Valuation of Securities: Bonds

Econ 422: Investment, Capital & Finance
University of Washington
Last updated: April 11, 2010

Reading

- BMA, Chapter 3
- http://finance.yahoo.com/bonds
- http://cxa.marketwatch.com/finra/MarketData/Default.aspx
- 422Bonds.xls
  – Illustration of Excel functions for bond pricing

Review of Roadmap

Intertemporal choice

Introduction of financial markets—borrowing & lending

Valuation of intertemporal cash flows – present value

Valuation of financial securities providing intertemporal cash flows

Choosing among financial securities

Bonds

- Bonds are a vehicle by which borrowing and lending is facilitated
- Bonds are a contract which obligates the issuer (borrower) to make a set of cash payments (interest payments) where the amount and timing is set by the contract
- The lender (the buyer of the contract) becomes the recipient of these payments
- The borrower/issuer is required to return the principal amount borrowed (Face Value) at maturity of the contract

Valuing Bonds

- Cash flow is contractually specified
  - Zero coupon bonds
  - Coupon bonds
- Determine cash flow from contract terms
- Compute present value of cash flow

Zero Coupon Bonds

- Issue no interest/coupon payments
- Referred to as pure discount bonds: they pay a predetermined Face Value amount at a specified date in the future (e.g., $1,000 or $10,000)
- Purchased at a price today that is below the face value
- The increase in value between purchase and redemption represents the interest earned (taxed as ‘phantom’ income)

Types of Zero Coupon Bonds

- US Treasury Bills
  - 1 month, 3 month, 6 month and 1 year maturities
- US Treasury Separate Trading of Registered Interest and Principal Securities (STRIPS)
  - 1 year and Beyond

See finance.yahoo.com/bonds for current price and yield data

Example: Zero Coupon Bonds

- The price is the present value of the future Face Value:
  $P = FV/(1+ r)^t$
- Suppose the government promises to pay you $1,000 in exactly 5 years and assume the relevant discount rate is 5%
- What is the value or price of this commitment the government is making to you?
  $P = \$1,000/(1+0.05)^5 = \$783.53$

Question: How is the interest rate determined?

Bond Price Sensitivity to Interest Rates

- 1 year zero coupon bond: $P = FV(1+r)^{-1}$

$$\begin{aligned}
\frac{dP}{dr} &= \frac{d}{dr} FV(1+r)^{-1} \\
&= -FV(1+r)^{-2} = -FV(1+r)^{-1}(1+r)^{-1} \\
&= -P(1+r)^{-1} \\
\Rightarrow \frac{dP}{P} &= -(1+r)^{-1} dr \approx \%\Delta P
\end{aligned}$$

Example: Bond Price Sensitivity to Interest Rates

Bond Maturity	Yield	Price	dP/P	dP/P = -T/(1+r) × dr
1	0.01	$ 990.10		
	0.02	$ 980.39	-0.98%	-0.98%
	0.03	$ 970.87	-0.97%	-0.97%
	0.04	$ 961.54	-0.96%	-0.96%
	0.05	$ 952.38	-0.95%	-0.95%
	0.06	$ 943.40	-0.94%	-0.94%
	0.07	$ 934.58	-0.93%	-0.93%
	0.08	$ 925.93	-0.93%	-0.93%
	0.09	$ 917.43	-0.92%	-0.92%

See Excel spreadsheet econ422Bonds.xls on class notes page

Bond Price Sensitivity to Interest Rates

- T year zero coupon bond: $P = FV(1+r)^{-T}$

$$\frac{dP}{dr} = \frac{d}{dr} FV(1+r)^{-T}$$
$$= -T \cdot FV(1+r)^{-T-1} = -T \cdot FV(1+r)^{-T}(1+r)^{-1}$$
$$= -T \cdot P(1+r)^{-1}$$
$$\Rightarrow \frac{dP}{P} = -T \cdot (1+r)^{-1} dr$$

ΔP/P for T year zero ≈ - maturity × ΔP/P for 1 year zero!

