Random Variable & Probability Review

Econ 422: Investment, Capital & Finance

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Why Probability Concepts Matter to Finance

- Financial values based on future cash flows
- Intertemporal decision-making
- Future cash flows uncertain
- Probability theory helps us to understand the set of possible outcomes and the likelihood of each occurrence

Random Events

- Outcomes associated with a random experiment
- Outcome associated with random “states of the world”
  - i.e., “carrying an umbrella” is an outcome associated with the state of the world “raining”

Sample Space

- The <u>*sample space*</u> for a random experiment is the set of all possible events.
- We denote the sample space by Ω

<u>Example</u>: Consider an experiment comprised of a single toss of a fair coin. The possible events or outcomes associated with this experiment are: Heads, Tails.

$\Omega = \{H, T\}$

Random Variable

- A random variable is a function that associates a number with each event in the sample space Ω.

- Example: X is a random variable associating the number of possible heads in two consecutive tosses of a fair coin.

  For Ω = {HH, HT, TH, TT}

  X = 0 {TT}

  X = 1 {HT} or {TH}

  X = 2 {HH}

Random Variable Distribution

The Distribution of a Random Variable specifies the following:

1. The set of possible values that the random variable can assume.

2. A function or list to associate a probability to each possible value.

Random Variable Type

- Discrete
  - Countable set of possible values for the random variable
  - Recall the coin toss example

- Continuous
  - The set of possible values for the random variable is not countable
  - The random variable can take values within a continuous interval

Example: *Discrete* RV Probability Distribution

Experiment: Roll of fair die.

Random variable: Let RV X = the number of dots on the top die face that results following the die toss.

RV Distribution:

Possible values of X	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Example: *Discrete* RV Probability Distribution

- Let RV X denote the price of Johnson & Johnson (JNJ) stock tomorrow
- Suppose unrealistically that JNJ can take on only four different values tomorrow
- RV distribution:

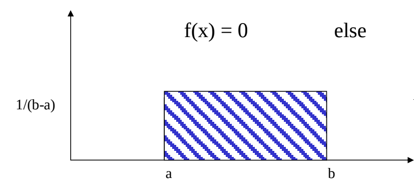
Possible values of x	\$40	\$45	\$50	\$60
Probability	0.25	0.30	0.40	0.05

Example: *Uniform* Continuous RV Probability Distribution

- A uniform RV X is a RV that takes on values in the interval $[a, b]$
- All values of X are equally likely
- The density function or probability distribution function for X is given by:

$$f(x) = 1/(b - a) \qquad \text{for } x \in [a, b]$$

$$f(x) = 0 \qquad \text{else}$$



Note: Area of shaded region = (b-a) * [1/(b-a)] = 1

Example: Continuous *Normal* RV Probability Distribution

- Let RV X denote the total rate of return on stock ABC over the next month

$$X = (P_{t+1} - P_t)/P_t + D_{t+1}/P_t$$

- Assume that X is Normal with mean μ and standard deviation σ
- X takes on the range of all real numbers, from $-\infty$ to $+\infty$
- The probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

The Probability Distribution Function

- Provides for both discrete and continuous random variables a compact representation of the random variable distribution
- Cumulates over the density function:

$$F_x(y) = P(X \leq y) \qquad \text{for all values of y}$$

The Probability Distribution Function

Example:

1. Die toss:

$F_x(y) = P(x \leq y)$ for all values of $y \in \{1,2,3,4,5,6\}$

$F_x(1) = P(x \leq 1) = 1/6$

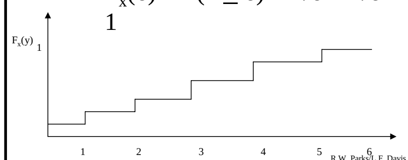
$F_x(2) = P(x \leq 2) = 1/6 + 1/6 = 2/6 = 1/3$

$F_x(3) = P(x \leq 3) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$

$F_x(4) = P(x \leq 4) = 1/6 + 1/6 + 1/6 + 1/6 = 4/6 = 2/3$

$F_x(5) = P(x \leq 5) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 5/6$

$F_x(6) = P(x \leq 6) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 6/6 = 1$



How do we describe the distribution of a random variable?

- The various attributes of a random variable are described by 'the moments' or parameters of the random variable distribution
- Two moments are especially useful:
  - First Moment = mean or expected value
  - Second Moment = variance
- These first two moments describe the central tendency and spread of a random variable distribution

Central Tendency or Measure of Location

- Mean or expected value
- Median = center of outcome values
- Mode = the value which occurs with the greatest frequency

Expected Value for Discrete Random Variable

The expected value or central tendency of a discrete random variable is the probability-weighted sum of all possible outcomes:

$E(X) = \Sigma x_i \, P(X = x_i)$ for all possible values of x

Example: The toss of a fair die once.

