

Economics 424/CFRM 462

Final Exam Solutions

This is a closed book and closed note exam. However, you are allowed one page of notes (8.5" by 11" or A4 double-sided) and the use of a calculator. Answer all questions and write all answers in the space provided. If you need additional space, you may attach extra sheets of paper with your calculations. Time limit is 2 hours and 10 minutes. Total points = 100.

I. Constant Expected Return Model (32 points, 4 points each)

Consider the constant expected return (CER) model

$$r_{it} = \mu_i + \varepsilon_{it}, \quad \varepsilon_{it} \sim iid N(0, \sigma_i^2)$$

$$cov(r_{it}, r_{jt}) = \sigma_{ij}, \quad cor(r_{it}, r_{jt}) = \rho_{ij}$$

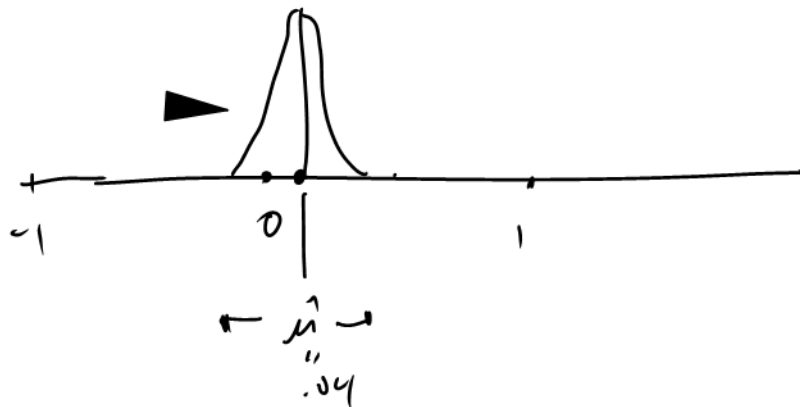
for the monthly *simple* returns on the Vanguard S&P500 index (VFINX) and Amazon stock (AMZN). The following R output gives the estimates of μ_i , σ_i , σ_{ij} and ρ_{ij} for the S&P 500 index (VFINX) and Amazon stock (AMZN) from the $T=60$ months of data:

```
> muhat.vals
  vfinx  amzn
0.016 0.036

> sigmahat.vals
  vfinx  amzn
0.044 0.085

> covhat.vals
vfinx, amzn
  0.001

> rhohat.vals
vfinx, amzn
  0.343
```



a. For each asset, compute estimated standard errors for $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\rho}$. Create a table showing the estimates in one row and the standard errors in another row. Are the estimates precise? Briefly justify your answers.

The standard estimated standard error formulas are:

$$SE(\hat{\mu}) = \frac{\hat{\sigma}}{\sqrt{T}}, \quad SE(\hat{\sigma}) = \frac{\hat{\sigma}}{\sqrt{2T}}, \quad SE(\hat{\rho}) = \frac{1 - \hat{\rho}^2}{\sqrt{T}}$$

```
> nobs = nrow(ret)
> se.muhat = sigmahat.vals/sqrt(nobs)
> se.sigma = sigmahat.vals/sqrt(2*nobs)
> se.rho = (1 - rhohat.vals^2)/sqrt(nobs)

> rbind(muhat.vals, se.muhat)
      vfinx  amzn
muhat.vals 0.0157 0.036
se.muhat   0.0057 0.011
```

```

> rbind(sigmahat.vals, se.sigma)
              vfinx    amzn
sigmahat.vals 0.04415 0.08489
se.sigma      0.00403 0.00775

> rbind(rhohat.vals, se.rho)
              vfinx, amzn
rhohat.vals   0.343
se.rho        0.114

```

Standard errors for means are about 1/3 the size of the estimates which indicates that the means are estimated with moderate imprecision. The standard errors for the volatilities are much smaller than the estimates which indicates precise estimation. The correlation standard error is about 1/3 the size of the estimate and indicates moderate imprecision.

b. For the S&P 500 index (VFINX) only, compute 95% confidence intervals for μ , σ . Briefly comment on the precision of the estimates. In particular, note if both positive and negative values are in the respective confidence intervals.