Note: Duration of T year zero = time to maturity = T

Bond Price Sensitivity to Interest Rates

Bond Maturity	Yield	Price	dP/P	dP/P = -T/(1+r) × dr
10	0.01	$ 905.29		
	0.02	$ 820.35	-9.38%	-9.80%
	0.03	$ 744.09	-9.30%	-9.71%
	0.04	$ 675.56	-9.21%	-9.62%
	0.05	$ 613.91	-9.13%	-9.52%
	0.06	$ 558.39	-9.04%	-9.43%
	0.07	$ 508.35	-8.96%	-9.35%
	0.08	$ 463.19	-8.88%	-9.26%
	0.09	$ 422.41	-8.80%	-9.17%
	0.1	$ 385.54	-8.73%	-9.09%

Coupon Bonds

- Coupon bonds have a Face Value (e.g. $1,000 or $10,000)
- Coupon bonds make periodic coupon interest payments based on the coupon rate, Face Value, and payment frequency
- The Face Value is paid at maturity

Coupon Payments

General formula for coupon payments:

Coupon payment =
[Annual Coupon rate*Face Value]/Coupon frequency per year

Determining Coupon Bond Value

Consider the following contractual terms of a bond:

- Face Value = $1,000
- 3 years to maturity
- Coupon rate = 7%
- Annual coupon payment = Coupon rate * Face Value = $70
- Relevant discount rate is 5%

Cash flow from coupon bond:

Year 1	7% * $1,000 = $70
Year 2	$70
Year 3	$70 + $1,000

Determining Coupon Bond Value Cont.

$PV = C/(1+r) + C/(1+r)^2 + (C+FV)/(1+r)^3$

$PV = 70/(1+0.05) + 70/(1+0.05)^2 + (70+1000)/(1+0.05)^3$

$PV = 66.667 + 63.492 + (60.469 + 863.838) = \$1054.465$

Short-cut: Coupon Bond = Finite Annuity plus discounted Face Value

$= [C/r]\ [1 - 1/(1+r)^T] + FV/(1+r)^T$

$= C^*PVA(r, T) + FV/(1+r)^T$

$= [70/0.05]\ [1-1/(1.05)^3] + 1000/(1.05)^3 = \$1054.465$

Example: Coupon Bond with Semi-Annual Coupons

Considering previous coupon bond, suppose the coupon frequency is semi-annual or twice per year:

- Face Value = $1,000
- 3 years to maturity
- Annual coupon rate = 7%
- Semi-Annual coupon payment = (Annual coupon rate/2)*Face Value
- Relevant discount rate is 5%

Cash flow:

Date 0.5	(7% * $1,000)/2 = $35
Date 1	$35 (Year 1 cash flows = $35 + $35 = $70)
Date 1.5	$35
Date 2	$35 (Year 2 cash flows = $35 + $35 = $70)
Date 2.5	$35
Date 3	$35 + $1,000 (Year 3 cash flows = $35 + $35 +$1,000 = $1,070)

Example continued

$$PV = C/(1+r/2)^1 + C/(1+r/2)^2 + C/(1+r/2)^3 + C/(1+r/2)^4 + C/(1+r/2)^5 + (C+FV)/(1+r/2)^6$$

$$PV = 35/(1+0.05/2)^1 + 35/(1+0.05/2)^2 + 35/(1+0.05/2)^3 + 35/(1+0.05/2)^4 + 35/(1+0.05/2)^5 + (35+1000)/(1+0.05/2)^6$$

$= \$1055.081$

Short Cut Formula

Coupon Bond Price = Finite Annuity plus discounted Face Value

$$= \left[\frac{C}{\frac{r}{n}}\right] \times \left[1 - \left(\frac{1}{1+\frac{r}{n}}\right)^{nT}\right] + \frac{FV}{\left(1+\frac{r}{n}\right)^{nT}}$$

$$= C \times PVA\left(\frac{r}{n}, nT\right) + \frac{FV}{\left(1+\frac{r}{n}\right)^{nT}}$$

where C = periodic coupon payment, n is the number of compounding periods per year, T is the maturity date in years.

