University of Washington
Department of Economics

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Economics 424

Final Exam Solutions

This is a closed book and closed note exam. However, you are allowed one page of handwritten notes. Answer all questions and write all answers in a blue book. Total points = 100.

I. Matrix Algebra and Portfolio Math (25 points)

Let R_i denote the continuously compounded return on asset i (i = 1, …, N) with $E[R_i] = \mu_i$, var(R_i) = σ_i^2 and cov(R_i, R_j) = σ_{ij}. Define the ($N \times 1$) vectors $\boldsymbol{R} = (R_1, \ldots, R_N)'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{m} = (m_1, \ldots, m_N)'$, $\boldsymbol{x} = (x_1, \ldots, x_N)'$, $\boldsymbol{y} = (y_1, \ldots, y_N)'$, $\boldsymbol{t} = (t_1, \ldots, t_N)'$, $\mathbf{1} = (1, \ldots, 1)'$ and the ($N \times N$) covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors $\boldsymbol{m}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ contain portfolio weights that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolio defined by the vector $\boldsymbol{x}$, give the expression for the portfolio return ($R_{p,x}$), the portfolio expected return ($\mu_{p,x}$), and the portfolio variance ($\sigma_{p,x}^2$).

$$R_{p,x} = \boldsymbol{R}'\boldsymbol{x},\ \mu_{p,x} = \boldsymbol{x}'\boldsymbol{\mu},\ \sigma_{p,x}^2 = \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}$$

b. Write down the optimization problem used to determine the global minimum variance portfolio assuming short sales are allowed. Let $\boldsymbol{m}$ denote the vector of portfolio weights in the global minimum variance portfolio.

$$\min_{m}\ \boldsymbol{m}'\boldsymbol{\Sigma}\boldsymbol{m} \text{ s.t. } \boldsymbol{m}'\boldsymbol{1} = 1$$

c. Write down the optimization problem used to determine an efficient portfolio with target return equal to μ_0 assuming short sales are allowed. Let $\boldsymbol{x}$ denote the vector of portfolio weights in the efficient portfolio.

$$\min_{x}\ \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \text{ s.t. } \boldsymbol{x}'\boldsymbol{1} = 1 \text{ and } \boldsymbol{x}'\boldsymbol{\mu} = \mu_0$$

d. Using the results from parts b. and c., briefly describe how you can compute the efficient frontier containing only risky assets (Markowitz bullet) using just two efficient portfolios. Give the expressions for the expected return and variance for portfolios on the efficient frontier.

Any portfolio on the Markowitz bullet can be computed as a convex combination of any other two portfolios on the bullet. For example, one portfolio could be the global minimum variance portfolio m and the other could be the portfolio x that has target return μ_0*. Then, any efficient portfolio has the form*

$$z = \alpha m + (1-\alpha)x$$
$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,x}$$
$$\sigma^2_{p,z} = \alpha^2\sigma^2_{p,m} + (1-\alpha)^2\sigma^2_{p,x} + 2\alpha(1-\alpha)\sigma_{mx}$$
$$\sigma_{mx} = m'\Sigma x$$

e. Write down the optimization problem used to determine the tangency portfolio, assuming short sales are allowed and the risk free rate is give by r_f. Let t denote the vector of portfolio weights in the tangency portfolio.

$$\max_{t} \frac{t'\mu - r_f}{(t'\Sigma t)^{1/2}} \text{ s.t. } t'1 = 1$$

II. Efficient Portfolios (40 points)

The graph below shows the efficient frontier computed from three Vanguard mutual funds: S&P 500 index (vfinx), European stock index (veurx), and long-term bond index (vbltx).