The rule-of-thumb formula for a 95% confidence interval for a parameter θ is

$$\hat{\theta} \pm 2 \times \widehat{SE}(\hat{\theta})$$

For VFINX, the estimated standard errors for μ and σ are

```

> # 95% ci for mu
> upper = muhat.vals + 2*se.muhat
> lower = muhat.vals - 2*se.muhat
> cbind(lower[1], upper[1])
      [,1] [,2]
vfinx 0.00431 0.0271

> # 95% ci for sigma
> upper = sigmahat.vals + 2*se.sigma
> lower = sigmahat.vals - 2*se.sigma
> cbind(lower[1], upper[1])
      [,1] [,2]
vfinx 0.0361 0.0522

```

The 95% CI for μ is fairly wide from an economic point of view (0.5% to 3%), whereas the 95% CI for σ is fairly tight from an economic point of view (3.6% to 5.2%). Only positive values are in the confidence intervals.

c. Compute a 95% confidence interval for $\rho_{vfinx, amzn}$. Use this interval to test the hypotheses $H_0 : \rho_{vfinx, amzn} = 0.5$ vs. $H_1 : \rho_{vfinx, amzn} \neq 0.5$ using a 5% significance level.

The 95% confidence intervals for $\rho_{vfinx, amzn}$ is

```
> upper = rhohat.vals + 2*se.rho
> lower = rhohat.vals - 2*se.rho
> cbind(lower, upper)
      lower upper
vfinx, amzn 0.116 0.571
```

Since zero is not in the 95% CI for ρ , we can reject the null hypothesis that $\rho = 0$ at the 5% level.

d. Let $W_0 = \$100,000$ be the initial wealth invested in each asset. Estimate the monthly 5% Value-at-Risk for each asset assuming each asset is normally distributed.

Because we have simple returns, the 5% VaR is computed using

$$Va\hat{R}_{.05} = (\hat{\mu} + \hat{\sigma} \times q_{.05}^Z) \times W_0$$

```
> VaR.05 = (muhat.vals + sigmahat.vals*qnorm(0.05))*100000
> VaR.05
  vfinx  amzn
-5692 -10366
```

e. Briefly explain how you can use the bootstrap to compute a standard error for the monthly 5% normal VaR for VFINX you computed in part d. above.

Bootstrapping involves sampling with replacement from the original data to create B different samples. On each sample the 5% VaR is computed, and the estimated bootstrap SE is the sample standard deviation of these B 5% VaR values.

f. For VFINX and AMZN Consider testing the hypotheses

$$H_0 : r_t \sim \text{normal vs. } H_1 : r_t \text{ is not normally distributed}$$

using a 5% significance level. The JB statistics are $JB_{vfinx} = 1.847$ and $JB_{amzn} = 1.512$. What do you conclude?

Under the null of normality, the JB statistic has a chi-square distribution with 2 degrees of freedom. The 5% critical value is the 95% quantile of the chi-square distribution with 2 degrees of freedom. This quantile is 6. Hence, we reject the null at the 5% significance level if $JB > 6$. Since both JB values are less than 6, we cannot reject the hypothesis that the returns on both assets are normally distributed.

g. The 24-month rolling estimates of μ and σ are illustrated in Figure 1 and Figure 2 below (on next page). The CER model assumes that μ and σ are constant over time. Is this a reasonable assumption? Why or why not.

For both VFINX and AMZN, the rolling mean and sd values behave similarly. The means do not move too much (they both dip a bit in late 2011 and then recover). The sd values, however, show a slow decline

over the sample reflecting a drop in market volatility over 2012-2014 compared with 2009-2011. The mean and sd values are not constant over time.

