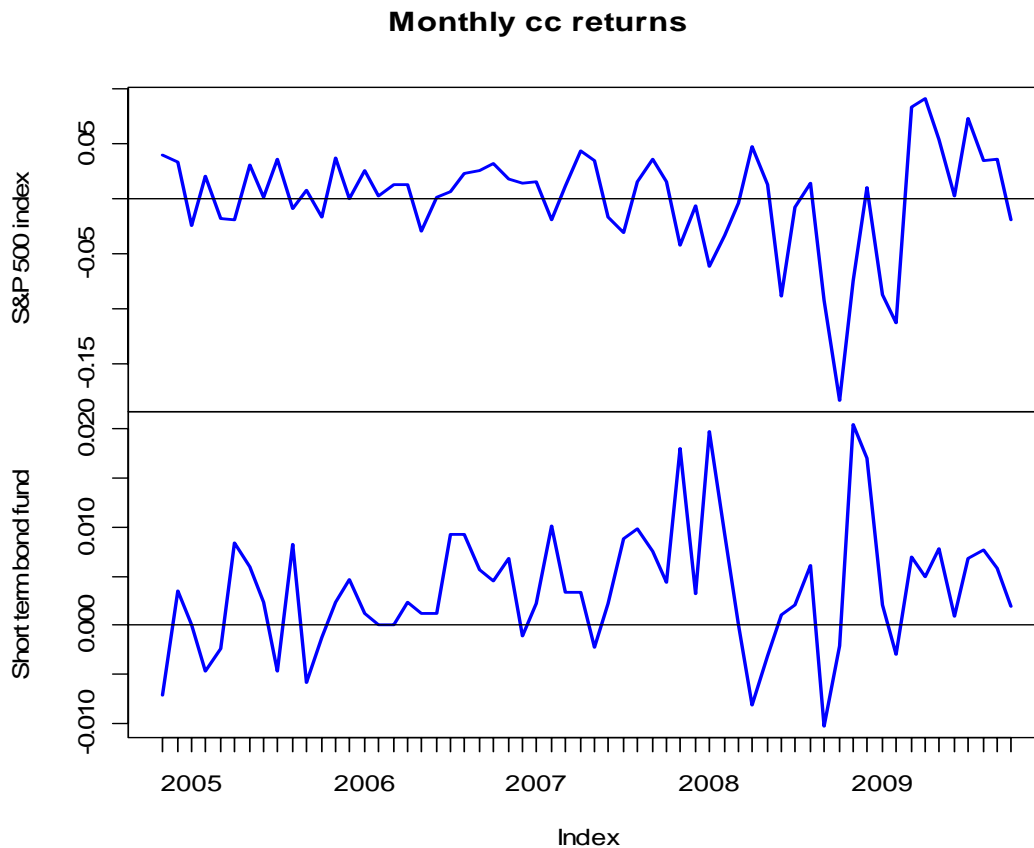
Figure 3: Monthly continuously compounded returns on two Vanguard Funds.

a. Do the monthly continuously compounded returns from the two funds look like realizations from a covariance stationary stochastic process? Why or why not?

Recall, covariance stationary processes have time invariant means, variances and autocorrelations that only depend on the time between observations. Consider first the S&P 500 index. The returns are not obviously trending so it looks like the mean is not changing over time. However, the volatility in the latter part of the sample appears to be much bigger than in the first part of the sample. This is evidence of non-stationarity. Next consider the Bond fund. The returns are not obviously trending up or down, but it looks like the mean is slightly higher in the latter part of the sample than in the first part which is evidence of non-stationarity. Also, the volatility of the bond returns appears to be slightly higher in the second half of the sample but the difference is not as large as it was for the S&P 500 returns. Overall, there is some evidence that the bond returns may be non-stationary.

