Present Value Methodology

Econ 422
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Present Value Concept

- Wealth in Fisher Model:
  $W = Y_0 + Y_1/(1+r)$
  The consumer/producer's wealth is their current endowment plus the future endowment discounted back to the present by the rate of interest (rate at which present and future consumption can be exchanged).
- Why do this?
  - Purpose of comparison—apples to apples (temporal) comparison with multiple agents or apples to apples comparison of investment/consumption opportunities
- Uniform method for valuing present and future streams of consumption in order for appropriate decision making by consumer/producer
- Useful concept for valuing multiple period investments and pricing financial instruments

Calculating Present Value

Present value calculations are the *reverse of compound growth calculations:*

Suppose V_0 = a value today (time 0)
r = fixed interest rate (annual)
T = amount of time (years) to future period

The value in T years we calculate as:

$$V_T = V_0 (1+r)^T \qquad \text{(Future Value)}$$

Example

- A $30,000 Certificate of Deposit with 5% annual interest in 10 years will be worth:
- $V^T = V_0 (1 + r)^T = 30{,}000 * (1 + 0.05)^{10} =$
  = $48,866.84
- Note: Computation is easy to do in Excel
  = `30,000 *(1 + 0.05)^10`

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Present Value

In reverse:

$V_0 = V_T / (1+r)^T$ (Present Value)

The present value amount is the future value discounted (divided) by the compounded rate of interest

<u>Example</u>: A $48,866.84 Certificate of Deposit received 10 years from now is worth today:

$V_0 = \$48{,}866.84/(1+0.05)^{10} =$ **$30,000**

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Exam Review

- Be able to calculate present and future values
- For any three of four variables: (V_0, r, T, V_T) you should be able to determine the value of the fourth variable.
- How do changes to r and T impact V_0 and V_T?

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Example: Rule of 70

- Q: How many years, T, will it take for an initial investment of V_0 to double if the annual interest rate is r?
- A: Solve $V_0 (1 + r)^T = 2V_0$
- => $(1 + r)^T = 2$
- => $T\ln(1 + r) = \ln(2)$
- => $T = \ln(2)/\ln(1+r)$
- $= 0.69/\ln(1 + r) \approx 0.70/r$ for r not too big

Present Value of Future Cash Flows

- A cash flow is a sequence of dated cash amounts received (+) or paid (-): $C_0, C_1, \ldots, C_T$
- Cash amounts received are positive; whereas, cash amounts paid are negative
- The present value of a cash flow is the sum of the present values for each element of the cash flow

Discount factors: Intertemporal Price of $1 with constant interest rate r

- $1/(1+r)$ = price of $1 to be received 1 year from today
- $1/(1+r)^2$ = price of $1 to be received 2 years from today
- $1/(1+r)^T$ = price of $1 to be received T years from today

Present Value of a Cash Flow

- $\{C_0, C_1, C_2, \ldots C_T\}$ represents a sequence of cash flows where payment
- C_i is received at time i. Let r = the interest or discount rate.

Q: What is the present value of this cash flow?

A: The present value of the sequence of cash flows is the sum of the present values:

$$PV = C_0 + C_1/(1+r) + C_2/(1+r)^2 + \ldots + C_T/(1+r)^T$$

Summation Notation

$$PV = \sum_{t=0}^{T} \frac{C_t}{(1+r)^t}$$

$$= C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

Example

You receive the following cash payments:

time 0: -$10,000 (Your initial investment)
time 1: $4,000
time 2: $4,000
time 3: $4,000

The discount rate = 0.08 (or 8%)

PV = -$10,000 + $4,000/(1+0.08)
+ $4,000/(1+0.08)² + $4,000/(1+0.08)³

= -$10,000 + $3,703.70 + $3,429.36 + $3,175.33
= $308.39

See econ422PresentValueProblems.xls for Excel calculations

PV Calculations in Excel

Excel function **NPV**:

```
NPV(rate, value1, value2, …, value29)
Rate = per period fixed interest rate
value1 = cash flow in period 1
value 2 = cash flow in period 2
…
value 29 = cash flow in 29th period
```

Note: NPV function does not take account of initial period cash flow!