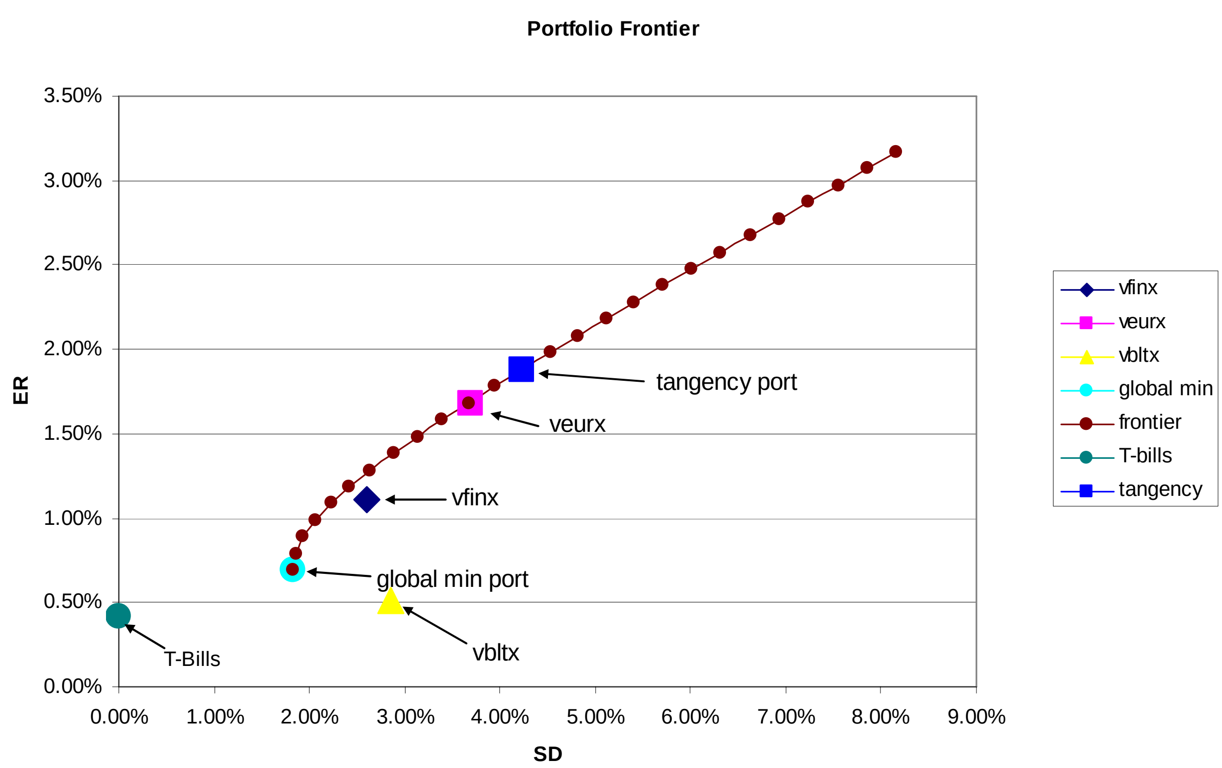


Figure 1 Markowitz Bullet

Expected return and standard deviation estimates for specific assets are summarized in the table below. These estimates are based on monthly continuously compounded return data over the three year period January 2003 – January 2006.

Table 1 Portfolio Statistics

Asset	**Mean (E[R])**	**Standard deviation (SD(R))**	**Weight in Global Min Portfolio**	**Weight in Tangency portfolio**
VFINX	1.11%	2.60%	83%	-44%
VEURX	1.68%	3.70%	-27%	139%
VBLTX	0.51%	2.85%	44%	5%
T-Bills	0.42%	0%		
Global Min Portfolio	0.69%	1.83%		
Tangency Portfolio	1.88%	4.23%		

The pair-wise scatter plots of the returns and sample correlation matrix are given below:

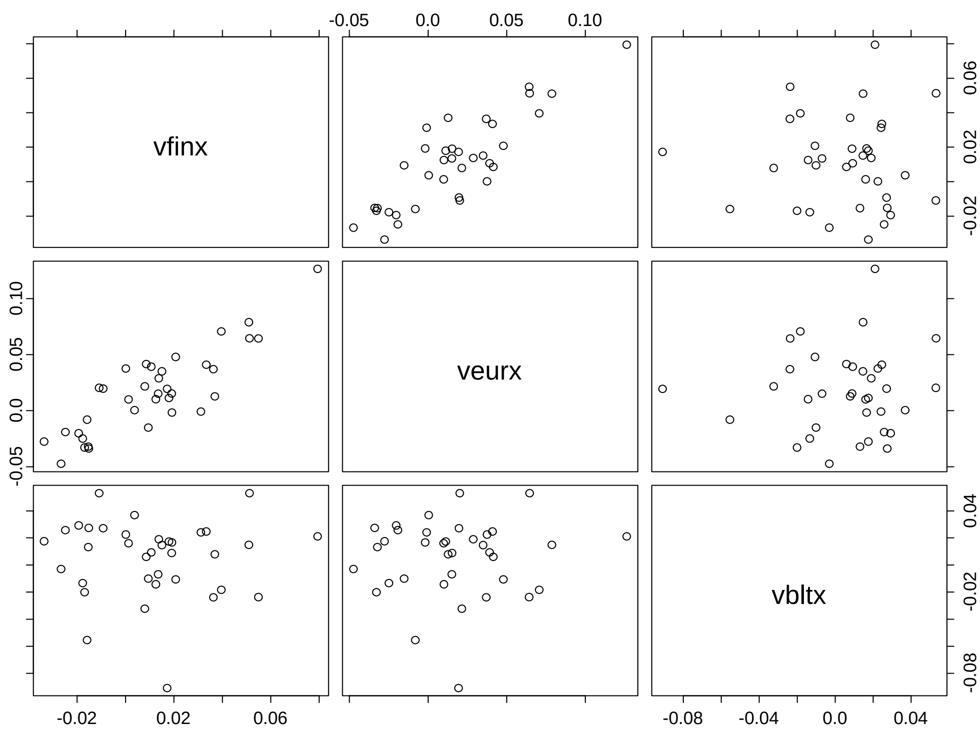


Figure 2 Return Scatter Plots

```
> print(cor.mat, digits=4)
         vfinx   veurx    vbltx
vfinx  1.00000 0.85441 -0.01635
```

```
veurx  0.85441 1.00000  0.04798
vbltx -0.01635 0.04798  1.00000
```

Using the above information, please answer the following questions.

a. Based on the pair-wise scatterplots and sample correlations, do you think there will be diversification benefits from forming portfolios of the three mutual funds? If yes, which asset should provide the largest diversification benefit? Briefly justify your answer.

Diversification works best when assets are not perfectly positively correlated. Here the 3 assets are imperfectly correlated so some diversification should be possible. Since vbltx is essentially uncorrelated with vfinx and veurx, the biggest diversification benefit should come from vbltx.

b. Find the efficient portfolio of risky assets only (e.g. a portfolio on the Markowitz bullet) that has an expected monthly return equal to 1.5%. In this portfolio, how much is invested in vfinx, veurx, and vbltx? (Hint: recall, any portfolio on the Markowitz bullet can be expressed as a convex combination of any other two portfolios on the Markowitz bullet)

Here, we make use of the fact that the global minimum variance portfolio and the tangency portfolio are on the Markowitz bullet. Therefore, the expected return for any portfolio on the Markowitz bullet can be expressed as

$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,t}$$

Setting $\mu_{p,z} = 0.015$ *and using* $\mu_{p,m} = 0.0069,\ \mu_{p,t} = 0.0188$ *we can solve for* α:

$$\alpha = \frac{\mu_{p,z} - \mu_{p,t}}{\mu_{p,m} - \mu_{p,t}} = \frac{0.015 - 0.018}{0.0069 - 0.018} = 0.32$$
$$1 - \alpha = 0.68$$

The weights on vfinx, veurx and vbltx in this portfolio are

$$\boldsymbol{z} = \alpha\boldsymbol{m} + (1-\alpha)\boldsymbol{x}$$
$$= (0.32)\begin{pmatrix} .83 \\ -.27 \\ .44 \end{pmatrix} + (0.68)\begin{pmatrix} -.44 \\ 1.39 \\ .05 \end{pmatrix} = \begin{pmatrix} 0.03 \\ 0.86 \\ 0.17 \end{pmatrix}$$

c. Transfer the graph of the Markowitz bullet to your blue book. On this graph indicate the efficient combinations of T-bills and risky assets. Also, indicate a portfolio that would be preferred by a very risk averse investor and a portfolio that would be preferred by a risk tolerant investor.

d. How much should be invested in T-bills and the tangency portfolio to create an efficient portfolio with the same expected return as vfinx? What is the standard deviation of this efficient portfolio? Transfer the graph of the Markowitz bullet to your blue book and indicate the location of this efficient portfolio on the graph.

$$x_{\tan} = \frac{\mu_{vfinx} - r_f}{\mu_{\tan} - r_f} = \frac{.0111 - 0.0042}{.0188 - 0.0042} = 0.47$$

$$x_f = 1 - x_{\tan} = 1 - 0.47 = 0.53$$

$$\sigma_{p,e} = x_{\tan}\sigma_{\tan} = (0.47)(0.0423) = 0.02$$

e. In the efficient portfolio you found in part d, what are the shares of wealth invested in vfinx, veurx, and vbltx?

vfinx	veurx	vbltx
0.47*(-0.44)=-0.21	0.47*1.39=0.66	0.47*(0.05)=0.02

f. Assume that the continuously compounded returns for all assets are normally distributed. For an initial investment of $100,000, compute the 1% value-at-risk (VaR) on vfinx and the efficient portfolio that has the same expected return as vfinx. (Hint: the 1% quantile of a standard normal random variable is -2.33). Which portfolio has the smallest 1% VaR?