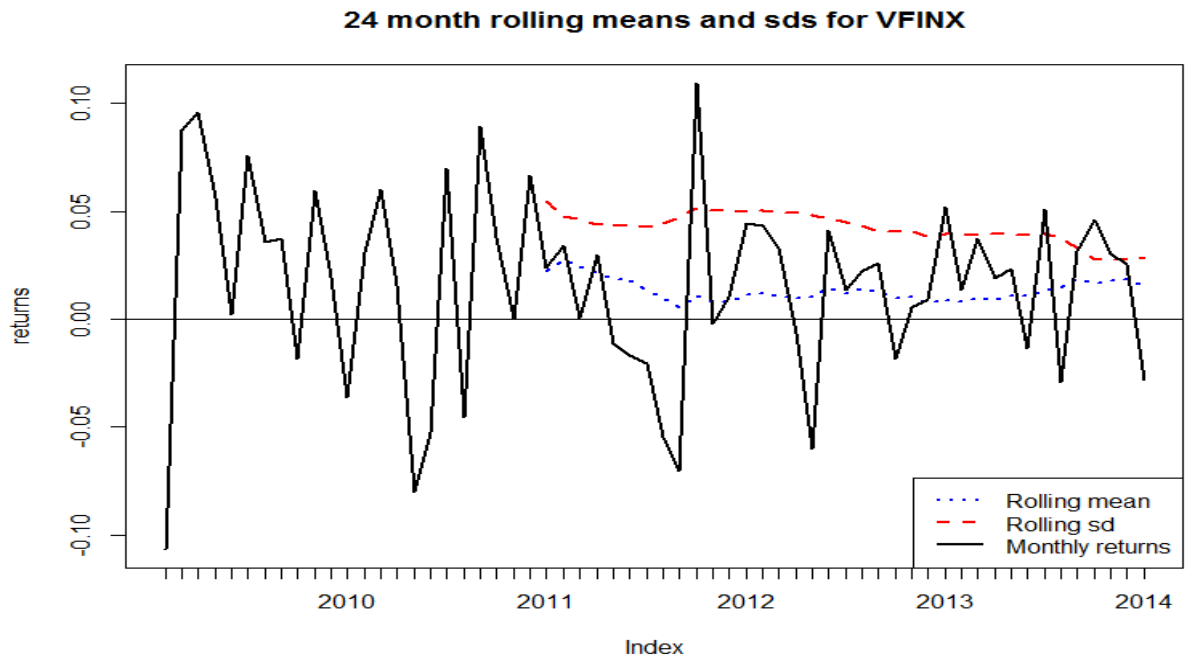


Figure 1 Rolling estimates for VFINX

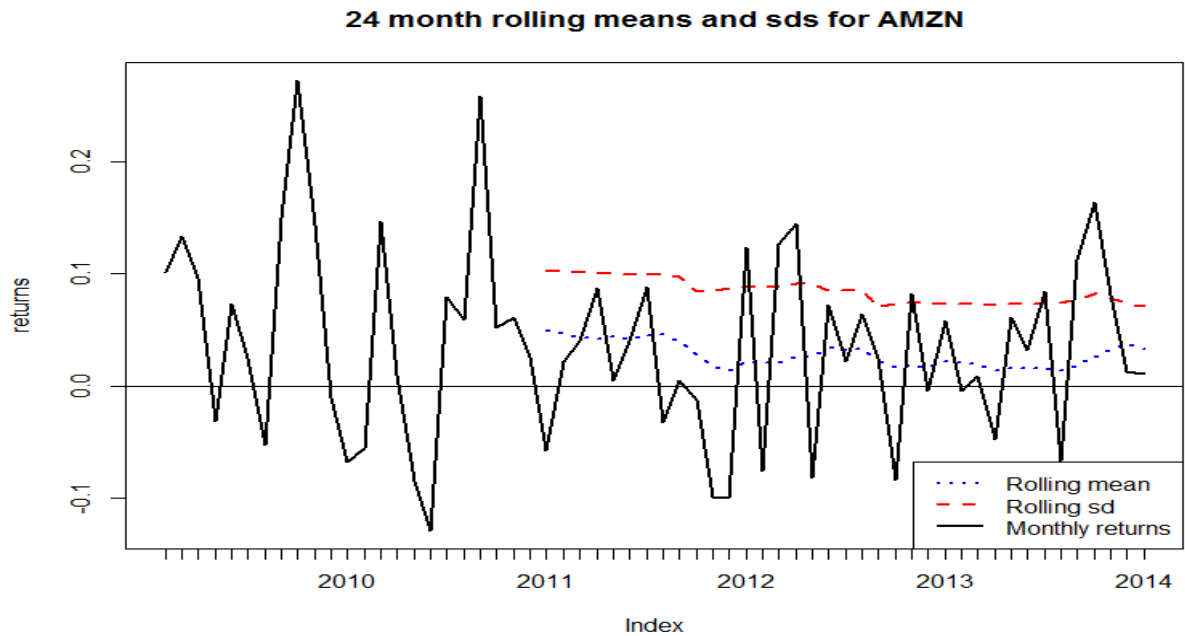


Figure 2 Rolling estimates for AMZN

h. The CER model can be used for continuously compounded returns or for simple returns. What are the advantages and disadvantages of using the CER model for continuously compounded returns? What are the advantages and disadvantages of using the CER model for simple returns?