Present Value Calculation Short-cuts

PERPETUITY:

A perpetuity pays an amount C ***starting next period*** and pays this same constant amount C in each period forever:

$C_1 = C, C_2 = C, C_3 = C, C_4 = C, \ldots.$

$$\text{PV(Perpetuity)} = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_t}{(1+r)^t} + \cdots$$

$$= \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = C \sum_{t=1}^{\infty} \frac{1}{(1+r)^t}$$

PV of Perpetuity

Based on the infinite sum property, we can write PV as:

PV = Initial Term/[1 – Common Ratio]

= C/(1 + r)/[1 - (1/(1 + r))]

= C/r

Initial Term = C/(1 + r)

Common Ratio = 1/(1 + r)

$PV(\text{Perpetuity}) = C/(1 + r) + C/(1 + r)^2 + C/(1 + r)^3 + \ldots + C/(1 + r)^t + \ldots$

Let $a = C/(1 + r)$ = initial term

$x = 1/(1 + r)$ = common ratio

Rewriting:

$$PV = a\,(1 + x + x^2 + x^3 + \ldots) \qquad (1.)$$

Post multiplying by x:

$$PVx = a(x + x^2 + x^3 + \ldots) \qquad (2.)$$

Subtracting (2.) from (1.):

$$PV(1 - x) = a \qquad \rightarrow PV = a/(1 - x)$$

$$PV(1 - 1/(1 + r)) = C/(1 + r)$$

Multiplying through by $(1 + r)$:

$$PV = C/r$$

Example

The *preferred stock* of a secure company will pay the owner of the stock $100/year forever, starting next year.

Q: If the interest rate is 5%, what is the share worth?

A: The share should be worth the value to you as an investor today of the future stream of cash flows.

This share of preferred stock is an example of a perpetuity, such that

PV(preferred stock) = $100/0.05 = $2,000

Example Continued

- Q: What if the interest rate is 10%?
- PV(preferred stock) = $100/0.10 = $1,000
- Notice: That when the interest rate doubled, the present value of the preferred stock decreased by ½.

Example Continued

The preferred stock of a secure company will pay the owner of the stock $100/year forever, starting this year.

Q: If the interest rate is 5%, what is the share worth?

A: The share should be worth the value to you as an investor today of the future stream of cash flows (perpetuity component) plus the $100 received this year.

PV(preferred stock) = $100 + $100/0.05 = $100 + $2,000 = $2,100

GROWING PERPETUITY

Suppose the cash flow starts at amount C at time 1, but grows at a rate of g thereafter, continuing forever:

$C_1 = C$, $C_2 = C\,(1+g)$, $C_3 = C(1+g)^2$, $C_4 = C(1+g)^3$, …

$$\text{PV(Perpetuity)} = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{t-1}}{(1+r)^t} + \cdots$$

$$= C\sum_{t=1}^{\infty}\frac{(1+g)^{t-1}}{(1+r)^t}$$

GROWING PERPETUITY

Based on the infinite sum property, we can write this as:

PV = Initial Term/[1 – Common Ratio]

= C/(1 + r)/[1- ((1 + g)/(1 + r))]
= C/(r - g)

Note: This formula requires r > g.

Example

- Your next year's cash flow or parental stipend will be \$10,000. Your parents have generously agreed to increase the yearly amount to account for increases in cost of living as indexed by the rate of inflation.
- Your parents have established a trust vehicle such that after their death you will continue to receive this cash flow, so effectively this will continue forever.
- Assume the rate of inflation is 3%.
- Assume the market interest rate is 8%.
- Q: What is the value to you today of this parental support?

Answer

This is a growing perpetuity with

C = \$10,000, r = 0.08, g = 0.03

Therefore,

PV = \$10,000/(0.08 – 0.03) = \$200,000

FINITE ANNUITY

A finite annuity will pay a constant amount C starting next period through period T, so that there are T total payments (e.g., financial vehicle that makes finite number of payments based on death of owner or joint death or term certain number of payments, etc.)

$$C_1 = C,\ C_2 = C,\ C_3 = C,\ C_4 = C,\ \ldots.\ C_T = C$$

$$\text{PV(Finite Annuity)} = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T}$$

$$= \sum_{t=1}^{T} \frac{C}{(1+r)^t} = C \cdot \sum_{t=1}^{T} \frac{1}{(1+r)^t}$$

Finite Annuity

Formula Result:

$$PV\ (\text{Finite Annuity}) = C^*(1/r)\ [1 - 1/(1 + r)^T]$$
$$= C^*PVA(r, T)$$

where

$$PVA(r, T) = (1/r)\ [1 - 1/(1+r)^T]$$
$$= \text{PV of annuity that pays \$1 for T years}$$

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Value of Finite Annuity = Difference Between Two Perpetuities

Consider the Finite Annuity cash flow: $C_1 = C, C_2 = C, C_3 = C, C_4 = C, \ldots. C_T = C$

Suppose you want to determine the present value of this future stream of cash.

Recall a perpetuity cash flow (#1):

$C_1 = C, C_2 = C, C_3 = C, C_4 = C, \ldots C_T = C, C_{T+1} = C, \ldots$

From our formula, the value today of this perpetuity = C/r

Consider a second perpetuity (#2) starting at time T+1:

$C_{T+1} = C, C_{T+2} = C, C_{T+3} = C, \ldots$

The value today of this perpetuity starting at T+1:

$= C/r\ [1/(1+r)^T]$ (why?)