First we find the 1% quantiles for the 2 portfolios

$$
\begin{aligned}
&vfinx: \ q_{.01} = 0.011 + (0.026)(-2.33) = -0.0495 \\
&port: \ q_{.01} = 0.011 + (0.020)(-2.33) = -0.0356
\end{aligned}
$$

Then we compute the 1% VaR

$$
\begin{aligned}
&vfinx: \ VaR_{.01} = \$100,000 \times \left(e^{0.0495} - 1\right) = -\$4,833 \\
&port: \ VaR_{.01} = \$100,000 \times \left(e^{0.0356} - 1\right) = -\$3,499
\end{aligned}
$$

g. How much should be invested in T-bills and the tangency portfolio to create an efficient portfolio with the same standard deviation as vfinx? What is the expected return on this portfolio? Transfer the above graph to your blue book and indicate the location of this efficient portfolio on the graph.

$$
\begin{aligned}
&x_{\tan} = \frac{\sigma_{vfinx}}{\sigma_{\tan}} = \frac{0.0260}{0.0423} = 0.62 \\
&x_f = 1 - x_{\tan} = 1 - 0.62 = 0.38 \\
&\mu_{p,e} = r_f + x_{\tan}(\mu_{\tan} - r_f) = 0.0042 + 0.62(0.0423 - 0.0042) = 0.0132
\end{aligned}
$$

h. The global minimum variance portfolio has a negative weight on veurx. What is the interpretation of a negative weight? Is the negative weight problematic? Why or why not?

A negative weight represents a short-sale of the asset. That is, you borrow the asset; sell it; use the proceeds to purchase more of the other assets. A short sale is sometime problematic. Sometimes you are prevented from short selling certain assets. For this example, short selling is problematic because you cannot short-sell a mutual fund.

III. Empirical Analysis of the single index model (35 points)

The single index model for asset returns has the form

$$
\begin{aligned}
&R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\
&i = 1, \ldots, n \text{ assets} \\
&t = 1, \ldots, T \text{ time periods}
\end{aligned}
$$

a. What are the assumptions of the SI model?

$\varepsilon_{it} \sim iid\ \mathrm{N}(0, \sigma^2_{\varepsilon_i})$

$R_{Mt} \sim iid\ \mathrm{N}(\mu_M, \sigma^2_M)$

$\mathrm{cov}(R_{Mt}, \varepsilon_{it}) = 0$

$\mathrm{cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0$ for all $i \neq j$

$\mathrm{cov}(\varepsilon_{it}, \varepsilon_{js}) = 0$ for all $i \neq j; t \neq s$

b. What is the interpretation of ε_{it} in the SI model?