Example Continued

P = [35/(0.05/2)] [1 - 1/(1 + 0.05/2)^6]
+ 1000/(1+0.05)^6

= $1055.081

Price vs. yield: Coupon Bond

Face value	Maturity	Coupon	Coupon $	Yield	Yield/2	Price
$ 1,000.00	3	0.05	$ 25.00	0.01	0.005	$ 1,117.93
				0.02	0.01	$ 1,086.93
				0.03	0.015	$ 1,056.97
				0.04	0.02	$ 1,028.01
				0.05	0.025	$ 1,000.00
				0.06	0.03	$ 972.91
				0.07	0.035	$ 946.71

If annual coupon rate = yield (r) then
bond price (PV) = face value

Bond Price Conventions: Coupon Bonds

- Price is often quoted as a percentage of "par value" = "face value"; e.g., P = 101.5 => PV is 101.5% of par value. If par value is \$1000, then PV = \$1015
- If coupon rate = yield (r), PV = Face Value and bond sells at 100% of par => P=100 and PV = 1000
- If coupon rate > yield (r), PV > Face Value and bond sells at more than 100% of par (sells at a premium) => P > 100 and PV > 1000
- If coupon rate < yield (r), PV < Face Value and bond sells at less than 100% of par (sells at a discount) => P < 100 and PV < 1000.

Coupon Bond Price Sensitivity to Interest Rates

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T} + \frac{FV}{(1+r)^T}$$

= portfolio of zero coupon bonds

$$\frac{d}{dr}P = \frac{d}{dr}\frac{C}{1+r} + \frac{d}{dr}\frac{C}{(1+r)^2} + \cdots + \frac{d}{dr}\frac{C}{(1+r)^T} + \frac{d}{dr}\frac{FV}{(1+r)^T}$$

For zero coupon bond with maturity M: $\frac{d}{dr}P = -M \times P \times \frac{1}{1+r}$

$$\frac{d}{dr}P = \frac{-1}{1+r}\left[\frac{1\times C}{1+r} + \frac{2\times C}{(1+r)^2} + \cdots + \frac{T\times C}{(1+r)^T} + \frac{T\times FV}{(1+r)^T}\right].$$

Coupon Bond Price Sensitivity to Interest Rates

$$\frac{dP}{P} = \frac{-1}{1+r}\left[w_1 + 2w_2 + \cdots + Tw_T\right]$$

where $w_k = \left(\frac{CP_k}{P}\right)$ $k = 1,\ldots,T-1$, $w_T = \left(\frac{(C+FV)P_k}{P}\right)$

$P_k = \frac{1}{(1+r)^k}$, $P =$ price of coupon bond

Duration of coupon bond = weighted average of the timing of the coupon payments

Yield to Maturity

- Calculation of bond price assumes relevant discount rate
- Bonds trade in secondary market in terms of price
- Price of bonds determined by intersection of supply and demand
- Given price of bond, we can determine the implicit discount rate or yield from setting our present value calculation = market price

Yield to Maturity Defined

A bond's ***yield to maturity*** (YTM or yield) is the discount rate that equates the present value of the bond's promised cash flow to its market price:

r = Yield to Maturity: $P_0 = \Sigma C/(1+r)^t + FV/(1+r)^T \quad t = 1, \ldots, T$

Equivalently, find r such that

$\Sigma C/(1+r)^t + FV/(1+r)^T - P_0 = 0$

Calculating Yield to Maturity

- For T = 1 or T = 2 you can solve using simple algebra
- For T = 2 you need to use the formula for the solution to a quadratic equation
- For T > 2 utilize numerical methods: plug & chug! (spreadsheets are a great tool!)