The advantages of the CER model for cc returns: (1) cc returns are more likely normally distributed than simple returns; (2) cc returns are additive over time. The main disadvantage of the CER model for cc returns involves the application to portfolios. The cc return on a portfolio is not a weighted average of the cc returns on the individual assets so the CER model does not hold exactly for portfolios. For simple returns, the main disadvantage is that multiperiod returns are not additive. The CER model does not hold for multiperiod returns. The main advantage, is that the portfolio return is a weighted average of asset returns. Here the CER model holds for portfolios.

II. Matrix Algebra and Portfolio Math (36 points, 4 points each)

Let R_i denote the continuously compounded return on asset i ($i = 1, \dots, N$) with $E[R_i] = \mu_i$, $\text{var}(R_i) = \sigma_i^2$ and $\text{cov}(R_i, R_j) = \sigma_{ij}$. Define the $(N \times 1)$ vectors $\mathbf{R} = (R_1, \dots, R_N)'$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N)'$, $\mathbf{m} = (m_1, \dots, m_N)'$, $\mathbf{x} = (x_1, \dots, x_N)'$, $\mathbf{y} = (y_1, \dots, y_N)'$, $\mathbf{t} = (t_1, \dots, t_N)'$, $\mathbf{1} = (1, \dots, 1)'$ and the $(N \times N)$ covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors \mathbf{m} , \mathbf{x} , \mathbf{y} and \mathbf{t} contain portfolio weights that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolios defined by the vectors \mathbf{x} and \mathbf{y} give the expression for the portfolio returns, ($R_{p,x}$ and $R_{p,y}$), the portfolio expected returns ($\mu_{p,x}$ and $\mu_{p,y}$), the portfolio variances ($\sigma_{p,x}^2$ and $\sigma_{p,y}^2$), and the covariance between $R_{p,x}$ and $R_{p,y}$ (σ_{xy}).

$$R_{p,x} = \mathbf{R}'\mathbf{x}, \mu_{p,x} = \mathbf{x}'\boldsymbol{\mu}, \sigma_{p,x}^2 = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$$

$$R_{p,y} = \mathbf{R}'\mathbf{y}, \mu_{p,y} = \mathbf{y}'\boldsymbol{\mu}, \sigma_{p,y}^2 = \mathbf{y}'\boldsymbol{\Sigma}\mathbf{y}$$

$$\sigma_{xy} = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}$$

b. For portfolio \mathbf{x} , derive the $n \times 1$ vector of marginal contributions to portfolio volatility defined by

$$\mathbf{MCR}^\sigma = \frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}}.$$

By the chain rule we have

$$\begin{aligned}\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}} (\mathbf{x}' \Sigma \mathbf{x})^{1/2} = \frac{1}{2} (\mathbf{x}' \Sigma \mathbf{x})^{-1/2} 2 \Sigma \mathbf{x} \\ &= \frac{\Sigma \mathbf{x}}{(\mathbf{x}' \Sigma \mathbf{x})^{1/2}} = \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})}\end{aligned}$$

c. Write down the optimization problem and give the Lagrangian used to determine the global minimum variance portfolio assuming short sales are allowed. Let \mathbf{m} denote the vector of portfolio weights in the global minimum variance portfolio.

$$\begin{aligned}\min_m \quad & \mathbf{m}' \Sigma \mathbf{m} \text{ s.t. } \mathbf{m}' \mathbf{1} = 1 \\ L(\mathbf{m}, \lambda) &= \mathbf{m}' \Sigma \mathbf{m} + \lambda (\mathbf{m}' \mathbf{1} - 1)\end{aligned}$$

d. Continuing with part b., derive the first order conditions for determining the minimum variance portfolio \mathbf{m} . Write these first order conditions as a system of linear equations in the form $\mathbf{A} \mathbf{z} = \mathbf{b}$ and show how the portfolio \mathbf{m} can be determined from this system.

The first order conditions are

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{m}} &= 2 \Sigma \mathbf{m} + \lambda \cdot \mathbf{1} = \mathbf{0} \\ \frac{\partial L}{\partial \lambda} &= \mathbf{m}' \mathbf{1} - 1 = 0\end{aligned}$$

These first order conditions can be written in matrix form as

$$\begin{bmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1}' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \text{ or } \mathbf{A} \mathbf{z} = \mathbf{b}$$

This linear system has solution

$$\mathbf{z} = \mathbf{A}^{-1} \mathbf{b}$$

The vector \mathbf{m} is the first n elements of the $n+1 \times 1$ vector \mathbf{z} .