Note: The Annuity = Perpetuity #1 – Perpetuity #2

$= C/r - C/r\ [1/(1+r)^T]$

$= C/r\ [1 - 1/(1+r)^T]$

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Alternative Derivation

$PV(\text{Finite Annuity}) = C/(1+r) + C/(1+r)^2 + C/(1+r)^3 + \ldots + C/(1+r)^{T-1}$

Let $a = C/(1+r)$

$x = 1/(1+r)$

Rewriting:

$$PV = a\ (1 + x + x^2 + x^3 + \ldots + x^{T-1}) \qquad (1.)$$

Multiplying by x:

$$PVx = a(x + x^2 + x^3 + \ldots + x^T) \qquad (2.)$$

Subtracting (2.) from (1.):

$PV(1-x) = a\ (1-x^T)$

$PV = a\ (1-x^T)\ /(1-x)$

$PV = C/(1+r)[(1-1/(1+r)^T)/(1-1/(1+r)]$

Multiplying the (1+r) in the denominator thru:

$$PV\ (\text{Finite Annuity}) = C/r\ [1 - 1/(1+r)^T]$$

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Example

Find the value of a 5 year car loan with annual payments of $3,600 per year starting next year (i.e, 5 payments of $3,600 in the future). The cost of capital or opportunity cost of capital is 6%.

PV = $3,600*PVA(5, 6%)

= $3,600*(1/0.06)[1 – 1/(1.06)^5]

= $15,164.51

Example Continued

Suppose you had also made a down-payment for the car of $5,000 to lower your monthly loan payments. The total cost/value of the car you purchased is then:

PV(down payment) + PV(loan annuity)

= $5,000 + $15,164.51

= $20,164.51

Computing Present Value of Finite Annuities in Excel

Excel function **PV**:

```
PV(Rate, Nper, Pmt, Fv, Type)
Rate = per period interest rate
Nper = number of annuity payments
Fv = cash balance after last payment
Type = 1 if payments start in first
period; 0 if payments start in initial
period
```

Example

- Borrow $200,000 to buy a house.
- Annual interest rate = 10%
- Loan is to be paid back in 30 years
- Q: What is the annual payment?
- PV = $200,000 = C*PVA(0.10, 30)
- => C = $200,000/PVA(0.10, 30)
- PVA(0.10, 30) = $(1/0.10)[1 - 1/(1.10)^{30}] = 9.427$
- => C = $200,000/9.427 = $21,215.85

Computing Payments from Finite Annuities in Excel

Excel function **PMT**:

```
PMT(Rate, Nper, Pv, Fv, Type)
Rate = per period interest rate
Nper = number of annuity payments
Pv = initial present value of annuity
Fv = future value after last payment
Type = 1 if payments are due at the
beginning of the period; 0 if payments
are due at the end of the period
```


Example

- You win the $5 million lottery!
- 25 annual installments of $200,000 starting next year
- Q: What is the PV of winnings if r = 10%?
- PV = $200,000 * PVA(0.10, 25)
- PVA = $(1/0.10)[1 - 1/(1.10)^{25}] = 9.07704$
- => PV= $200,000 * (9.07704) = $1,815,408 < $5M!

Future Value of an Annuity

- Invest \$C every year, starting next year, for T years at a fixed rate r
- How much will investment be worth in year T?
- Trick: $FVA(r,T) = PVA(r,T)*(1+r)^T$
  - $= (1/r)\,[1 - 1/(1+r)^T]*(1+r)^T$
  - $= (1/r)[(1+r)^T - 1]$
- Therefore
- FV = C*FVA(r, T)
- where FVA(r, T) = FV of \$1 invested every year for T years at rate r

Example

- Save \$1,000 per year, starting next year, for 35 years in IRA
- Annual rate = 7%
- Q: How much will you have saved in 35 years?
- FV = \$1,000*FVA(0.07, 35)
- $FVA(0.07, 35) = (1/0.07)*[(1.07)^{35} - 1] = 138.23688$
- => FV = \$1,000*(138.23688) = \$138,236.88

Computing Future Value of Finite Annuities in Excel

Excel function **FV**:

```
FV(Rate, Nper, Pmt, Pv, Type)
Rate = per period interest rate
Nper = number of annuity payments
Pmt = payment made each period
Pv = present value of future payments
Type = 1 if payments start in first
period; 0 if payments start in initial
period
```

Finite Growing Annuities

- Similar to how we amended the Perpetuity formula for 'Growing' Perpetuities, we can amend the Annuity formula to account for a 'Growing' Annuity.
- The cash flow for a finite growing annuity pays an amount C, starting next period, with the cash flow growing thereafter at a rate of g, through period T:

$PV = C/(1+r) + C(1+g)/(1+r)^2 + C(1+g)^2/(1+r)^3 + \ldots + C(1+g)^{T-1}/(1+r)^T$

$= \Sigma\, C(1+g)^{t-1}/(1+r)^t$ for $t = 1,\ldots, T$

` $= C/(r-g)\,[1- (1+g)^T/(1+r)^T]$

Class Example

- An asset generates a cash flow that is $1 next year, but is expected to grow at 5% per year indefinitely.
- Suppose the relevant discount rate is 7%.