Yield to Maturity Problems

1. Suppose a 2 year zero (zero coupon bond) is quoted at P = $90.70295 (i.e., FV = 100). What is the Yield to Maturity?

2. Suppose the 2-year is not a zero; i.e., it pays an annual coupon of 4%, is quoted at $98.14. Find the Yield to Maturity.

Treasury Quote Data

- New Treasury securities are sold at auction through the Federal Reserve (www.treasurydirect.gov)
- Existing treasury securities are sold over-the-counter (OTC) through individual dealers.
- In 2002 bond dealers are required to report bond transactions to the Transaction Report and Compliance Engine (TRACE). See www.nasdbondinfo.com or the Wall Street Journal website (www.wsj.com)
- Yahoo! Finance (http://finance.yahoo.com/bonds)

Types of Treasury Securities

- U.S. government Treasury issues are exempt from state taxes but not Federal taxes
  - After tax yield: $(1 - t)*r$, t = tax rate
- State and local government issues are called municipal bonds (munis)
  - Exempt from both state and federal taxes
  - Yields are typically lower than Treasury issues

Treasury Bonds and Notes: Accrued Interest

- Buyer pays and seller receives accrued interest.
- Accrued interest assumes that interest is earned continually (although paid only every six months)
- Accrued interest=((#days since last pmt)/(number of days in 6 month interval))*semi-annual coupon pmt

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Accrued Interest: Example

- Quote: (9/23/03 data)

Rate	maturity Mo/Yr	Asked	Asked Yield
4.25	**Aug 13n**	**100.3175**	**7.11**

- Clean price: Pay 100.375% of face value or \$1003.175 plus accrued interest
- **AI = (0.0425*1000/2)*(39/184)=\$4.50**
  Assumes that bond is bought 9/23/03
  Last coupon 8/15/03, next 2/15/04

• Dirty Price: Clean Price + Accrued Interest = 1003.175 +4.50 = 1007. 675

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Corporate Bonds

- Corporations use bonds (debt) to finance operations
- Debt is not an ownership interest in firm
- Corporation's payment of interest on debt is a cost of doing business and is tax deductible (dividends paid are not tax deductible)
- Unpaid debt is a liability to the firm. If it is not paid, creditors can claim assets of firm

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Bond Ratings

- Firms frequently pay to have their debt rated
  - Assessment of the credit worthiness of firm
- Leading bond rating firms
  - Moodys, Standard & Poor's, Fitch
- Ratings assess possibility of default
  - Default risk is hot topic these days!

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Bond Ratings

	Investment Quality Rating				Low-Quality, Speculative, and/or "Junk" Bond Ratings				
	High Grade		Medium Grade		Low Grade		Very Low Grade		
S&P	AAA	AA	A	BBB	BB	B	CCC	CC	C
Moody's	Aaa	Aa	A	Baa	Ba	B	Caa	Ca	C

Market prices and yields are directly influenced by credit ratings:

The lower the credit rating, the lower the price and higher the yield

Credit spread: = corporate bond yield – U.S. Treasury bond yield

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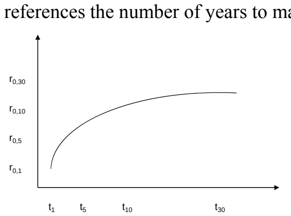
'Term Structure' of Interest Rates

Refers to a set of interest rates observed at one point in time for varying maturities:

$$\{r_{0,1}, r_{0,2}, r_{0,3}, r_{0,4}, r_{0,5}, r_{0,10}, r_{0,15}, r_{0,20}, r_{0,30}\}$$

where the first subscript references today and the second references the number of years to maturity