$E(X) = 1*1/6 + 2*1/6 + 3*1/6 + 4*1/6 + 5*1/6 + 6*1/6$

$= 1/6 + 1/3 + ½ + 2/3 + 5/6 + 1 = 3 ½$

Expected Value for Discrete Random Variable

Example: The toss of two fair die once (Craps)
Random variable x = value of the two die
36 Possible outcomes:

{(1,1), (1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (1,5), (5,1), (1,6), (6,1), (2,2), (2,3), (3,2), (2,4), (4,2), (2,5), (5,2), (2,6), (6,2), (3,3), (3,4), (4,3), (3,5), (5,3), (3,6), (6,3), (4,4), (4,5), (5,4), (4,6), (6,4), (5,5), (5,6), (6,5), (6,6)}

Possible values of x		2	3	4	5	6	7	8	9	10	11	12
Probability		1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

E(X) = 2*1/36 + 3*2/36+ 4*3/36 + 5*4/36+ 6*5/36 + 7*6/36 + 8* 5/36 + 9* 4/36 + 10* 3/36 + 11*2/36 + 12*1/36

E(X) = (2+6+12+20+30+42+ 40+ 36+ 30+ 22+12)/36 = 282/36 = **7 5/6**

Expected Value for a Continuous Random Variable

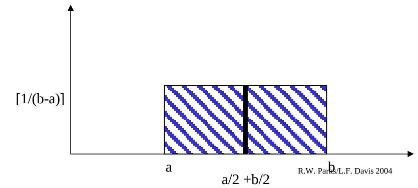
$E(X) = \int x\, f(x)\, dx$ x bounded between $-\infty$ and $+\infty$

Example: Uniform distribution

$E(X) = \int x\, 1/(b - a)\, dx = 1/(b - a) \int x\, dx$ x bounded between a and b

$= ½\, [1/(b - a)]\, (b^2 - a^2)$

$= ½\, [1/(b - a)](b - a)(b + a) = ½\,(b + a) = a/2 + b/2$



Helpful Algebraic Properties of Expectations

Let a and b be constants and x is a random variable:

1. $E(a) = a$
2. $E(a + x) = a + E(x)$
3. $E(bx) = bE(x)$
4. $E(a + bx) = a + bE(x)$
5. $E(x_1 + x_2 + \ldots + x_n) = E(x_1) + E(x_2) + \ldots + E(x_n)$
6. $E(\Sigma a x_i) = a\, \Sigma E(x_i)$ for i =1, . . ., n

Example

- Let R denote the random return on Microsoft stock over the next year and assume $R \sim N(0.10, 0.20^2)$
- Let W_0 = \$10,000 denote the initial investment in Microsoft
- Q: What is the expected wealth at the end of the year? That is, what is $E[W_1] = E[W_0(1+R)]$?

Measures of Spread or Dispersion

- Variance
  - Squared deviation from mean value
- Standard Deviation
  - Square root of variance
  - In same units as the random variable
  - Typical size of deviation from mean value

Variance and Standard Deviation

- Variance (Second Moment)

$V(x) = E[(x-E(x))^2] \geq 0$

$V(x) = \Sigma(x_i - E(x_i))^2 P(x = x_i)$ for all possible values of x

- Standard Deviation (on same scale as x)

$SD(x) = (V(x))^{1/2} = (E[(x-E(x))^2])^{1/2}$

- Alternative form for Variance:

$$\begin{aligned} V(x) &= E[x^2 - 2xE(x) + E(x)^2] \\ &= E[x^2] - 2E(x)^2 + E(x)^2 \\ &= E[x^2] - E(x)^2 \geq 0 \end{aligned}$$

Helpful Algebraic Properties of Variance

- Let *a* and *b* be constants and let X be a random variable:

1. $V(a) = 0$
2. $V(bX) = b^2V(X)$
3. $V(a + bX) = b^2V(X)$

Example Continued

- Let R denote the random return on Microsoft stock over the next year and assume $R \sim N(0.10, 0.20^2)$
- Let W_0 = \$10,000 denote the initial investment in Microsoft
- What is the standard deviation of wealth at the end of the year? That is, what is $SD[W_1] = SD[W_0(1+R)]$?
- Sketch the distribution of W_1
- Compute $Pr(W_1 < 10,000)$