ε_{it} represents *random news that is unrelated to the market.*

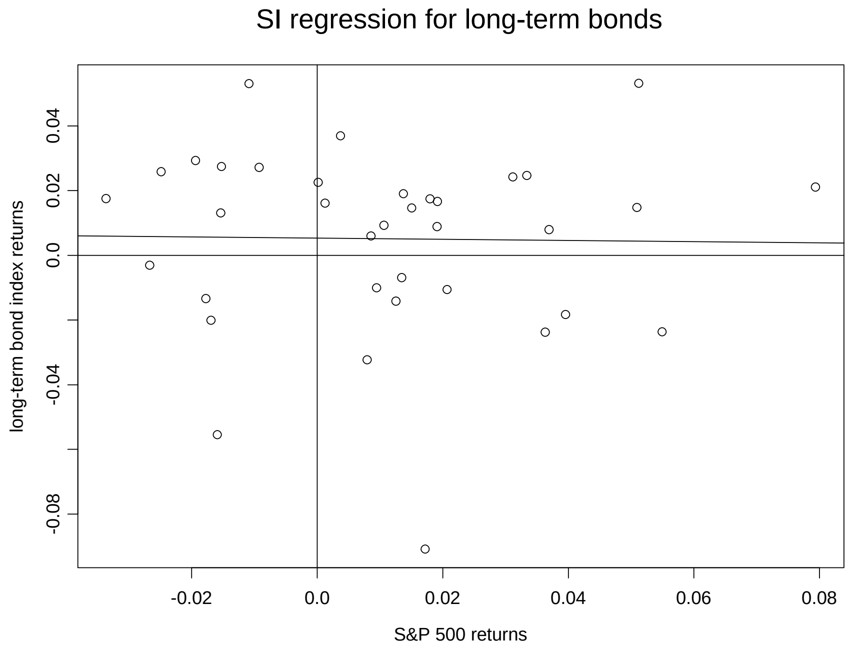
The following represents S-PLUS linear regression output from estimating the single index model for the Vanguard long-term bond index (vbltx) and the Vanguard European Equity index (veurx) using monthly continuously compounded return data over the 3 year period January 2003 –January 2006. In the regressions, the market index is the Vanguard S&P 500 index (vfinx).

```
> summary(vbltx.fit,cor=F)

Call: lm(formula = vbltx ~ vfinx, data = final.ts)
Residuals:
      Min       1Q   Median       3Q     Max
 -0.09582 -0.0164 0.008568 0.01776 0.04875

Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept)  0.0053  0.0053     1.0080  0.3206
      vfinx -0.0179  0.1880    -0.0953  0.9246

Residual standard error: 0.02894 on 34 degrees of freedom
Multiple R-Squared: 0.0002673
F-statistic: 0.009091 on 1 and 34 degrees of freedom, the p-value
is 0.9246
```


SI regression for long-term bonds
long-term bond index returns
0.04
0.02
0.0
-0.04
-0.08
-0.02
0.0
0.02
0.04
0.06
0.08
S&P 500 returns

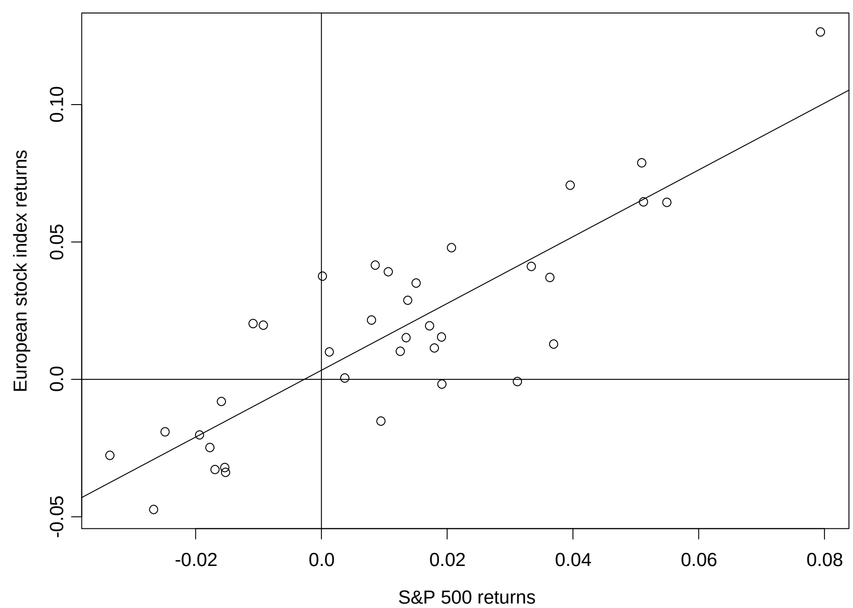
```
> summary(veurx.fit,cor=F)

Call: lm(formula = veurx ~ vfinx, data = final.ts)
Residuals:
      Min       1Q    Median     3Q     Max
 -0.04203 -0.01184 -0.001937 0.0134 0.03397

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept) 0.0033 0.0035     0.9411  0.3533
      vfinx 1.2155 0.1268     9.5881  0.0000

Residual standard error: 0.01952 on 34 degrees of freedom
Multiple R-Squared: 0.73
F-statistic: 91.93 on 1 and 34 degrees of freedom, the p-value is
3.387e-011
```

SI regression for European stock index



c. For the long-term bond index and the European equity index, what are the estimated values of α and β and what are the estimated standard errors for these estimates? Is β for the European equity index estimated more precisely than β for the long-term bond index? Why or why not?