Q: After receiving the third payment, what can you expect to sell the asset for?

Q: What is the present value of the asset you held?

Compounding Frequency

- Cash flows can occur annually (once per annum), semi-annually (twice per annum), quarterly (four times per annum), monthly (twelve times per annum), daily (365 times per annum), etc.
- Based on the cash flows, the formulas for compounding and discounting can be adjusted accordingly:

General formula: For stated annual interest rate r compounded for T years n times per year:

$$FV = V_0 * [1 + r/n]^{nT}$$

Compounding Frequency

Effective Annual Rate (annual rate that gives the same FV with compounding n times per year):

$$[1 + r_{EAR}]^T = [1 + r/n]^{nT}$$

$$=> \ r_{EAR} = [1 + r/n]^n - 1$$

Example

- Invest $1,000 for 1 year
- Annual rate (APR) r = 10%
- Semi-annual compounding: semi-annual rate = 0.10/2 = 0.05
- FV = $1,000*$(1 + r/2)^{2*1}$ = $1,000*$(1.05)^2$ = $1102.50
- Note: $1,000*$(1 + 0.05)^2$ = $1,000*$(1 + 2*(0.05) + (0.05)^2)$
- = $1,000 + $100 + $2.5
- = principal + simple interest + interest on interest
- Effective annual rate:
- $(1 + r_{EAR}) = (1 + APR/2)^2$
- => $r_{EAR} = (1.05)^2 – 1 = 0.1025$ or 10.25%

Example: The Difference In Compounding

Annual rate of Interest 5%

T = 1 Year

Compounding Frequency	Times Per Annum	One plus Effective Rate
Yearly	1	1.05
Semi-Annual	2	1.050625
Quarterly	4	1.050945337
Monthly	12	1.051161898
Daily	365	1.051267496
Hourly	8,760	1.051270946
By the minute	525,600	1.051271094
By the second	31,536,000	1.051271093

Example

- Take out (borrow) $300,000 30 year fixed rate mortgage
- Annual rate = 8%, monthly rate = 0.08/12 = 0.0067
- 30*12 = 360 monthly payments
- Q: What is the monthly payment?

- PV = $300,000 = C*PVA(0.08/12, 360)
- PVA(0.0067, 360) = 136.283
- => C = $300,000/136.283 = $2,201.30
- Note: total amount paid over 30 years is
- 360*$2,201.30 = $792,468

Example

- Consider previous 30 year mortgage
- Suppose the day after the mortgage is issued, the annual rate on new mortgages shoots up to 15%
- Q: How much is the old mortgage worth?
- PV = $2,201*PVA(0.15/12, 360)
- PVA(0.15/12, 360) = 79.086
- => PV = $2,201*79.086 = $174,092 < $300,000!

Continuous Compounding

Increasing the frequency of compounding to continuously:

$$\lim_{n\to\infty} [1 + r/n]^{nT} = (2.718)^{rT} = e^{rT}$$

Effective Annual Rate:

$$[1 + r_{EAR}]^{T} = e^{rT}$$
$$\Rightarrow r_{EAR} = e^{r} - 1$$

Example

- r = annual (simple) interest rate = 10%, T = 1 year
- FV of 1$ with annual compounding:
- FV = $1(1+r) = $1.10
- FV of 1$ with continuous compounding:
- FV = $1*$e^r$ = $2.7180^{0.10}$ = $1.10517
- Effective annual rate
- 1 + r_{EAR} = 1.10517 => r_{EAR} = 0.10517 = 10.517%

Further Insight on Continuous Compounding

Example: Invest $$V_0$ for 1 year with annual rate r and continuous compounding

$$V_1 = V_0 e^{r\times 1} \Rightarrow \left(\frac{V_1}{V_0}\right) = e^r$$

$$\Rightarrow \ln\left(\frac{V_1}{V_0}\right) = r$$

$$\Rightarrow \ln V_1 - \ln V_0 = r$$

Test/Practical Tips

- General formula will always work by may be tedious
- Short-cuts exist if you can recognize them
- Use short-cuts!
- Break down complicated problems into simple pieces