In Class Problem #1

1. For the random variable x presented below, compute E(x)
2. Compute V(x)
3. What is the probability distribution of y = (2x-1)? Complete the table below.
4. Compute E(y)
5. Compute V(y)
6. Compute $E(4x^2-4x+1)$

x outcomes	-1	0	1	2
Probs	0.25	0.10	0.25	0.40
y outcomes				
y^2				

Joint Distribution: Example

Suppose two random variables are jointly distributed. Random variable X is Firm X's next period stock price and random variable Y is Firm Y's next period stock price. There are 9 possible states of the world next period distinguished by the x and y values. The joint probability distribution is given by:

		Y			Marginal
		$20	$25	$35	Dist. X
X	$20	0.05	0.10	0.20	0.35
	$25	0.20	0.05	0.10	0.35
	$35	0.10	0.10	0.10	0.30
Marginal Dist. Y		0.35	0.25	0.40	1.00

Marginal Distributions

- The marginal probability distribution for X shows for each X value the probability of this occurrence without regard to the value of Y. You can determine the marginal distribution of X by compressing the table in the Y direction, i.e., by adding the row probabilities.

- The marginal probability distribution for Y shows for each Y value the probability of this occurrence without regard to the value of X. You can determine the marginal distribution of Y by compressing the table in the X direction, i.e., by adding the column probabilities.

Covariance

- For the random variables X and Y, we define the covariance between X and Y as:

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$
$$= E[XY – 2E(X)E(Y) + E(X)E(Y)]$$
$$= E[XY] – E(X)\,E(Y)$$

- Covariance measures direction but not magnitude of linear association

- *Note: Variance is the covariance of random variable x with itself*

$$Cov(X,X) = var(X)$$

Correlation Coefficient

- Scaled or unit-less measure of covariance:

$$\text{Corr}(X,Y) = \rho_{xy} = [\text{Cov}(X,Y)/ (\text{SD}(X)*\text{SD}(Y))]$$

$$-1 \leq \rho_{xy} \leq 1$$

- Measures both direction and strength of linear association
- Note: $\text{Cov}(X,Y) = \rho_{xy} * \text{SD}(X)*\text{SD}(Y)$

Variance of a Sum of Random Variables

- $V(X + Y) = V(X) + V(Y) + 2\,\text{Cov}(X,Y)$
  $= V(X) + V(Y) + 2\,\rho_{xy} * \text{SD}(X)*\text{SD}(Y)$
- $\text{Cov}(X,Y) = E(XY) – E(X)* E(Y)$
- $V(X + Y) = V(X) + V(Y) + 2(E(XY) – E(X) *E(Y))$
- $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab*\text{Cov}(X,Y)$
  $= a^2V(X) + b^2V(Y) + 2ab*\text{SD}(X)\text{SD}(Y)* \rho_{xy}$

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X	$20	0.05	0.10	0.20	0.35
	$25	0.20	0.05	0.10	0.35
	$35	0.10	0.10	0.10	0.30
Marginal Dist. Y		0.35	0.25	0.40	1.00

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Example: Portfolio X+ Y

Suppose you own one share of each Firm X and Firm Y stock. Your portfolio value next period is the random variable X + Y.

From the joint distribution there are only 9 possible states of the world next period

The possible portfolio outcomes and the respective portfolio values are provided on the left:

State	X + Y	Probability
1	$40	0.05
2	$45	0.10
3	$55	0.20
4	$45	0.20
5	$50	0.05
6	$60	0.10
7	$55	0.10
8	$60	0.10
9	$70	0.10
		1.00

X + Y	Probability
$40	0.05
$45	0.30
$50	0.05
$55	0.30
$60	0.20
$70	0.10
	1.00

Combining duplicate portfolio values X+Y (some states of the world result in the same portfolio value) the portfolio value distribution is provided on the right.

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Portfolio Distribution

- What is E[X+Y]?
- What is cov(X,Y)?
- What is corr(X,Y)?
- What is V(X+Y)?

- See Excel spreadsheet for computations

In Class Problem #2

Suppose an investor can choose between two assets 1 and 2. The probability distributions for the rates of return for each asset are provided below.

a. Compute $E(r_1)$ and $E(r_2)$
b. Compute $V(r_1)$ and $V(r_2)$
c. Based on expected return and the variance of return, which asset do you prefer and why? On what other basis might you prefer one asset to another?
d. Assume you hold a portfolio with 50% held in asset 1 and 50% held in asset 2. What is the expected return and variance of your portfolio?

r_1 outcomes	5%	8%	12%	15%
r_2 outcomes	15%	12%	8%	5%
Probs	0.25	0.25	0.25	0.25