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Spot Rates

Spot rates are derived from zero coupon U.S. treasury bond prices

$$P_{0,1} = \frac{1,000}{(1+r_{0,1})} \Rightarrow r_{0,1} = \left(\frac{1,000}{P_{0,1}}\right) - 1$$

$$P_{0,2} = \frac{1,000}{(1+r_{0,2})^2} \Rightarrow r_{0,2} = \left(\frac{1,000}{P_{0,2}}\right)^{1/2} - 1$$

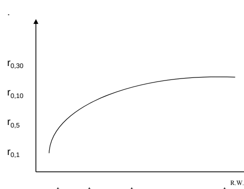
$$P_{0,T} = \frac{1,000}{(1+r_{0,T})^T} \Rightarrow r_{0,T} = \left(\frac{1,000}{P_{0,T}}\right)^{1/T} - 1$$

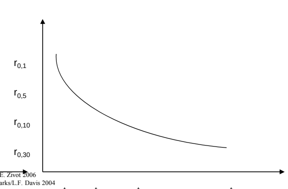
In Class Example

- Zero coupon bond prices of maturity 1, 2 and 3 years: P_1 = \$909.09, P_2 = \$900.90, P_3 = \$892.86. Face value = \$1,000.
- Derive spot rates $r_{0,1}$, $r_{0,2}$, and $r_{0,3}$
- Plot term structure

Term Structure/Shape of the Yield Curve

- The term structure can have various shapes
- Increasing rates are most common-upward sloping yield curve, but the plot can show a decrease or a hump
- Yield curve is representative of the term structure. The yield curve plots the going rates for various maturity Government securities—Tbills, Tnotes and Tbonds
- ***See Yahoo! Bond page for current yield curve.***

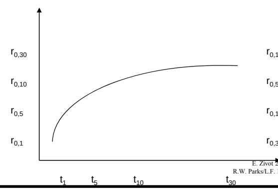


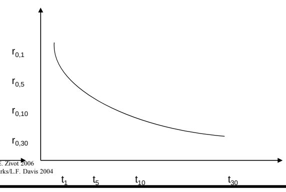


Term Structure/Shape of the Yield Curve

Why does the term structure change?

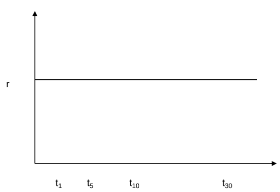
Expectation Hypothesis: The term structure rises or falls because the market expects interest rates in the future to be higher or lower.



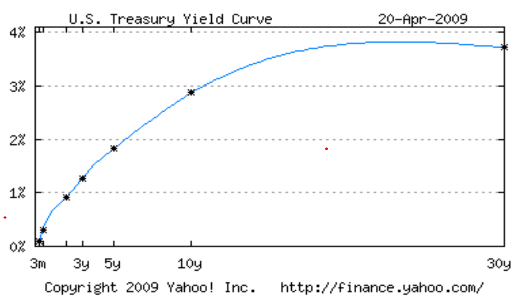


Constant Discount Rate: Flat 'Term Structure' of Interest Rates

Note: In the previous present value examples we were implicitly assuming a flat term structure: $r_{0,1} = r_{0,2} = \ldots = r_{0,T} = r$. This simplifies calculations, but may be highly unrealistic in practice.



Current Term Structure



PV Calculations with the Term Structure of Interest Rates

$$PV = C_0 + \frac{C_1}{1+r_{0,1}} + \frac{C_2}{\left(1+r_{0,2}\right)^2} + \cdots + \frac{C_T}{\left(1+r_{0,T}\right)^T}$$

Note: If term structure is flat then $r_{0,1} = r_{0,2} = \ldots = r_{0,T} = r$ and we get the simple formula

$$PV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{\left(1+r\right)^2} + \cdots + \frac{C_T}{\left(1+r\right)^T}$$

Arbitrage

- *Arbitrage Opportunity*: A riskless trading strategy that costs nothing to implement but generates a positive profit (free money machine!)
- In well functioning financial markets, there should not be arbitrage opportunities. If arbitrage opportunities occur, then market prices adjust until the arbitrage opportunities disappear. This should happen quickly.