e. Write down the optimization problem and give the Lagrangian used to determine an efficient portfolio with target return equal to μ_0 assuming short sales are allowed. Let \mathbf{x} denote the vector of portfolio weights in the efficient portfolio.

$$\min_{\mathbf{x}} \mathbf{x}'\Sigma\mathbf{x} \text{ s.t. } \mathbf{x}'\mathbf{1} = 1 \text{ and } \mathbf{x}'\boldsymbol{\mu} = \mu_0$$

$$L(\mathbf{x}, \lambda_1, \lambda_2) = \mathbf{x}'\Sigma\mathbf{x} + \lambda_1(\mathbf{x}'\mathbf{1} - 1) + \lambda_2(\mathbf{x}'\boldsymbol{\mu} - \mu_0)$$

f. Briefly describe how you would compute the efficient frontier containing only risky assets (Markowitz bullet) when short sales are allowed.

Because short sales are allowed, the Markowitz bullet can be constructed using a convex combination of any two efficient (frontier) portfolio: $\mathbf{z} = \alpha \cdot \mathbf{m} + (1 - \alpha) \cdot \mathbf{x}$, where \mathbf{m} and \mathbf{x} are any two frontier portfolios. To get a nice picture, choose portfolio \mathbf{m} to be the global minimum variance portfolio and choose the other efficient portfolio \mathbf{x} to be the efficient portfolio with target return equal to the highest expected return of the available assets. Then, for example, vary α from 1 to -1 in increments of 0.1 and compute \mathbf{z} . For each \mathbf{z} , compute $\mu_{p,z} = \mathbf{z}'\boldsymbol{\mu}$ and $\sigma_{p,z} = (\mathbf{z}'\Sigma\mathbf{z})^{1/2}$ and then plot these pairs of points.

g. Write down the optimization problem used to determine the tangency portfolio, assuming **short sales are not allowed** and the risk free rate is given by r_f . Let \mathbf{t} denote the vector of portfolio weights in the tangency portfolio.

$$\max_{\mathbf{t}} \frac{\mathbf{t}'\boldsymbol{\mu} - r_f}{(\mathbf{t}'\Sigma\mathbf{t})^{1/2}} \text{ s.t. } \mathbf{t}'\mathbf{1} = 1 \text{ and } t_i \geq 0, i = 1, \dots, N$$

h. Continuing with part g., is there an analytical solution (i.e., matrix algebra mathematical formula) for the tangency portfolio when short sales are not allowed?

There is no analytical solution for the tangency portfolio when short sales are not allowed. The inequality constraints do not allow the method of Lagrange multipliers to be used to determine an analytic solution using matrix algebra.

i. Write down the equations for the expected return (μ_p^e) and standard deviation (σ_p^e) of efficient portfolios consisting of the tangency portfolio and T-bills, where the T-bill rate (risk-free rate) is given by r_f and \mathbf{t} denotes the vector of portfolio weights in the tangency portfolio.

$$\mu_p^e = r_f + x_{\text{tan}}(\mu_{\text{tan}} - r_f), \mu_{\text{tan}} = \mathbf{t}'\boldsymbol{\mu}$$

$$\sigma_p^e = x_{\text{tan}}\sigma_{\text{tan}}, \sigma_{\text{tan}}^2 = \mathbf{t}'\Sigma\mathbf{t}$$

III. Efficient Portfolios (32 points, 4 points each)

The graph below shows the efficient frontier computed from three Vanguard mutual funds: S&P 500 Index (vfinx), US Long Term Bond Index (vbltx), and European Stock Index (veurx).