The figures below gives some graphical diagnostics of the return distributions for the S&P 500 index and the bond fund.
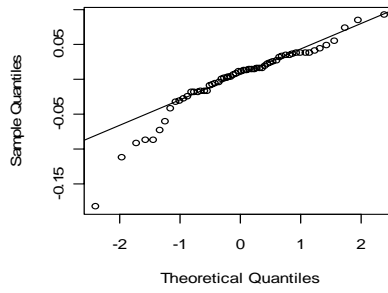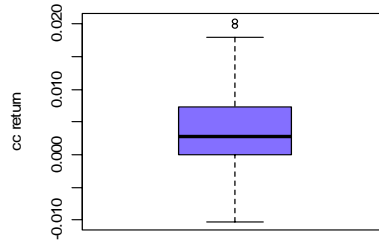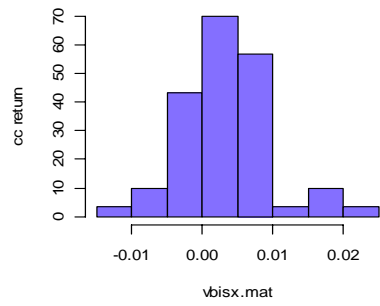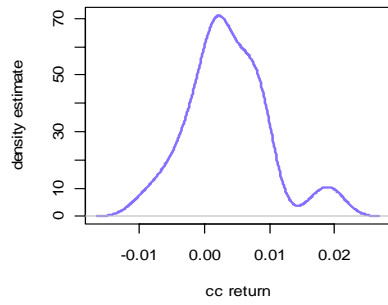
S&P 500 monthly cc returns
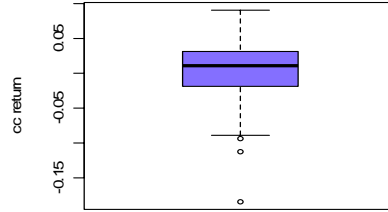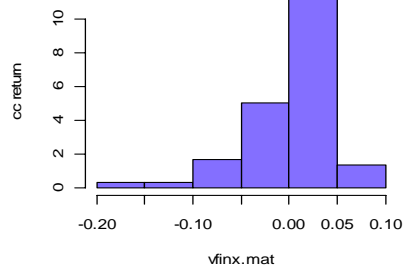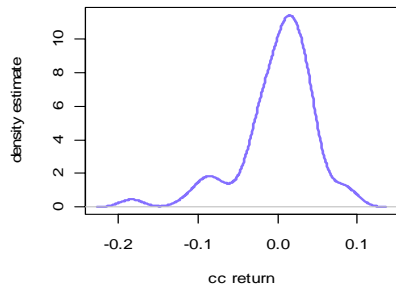
Smoothed density
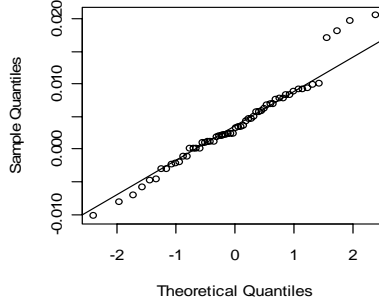

Normal Q-Q Plot

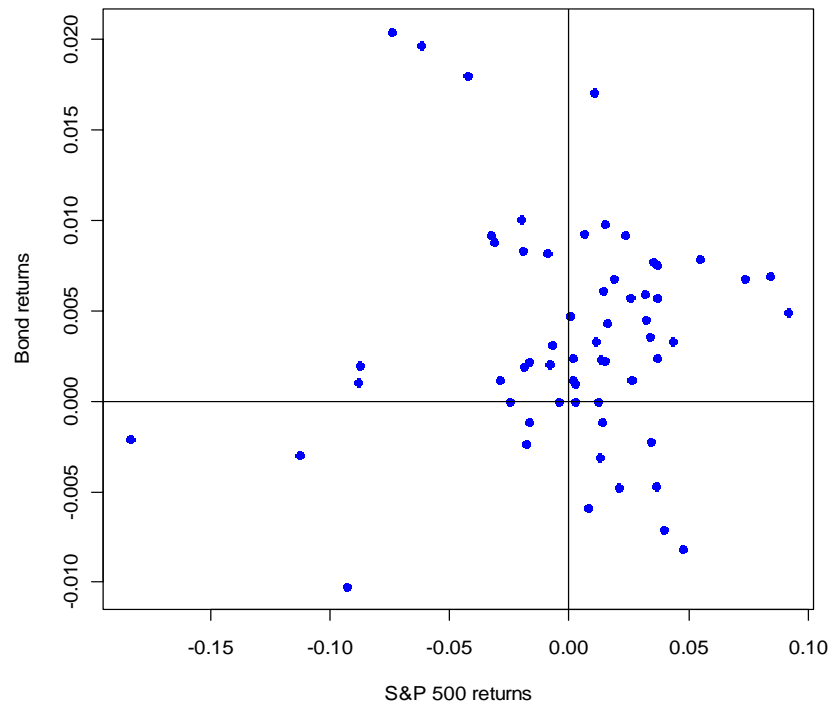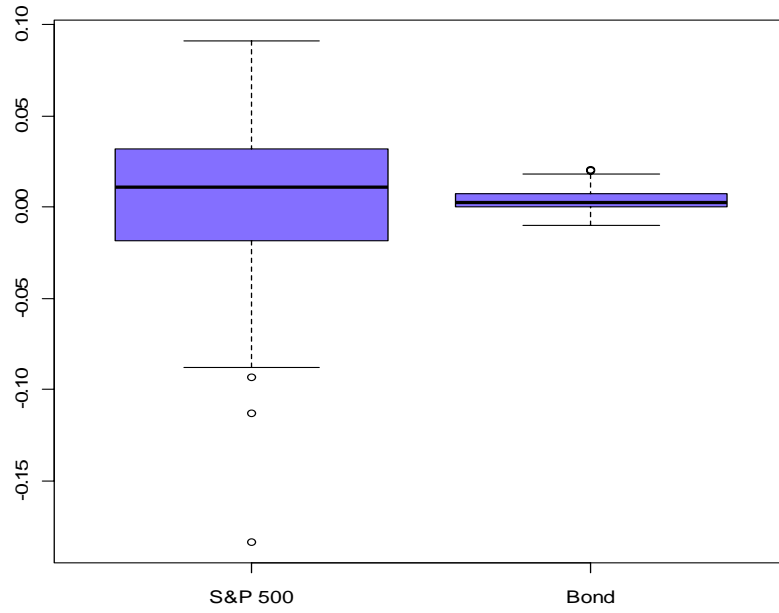## Bond fund monthly cc returns