Example: Cross Listed Stocks

- IBM sells on NYSE for $101
- IBM sells on NASDAQ for $100
- Arbitrage: buy low, sell high
  - Assume no transactions costs
  - Short sell IBM on NYSE for $101
  - Use proceeds to buy IBM on NASDAQ for $100
  - Close out short position on NYSE
  - Cost: 0, Profit: $1

Example Cross Listed Stocks

- Q: What happens to the price of IBM in the NYSE and NASDAQ in well functioning markets?
- A: They converge to the same value to eliminate the arbitrage opportunity

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Example Cross Listed Stocks

The existence of an arbitrage opportunity creates trades that cause the price of IBM in the NYSE to fall, and the price of IBM in the NASDAQ to rise until the arbitrage opportunity disappears. When there are no arbitrage opportunities, the price of IBM in both markets must be the same. This is the ***Law of One Price***.

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Example: Zero Coupon Bond Prices

- Suppose the 2 yr. spot rate is 0.05 ($r_{0,2} = 0.05$). That is, you can borrow and lend risklessly for 2 years at an annual rate of 0.05.
- The no-arbitrage price of a 2 yr. zero coupon bond with face value \$100 is $P_0 = \$100/(1.05)^2 = \$90.70$
- Now suppose that the current market price of the 2 yr. zero is \$85. Then there is an arbitrage opportunity.

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Example: Zero Coupon Bond Prices

Time 0 Transaction	Cash Flow	Time 2 Transaction	Cash Flow
Borrow \$85 at $r_{0,2}$=0.05	+\$85	Repay loan	-\$85(1.05)^2 = -\$93.71
Buy 2yr zero	-\$85	Receive FV	\$100
Net cash flow:	\$0	Net cash flow:	\$6.29

Free money!

Example: Zero Coupon Bond Prices

- Existence of arbitrage opportunity causes the demand for the under-priced 2 yr. zero to increase, which causes the price of the 2 yr. zero to increase
- Price will increase until the arbitrage opportunities disappear; that is, the price will increase until it equals the no-arbitrage price of \$90.70.

In Class Example

- Suppose the 1 yr spot rate, $r_{0,1}$, is 0.20. You can borrow and lend for 1 year at this rate.
- Suppose the 2 yr spot rate, $r_{0,2}$, is 0.05. You can borrow and lend for 2 years at this annual rate.
- Show that there is an arbitrage opportunity.

No Arbitrage Relationship Between Spot Rates in a World of Certainty

- Consider 2 investment strategies
  - Invest for 2 years at $r_{0,2}$.
    - \$1 grows to \$1$(1+r_{0,2})^2$
  - Invest for 1 year at $r_{0,1}$, and then roll over the investment for another year at $r_{1,2}$. The spot rate $r_{1,2}$ is the 1 yr spot rate between years 1 and 2 which is assumed to be known.
    - \$1 grows to \$1$(1+r_{0,1})(1+r_{1,2})$

No arbitrage condition in a world of certainty

$$(1 + r_{0,2})^2 = (1+r_{0,1})(1+r_{1,2})$$

That is, in a world of certainty, the return from holding a 2-year bond should be exactly the same as the return from rolling over 2 1-year bonds.

You can solve for $r_{0,2}$:

$$r_{0,2} = ((1+r_{0,1})(1+r_{1,2}))^{1/2} - 1$$

Term Structure Example

Suppose you know with *certainty* the following sequence of one year rates:

$r_{0,1} = 7\%$
$r_{1,2} = 9\%$

Calculate the no-arbitrage two year rate.