	$\hat{\alpha}$	$SE(\hat{\alpha})$	$\hat{\beta}$	$SE(\hat{\beta})$
Long-term bond	0.0053	0.0053	-0.0179	0.1880
European equity	0.0033	0.0035	1.2155	0.1268

The value of $SE(\hat{\beta})$ is smaller for the European stock index, which indicates that β_{veurx} is estimated more precisely. This is to be expected since the European stock index is a well diversified portfolio whereas the long-term bond index is not very diversified. Diversification concentrates the risk of the European stock index on market risk and forces the returns to more closely follow the market index return. This leads to a smaller error variance and hence a more precise estimate of β.

d. Based on the estimated values of β, what can you say about the risk characteristics of the long-term bond index and the European equity index relative to the market index?

The estimated beta for the long-term bond is negative and close to zero. This means that adding the short-term bond index to the market portfolio will lower the variability of the market index. Thus the long-term bond is much less risky than the extended market index. Since the European equity beta is about 1.2, adding it to the market index will raise the variability. Thus the European equity index is more risky than the market index.

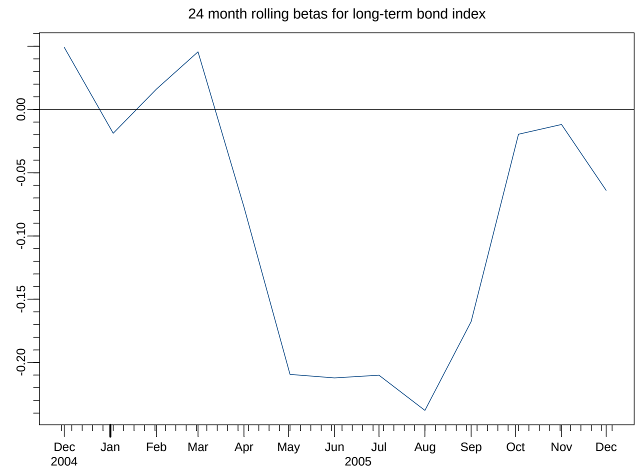
e. For each regression, what is the proportion of market or systematic risk and what is the proportion of firm specific or unsystematic risk?

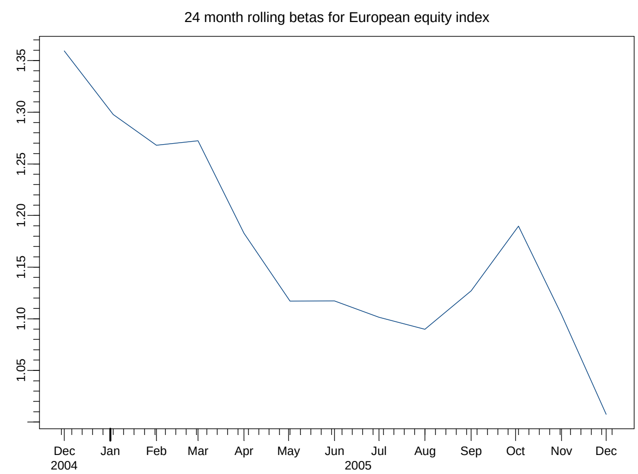
	R-square (% market risk)	*1-(R-square) (% non-market risk)*
Long-term bond	0.00	1.00
European Equity	0.73	0.27

f. Why should the European equity index have a greater proportion of market risk than the long-term bond index?

The European equity index is a large diversified portfolio of stocks, whereas the long-term bond index is a small, not diversified, index of long-term bonds. The diversification effect concentrates the market risk of the European equity index and reduces the amount of non-market risk. This tends to increase the R-square (% of market risk). Long-term bonds generally do not have much market risk since long-term interest rates are fairly stable. Hence the R-square of the bond index is expected to be low (close to zero). This is what we see in the data.

g. The following graphs show the 24-month rolling estimates of β for the SI models for the long-term bond index and the European equity index. Using these graphs, what can you say about the stability of β over time?





The rolling estimates reveal instability in the estimated b's. The rolling b's on the long-term bond index start out close to zero, dip to negative values in the middle of the sample, and then return to zero at the end of the sample. The rolling b's for the European stock index steadily decline over the sample from about 1.3 down to 1.