Smoothed density

Normal Q-Q Plot

b. Do the returns on the S&P500 index and the bond fund look normally distributed? Briefly justify your answer.

Consider first the S&P 500 index. The returns exhibit non-normal behavior. The histogram shows a long left tail (negative skewness) and the boxplot indicates a some large negative outliers (fat tails). The normal QQ-plot is not linear in the left tail. All of this is strong evidence against the normal distribution.

Next consider the bond fund. The histogram and smoothed histogram are somewhat symmetric, but show an interesting bi-modality (e.g. two modes). The second mode is distinctly positive around 2%. Recall, the time plot of the bond returns looks to have a higher mean in the second part of the sample and this appears to be showing up in the histogram as two modes. The boxplot shows some positive skewness and a few large positive values. The QQ-plot is linear in the left tail but not linear in the right tail. All of this is evidence that the bond returns are also not normally distributed.

c. Which asset appears to be riskier? Briefly justify your answer.

If we measure risk by return volatility, then the S&P 500 index is clearly more risky than the short term bond fund. The figure showing the side-by-side boxplots clearly shows that the bond fund returns are substantially less volatile than the S&P 500 index returns.

d. Based on the scatterplot of returns, does there appear to be any linear dependence between the returns on the S&P500 index and the bond fund? Briefly justify your answer.

The scatterplot of returns does not show any obvious linear pattern. The points are fairly equally distributed in the four quandrants which would indicate that the covariance and correlation between the returns is close to zero (we will see this in the sample statistics in the next problem). Hence, the S&P 500 and bond returns are not linearly related.

V. Constant Expected Return Model (20 points, 4 points each)

Consider the constant expected return model

$$r_{it} = \mu_i + \varepsilon_{it}, \; \varepsilon_{it} \sim iid \; N(0, \sigma_i^2)$$
$$\mathrm{cov}(r_{it}, r_{jt}) = \sigma_{ij}, \; cor(r_{it}, r_{jt}) = \rho_{ij}$$

for the monthly continuously compounded returns on the Vanguard S&P500 index and the Vanguard short term bond fund presented in part IV. The following R output gives the estimates of $\mu_i, \sigma_i, \sigma_{ij}$ and ρ_{ij} for the S&P 500 index (VFINX) and the bond fund (VBISX) from the $T=60$ months of data:

```
> muhat.vals
    sp500       bond
0.0002158 0.0035687

> sigmahat.vals
   sp500      bond
0.047410 0.006237

> covhat.vals
sp500,bond
 3.236e-06

> rhohat.vals
sp500,bond
   0.01094
```

a.   Using the above output, compute for both assets estimated standard errors for μ and σ , and compute an estimated standard error for ρ . Briefly comment on the precision of the estimates.

```
> nobs = nrow(ret.z)
> se.muhat = sigmahat.vals/sqrt(nobs)
> se.sigma = sigmahat.vals/sqrt(2*nobs)
> se.rho = (1 - rhohat.vals^2)/sqrt(nobs)
> se.muhat
    sp500       bond
0.0061206 0.0008052
> se.sigma
    sp500       bond
0.0043279 0.0005694
> se.rho
sp500,bond
    0.1291
> nobs
[1] 60
```

The means are estimated less precisely than the standard deviation, and the mean and standard deviation are estimated more precisely for the bond fund than for the S&P 500 index. The correlation is not estimated very precisely as the SE is much larger than the estimate.

b. For the S&P 500 index, compute 95% confidence intervals for μ, σ. Briefly comment on the precision of the estimates. In particular, note if both positive and negative values are in the respective confidence intervals.

```
> upper = muhat.vals + 2*se.muhat
> lower = muhat.vals - 2*se.muhat
> cbind(lower[1],upper[1])
           [,1]     [,2]
sp500 -0.01203 0.01246
```

```
# 95% ci for sigma
> upper = sigmahat.vals + 2*se.sigma
> lower = sigmahat.vals - 2*se.sigma
> cbind(lower[1],upper[1])
          [,1]     [,2]
sp500 0.03875 0.05607
```

The 95% confidence interval for μ contains both negative and positive values and indicates that the mean could be as low as -1.2% or as high as 1.2% per month. This is a fairly wide range and indicates much uncertainty about the true value of the mean.