By our arbitrage condition:

$$r_{0,2} = [(1+r_{0,1})(1+r_{1,2})]^{1/2} - 1$$
$$r_{0,2} = [(1 + 0.07)(1 + 0.09)]^{1/2} - 1$$
$$r_{0,2} = 0.079954 = 7.9954\%$$

No Arbitrage Condition in a World of Certainty

A similar relationship will hold for the T[th] period interest rate:

$$r_{0,T} = ((1+r_{0,1})(1+r_{1,2})\ldots(1+r_{T-1,T}))^{1/T} - 1$$

T year spot rate is a geometric average of 1 year spot rates.

Forward Interest Rates

- In our previous example, $r_{1,2}$ is known with certainty.
- In reality, we do not know with certainty any future interest rates; i.e., $r_{t-1,t}$ for $t > 1$ is unknown
- We can use today's term structure, however, to infer something about future rates
- Future rates inferred from today's spot rates are called *implied forward rates*

Implied Forward Rates and Term Structure

Recall our No Arbitrage Condition in a world of certainty:

$$(1+r_{0,2})^2 = (1+r_{0,1})(1+r_{1,2})$$

If we don't know $r_{1,2}$, then we can infer what the market thinks $r_{1,2}$ will be. This is the ***implied forward rate*** $f_{1,2}$. It is the one period rate implied by the no arbitrage condition:

$$(1+r_{0,2})^2 = (1+r_{0,1})(1+\boldsymbol{f_{1,2}})$$

Solving for $f_{1,2}$:

$$(1+f_{1,2}) = [(1+r_{0,2})^2/(1+r_{0,1})]$$

$$f_{1,2} = [(1+r_{0,2})^2/(1+r_{0,1})] - 1$$

The implied forward rate is the additional interest that you earn by investing for two years rather than one.

Alternative Derivation

$$P_{0,1} = \frac{FV}{1+r_{0,1}},\ P_{0,2} = \frac{FV}{\left(1+r_{0,2}\right)^2}$$

$$\frac{P_{0,1}}{P_{0,2}} = \frac{FV/1+r_{0,1}}{FV/\left(1+r_{0,2}\right)^2} = \frac{\left(1+r_{0,2}\right)^2}{1+r_{0,1}}$$

$$\Rightarrow \frac{P_{0,1}}{P_{0,2}} - 1 = f_{1,2}$$

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Forward Rates and Term Structure

Consider determining $f_{2,3}$, the 1 period rate between years 2 and 3.

The no-arbitrage condition defining the forward rate is

$$(1+ r_{0,3})^3 = (1+r_{0,2})^2(1+\boldsymbol{f_{2,3}})$$

Solving for $f_{2,3}$:

$$(1+f_{2,3}) = [(1+ r_{0,3})^3 /(1+r_{0,2})^2]$$

$$\Rightarrow \quad f_{2,3} = [(1+ r_{0,3})^3 /(1+r_{0,2})^2] - 1$$

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Alternative Derivation

$$P_{0,2} = \frac{FV}{\left(1+r_{0,2}\right)^2},\ P_{0,3} = \frac{FV}{\left(1+r_{0,3}\right)^3}$$

$$\frac{P_{0,2}}{P_{0,3}} - 1 = \frac{\left(1+r_{0,3}\right)^3}{\left(1+r_{0,2}\right)^2} - 1 = f_{2,3}$$

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R.W. Parks/L.F. Davis 2004

Forward Rates and Term Structure

The following general formula allows us to determine the forward rate for any future period between periods t-1 and t:

$$(1 + f_{t-1,t}) = [(1 + r_{0,t})^t /(1 + r_{0,t-1})^{t-1}]$$

$$\Rightarrow \quad f_{t-1,t} = [(1 + r_{0,t})^t /(1 + r_{0,t-1})^{t-1}] - 1$$

E. Zivot 2006
R.W. Parks/L.F. Davis 2004

Examples

- Flat term structure

$r_{0,1} = 2\%$, $r_{0,2} = 2\%$, $r_{0,3} = 2\%$,

$$f_{1,2} = [(1+ r_{0,2})^2 /(1+r_{0,1})] – 1 = [(1.02)^2/(1.02)] – 1 = 0.02 = 2\%$$

$$f_{2,3} = [(1+ r_{0,3})^3 /(1+r_{0,2})^2] – 1 = [(1.02)^3/(1.02)^2] – 1 = 0.02 = 2\%$$