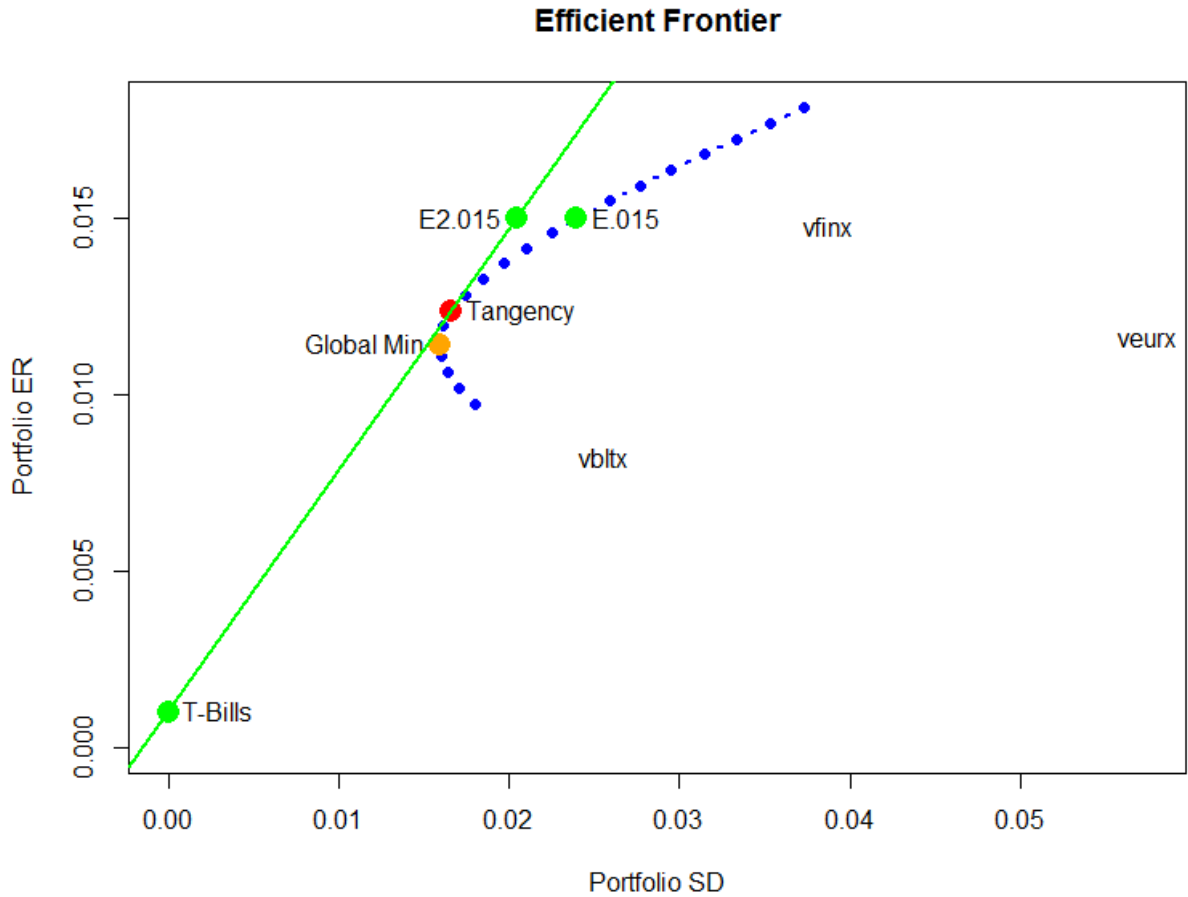


Figure 3 Markowitz Bullet

Expected return and standard deviation estimates for specific assets are summarized in the table below. These estimates are based on monthly continuously compounded return data over the five year (60 month) period January 2009 – January 2014 (same data as in the class project).

Table 1 Portfolio Statistics

Asset	Mean (E[R])	Standard deviation (SD(R))	Weight in Global Min Portfolio	Weight in Efficient Portfolio Mean = 1.48%	Weight in Tangency portfolio
VFINX	0.0148	0.0387	0.58	1.35	0.80
VEURX	0.0116	0.0574	-0.19	-0.69	-0.33
VBLTX	0.0083	0.0255	0.61	0.34	0.53
T-Bills	0.0010	0.0000			
Global Min Portfolio	0.0114	0.0159			
Efficient Portfolio with Mean=0.0148	0.0148	0.0232			
Tangency Portfolio	0.0124	0.0166			

Using the above information, please answer the following questions.

a. Compute annualized means and standard deviations from the monthly statistics in Table 1 for the three portfolios vfinx, vbltx, and veurx using the square-root-of-time-rule (this is only an approximation because we have simple returns). Also, compute the annualized T-Bill rate.

The annualized cc mean return is $\mu_A = 12 \cdot \mu_m$. For the three portfolios we have

$$\mu_{A,vfinx} = 12 \cdot (.0148) = .177$$

$$\mu_{A,veurx} = 12 \cdot (.0116) = .139$$

$$\mu_{A,vbltx} = 12 \cdot (.0083) = .099$$

The annualized cc standard deviation is $\sigma_A = \sqrt{12}\sigma_m$

$$\sigma_{A,vfinx} = \sqrt{12} \cdot (.0387) = .134$$

$$\sigma_{A,veurx} = \sqrt{12} \cdot (.0574) = .199$$

$$\sigma_{A,vbltx} = \sqrt{12} \cdot (.0255) = .088$$

The annualized T-Bill rate is $r_{A,f} = 12 \cdot 0.001 = 0.012$

b. Using the annualized information from part a., compute the annualized Sharpe ratios/slopes for each of the three portfolios. Which portfolio is ranked best using the Sharpe ratio?