The 95% confidence interval for σ only contains positive values (which is good because σ must be positive) and is fairly narrow: from 3.8% to 5.6%.

c. Suppose you currently hold $2M (million) in the S&P 500 Index. That is, your initial wealth at the beginning of the month is $W_0 = \$2M$. Using the estimates from the CER model compute the 1% and 5% value-at-risk (VaR) associated with a one-month investment in the S&P 500 index. Hint: the 1% and 5% quantiles of the standard normal distribution are -2.326 and -1.645, respectively.

```
> W0 = 2
> q.05 = muhat.vals + sigmahat.vals*qnorm(0.05)
> q.01 = muhat.vals + sigmahat.vals*qnorm(0.01)
> VaR.05 = (exp(q.05[1]) - 1)*W0
> VaR.01 = (exp(q.01[1]) - 1)*W0
> VaR.05
   sp500
-0.1496
> VaR.01
   sp500
-0.2085
```

With 5% probability, you can lose $149,600 or more, and with 1% probability you can lose $208,500 or more.

d. Describe briefly how you could use the bootstrap to compute an estimated standard error for the 5% VaR found in part c above.

Bootstrapping involves sampling with replacement from the original data to create B different samples. On each sample the 5% VaR is computed, and the estimated bootstrap SE is the sample standard deviation of these B 5% VaR values.

e) Below are the sample autocorrelation functions for the S&P 500 and the bond fund. Using the information in these graphs, would you say that the CER model assumption that returns are uncorrelated over time is appropriate? Briefly justify your answer.

Vanguard S&P 500 Index
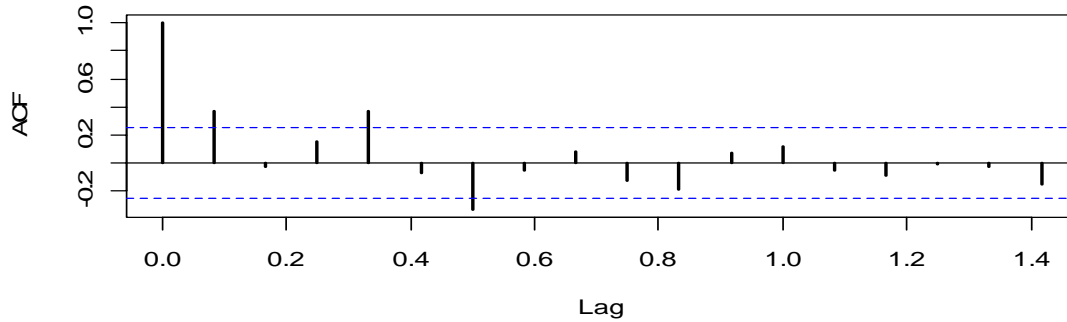
Vanguard Short Term Bond
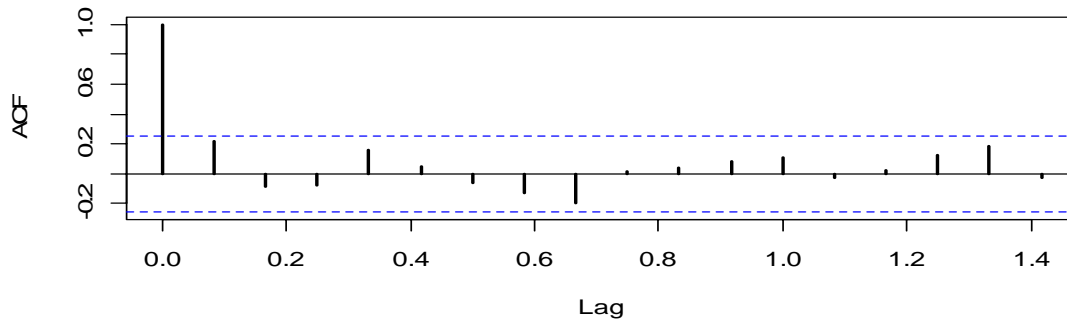
The returns on the S&P 500 index display some large positive autocorrelations at lags 1 and 4. This is evidence against the CER model assumption of uncorrelated returns. The bond returns, in contrast, show no evidence of autocorrelation and exhibit behavior consistent with the CER model assumption of no autocorrelation.