E. Zivot 2006
R.W. Parks/L.F. Davis 2004

Examples

- Upward sloping term structure

$r_{0,1} = 2\%$, $r_{0,2} = 3\%$, $r_{0,3} = 4\%$,

$$f_{1,2} = [(1+r_{0,2})^2 /(1+r_{0,1})] – 1 = [(1.03)^2/(1.02)] – 1 = 0.04 = 4\%$$

$$f_{2,3} = [(1+ r_{0,3})^3 /(1+r_{0,2})^2] – 1 = [(1.04)^3/(1.03)^2] – 1 = 0.06 = 6\%$$

E. Zivot 2006
R.W. Parks/L.F. Davis 2004

Examples

- Downward sloping term structure

$r_{0,1} = 4\%$, $r_{0,2} = 3\%$, $r_{0,3} = 2\%$,

$$f_{1,2} = [(1+ r_{0,2})^2 /(1+r_{0,1})] - 1$$
$$= [(1.03)^2/(1.04)] - 1 = 0.02 = 2\%$$

$$f_{2,3} = [(1+ r_{0,3})^3 /(1+r_{0,2})^2] - 1$$
$$= [(1.02)^3/(1.03)^2] - 1 = 0.0003 = 0.3\%$$

Forward Rates as Forecasts of Future Spot Rates

The forward rate can be viewed as a forecast of the future spot rate of interest:

$$r_{t-1,t} = f_{t-1,t} + \varepsilon_t$$

ε_t = forecast error

That is, the implied forward rate $f_{t-1,t}$ is the current market forecast of the future spot rate $r_{t-1,t}$

Question: How good is this forecast?

Returning to the Expectations Hypothesis

Expectation Hypothesis: The term structure rises or falls because the market expects interest rates in the future to be higher or lower, where:

$$r_{t-1,t} = f_{t-1,t} + \varepsilon_t$$

- If the Expectations Hypothesis is true, then the forecast error has mean zero ($E(\varepsilon_t) = 0$) and is uncorrelated with the forward rate. In other words, the forecast error is white noise or has no systematic component that can improve the forecast of the future spot rate.

Implication of the Expectations Hypothesis

If the Expectations Hypothesis is true, then investing in a succession of short term bonds is the same, on average, as investing in long term bonds.

Expectations Hypothesis of the Term Structure

- Empirically, Eugene Fama of the University of Chicago found that expectation hypothesis is not an exact depiction of real world; i.e, when forward rates exceed the spot rates, future spot rates rise ***but by less than predicted*** by the theory.
- Implication: investing in long term bond tends to give higher return than rolling over series of short term bonds.
- The failure of the expectations hypothesis may be due to risk averse behavior of investors

Modifying the Expectations Hypothesis

- ***Liquidity Preference Theory***: Expectation Hypothesis omits fundamental notion that there is risk associated with longer term investments. A long term treasury bond is more risky than a short term treasury bill—longer time over which interest rate changes can occur impacting price.
- Liquidity preference suggests that ($r_{t-1,t}$ - $f_{t-1,t}$) may not be zero if investors require additional interest to account for additional risk to hold longer term investments. That is, the higher risk of longer term investments makes spot rates for longer maturities higher than spot rates for shorter maturities.

Modifying the Expectations Hypothesis

- The liquidity preference theory suggests an upward sloping term structure, which implies $f_{t-1,t} - r_{t-1,t} > 0$. The difference between the higher forward rate and the spot rate is termed the ***liquidity premium***.