The annualized Sharpe ratio is

$$SR_A = \frac{\mu_A - r_{f,A}}{\sigma_A}, \quad r_{f,A} = 12 \cdot r_{f,m}$$

Using the results from part a., we get

$$SR_{A,vfinx} = \frac{0.177 - 0.012}{0.134} = 1.233$$

$$SR_{A,veurx} = \frac{0.139 - 0.012}{0.199} = 0.640$$

$$SR_{A,vbltx} = \frac{0.099 - 0.012}{0.088} = 0.986$$

The asset with the highest annual Sharpe ratio is vfinx.

c. Find the efficient portfolio of risky assets only (i.e., a portfolio on the Markowitz bullet) that has an expected monthly return equal to 0.015 (1.5%). In this portfolio, how much is invested in vfinx, vbltx, and veurx? Show this portfolio on Figure 3.

Here, we make use of the fact that the global minimum variance portfolio and the tangency portfolio are on the Markowitz bullet. Therefore, the expected return for any portfolio on the Markowitz bullet can be expressed as

$$\mu_{p,z} = \alpha \mu_{p,m} + (1 - \alpha) \mu_{p,t}$$

Setting $\mu_{p,z} = 0.015$ and using $\mu_{p,m} = 0.0114$, $\mu_{p,t} = 0.0124$ we can solve for α :

$$\alpha = \frac{\mu_{p,z} - \mu_{p,t}}{\mu_{p,m} - \mu_{p,t}} = \frac{0.015 - 0.0124}{0.0114 - 0.0124} = -2.71$$

$$1 - \alpha = 3.71$$

The weights on vfinx, veurx and vbltx in this portfolio are

$$\mathbf{z} = \alpha \mathbf{m} + (1 - \alpha) \mathbf{t}$$

$$= (0 - 2.71) \begin{pmatrix} .58 \\ -.19 \\ .61 \end{pmatrix} + (3.71) \begin{pmatrix} .80 \\ -.33 \\ .53 \end{pmatrix} = \begin{pmatrix} 1.40 \\ -.72 \\ .32 \end{pmatrix}$$

This portfolio is plotted as the point E.015 in Figure 3.

We can get the same answer if we use the global minimum variance portfolio and the efficient portfolio with expected return equal to 1.48%. The calculations are essentially the same:

$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,e}$$

Setting $\mu_{p,z} = 0.015$ and using $\mu_{p,m} = 0.0114$, $\mu_{p,e} = 0.0148$ we can solve for α :

$$\alpha = \frac{\mu_{p,z} - \mu_{p,e}}{\mu_{p,m} - \mu_{p,e}} = \frac{0.015 - 0.0148}{0.0114 - 0.0148} = -0.06$$

$$1 - \alpha = 1.06$$

The weights on vfinx, veurx and vbltx in this portfolio are

$$\mathbf{z} = \alpha\mathbf{m} + (1-\alpha)\mathbf{x}$$

$$= (-0.06) \begin{pmatrix} .58 \\ -.19 \\ .61 \end{pmatrix} + (1.06) \begin{pmatrix} 1.35 \\ -.69 \\ .34 \end{pmatrix} = \begin{pmatrix} 1.40 \\ -.72 \\ 0.32 \end{pmatrix}$$

d. How much should be invested in T-bills and the tangency portfolio to create an efficient portfolio with expected return equal to 0.015 (1.5%)? What is the standard deviation of this efficient portfolio? Indicate the location of this efficient portfolio on Figure 3.

$$x_{\tan} = \frac{.015 - r_f}{\mu_{\tan} - r_f} = \frac{.01 - 0.001}{.0124 - 0.001} = 1.23$$

$$x_f = 1 - x_{\tan} = 1 - 1.23 = -0.23$$

$$\sigma_{p,e} = x_{\tan} \sigma_{\tan} = (1.23)(0.0166) = 0.024$$

This portfolio is plotted as the point E2.015 in Figure 3.

e. In the efficient portfolio you found in part d, what are the shares of wealth invested in T-Bills, vfinx, vbltx, and veurx?

vfinx	veurx	vbltx	T-Bills
1.23*(0.80)=0.99	1.23*(-0.33)=-0.41	1.23*(0.53)=0.65	-0.23

f. Assuming an initial \$100,000 investment for one month, compute the 5% value-at-risk based on the normal distribution for the global minimum variance portfolio.

First we find the 5% quantiles for the global minimum variance portfolio

$$port: q_{.05} = 0.0114 + (0.0159)(-1.645) = -0.0147$$

Then we compute the 5% VaR

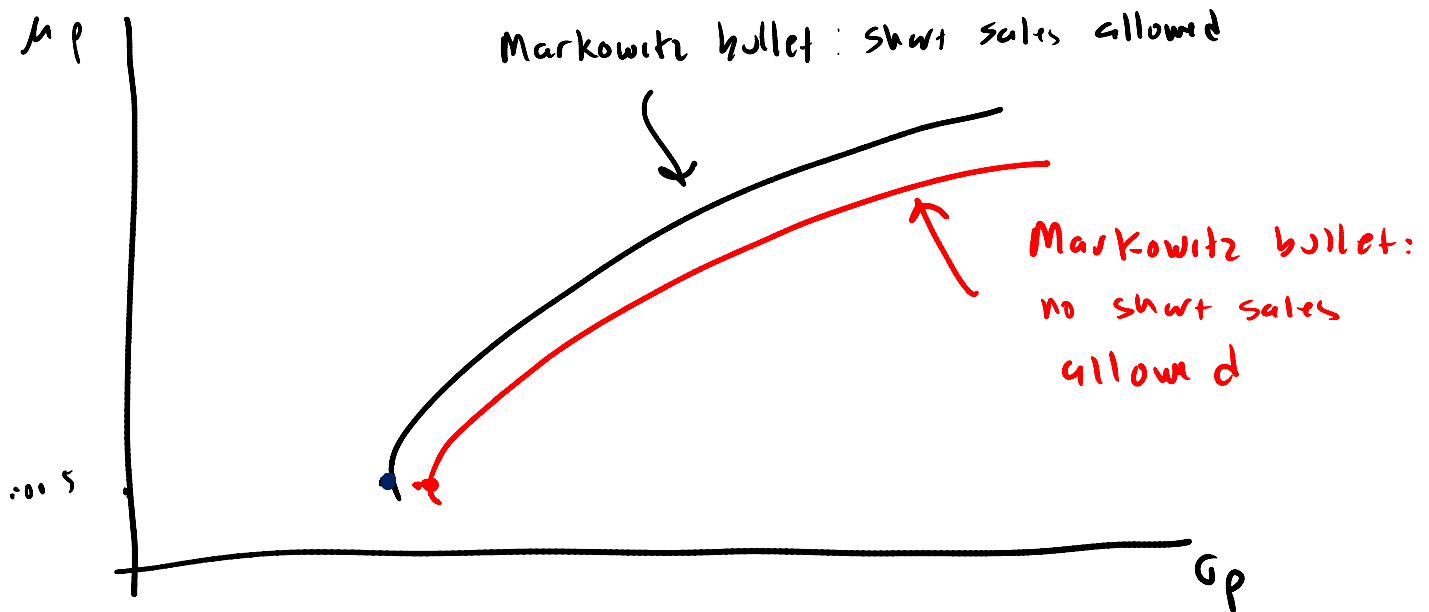
$$\text{port: } \text{VaR}_{.05} = \$100,000 \times (-0.0147) = -\$1,473$$

```

> w0 = 100000
> q.05 = gmin.port$er + gmin.port$sd*qnorm(0.05)
> q.05
[1] -0.0147
> VaR.05 = q.05*w0
> VaR.05
[1] -1473

```

g. The efficient frontier of risky assets shown in Figure 3 allows for short sales (see the weights in the portfolios listed in Table 1). Using the space below, copy the efficient frontier from Figure 3 and indicate roughly the location of the efficient frontier of risky assets that *does not* allow short sales.



h. Suppose you want to find the efficient portfolio of risky assets only that has an expected monthly return equal to 0.02 (2%) but that you are prevented from short selling. Is it possible to find such an efficient portfolio? Briefly explain why or why not.

It is not possible to find an efficient portfolio of risky assets only that has an expected return of 2%. To see this, from Table 1 above notice that the efficient portfolio with expected return equal to 1.48% has a short position in veurx and the tangency portfolio (with mean 1.24%) has a short position in veurx as well. Hence, for an efficient portfolio with mean 2% we would need a short position in veurx.