

Market Risk Modeling in S-PLUS

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Created: May 11, 2005

Revised: June 3, 2005.

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Abstract

This document summarizes the steps for calculating Value-at-Risk (VaR) for a portfolio of equity assets using S-PLUS 7.0 and S+FinMetrics 2.0. Unconditional VaR is computed using empirical quantiles, the normal distribution, and an extreme value theory (EVT) Pareto tail distribution. Conditional VaR is computed using GARCH models, GARCH+EVT. The VaR models are evaluated by examining VaR violations in a backtesting environment.

1 Basic Concepts

This section reviews some basic concepts of asset returns and portfolios, and defines the market risk concepts *value-at-risk* (VaR) and expected tail loss (ETL) (which is also called *expected shortfall* (ES)).

1.1 Asset Returns

The portfolio consists of $i = 1, \dots, N$ equity assets. Let P_{it} denote the price of asset i at time t . The one-period simple return on asset i between times $t - 1$ and t is

$$R_t = \frac{P_{it} - P_{it-1}}{P_{it-1}}$$

The one-period continuously compounded (log) return is

$$r_{it} = \ln(1 + R_{it}) = \ln\left(\frac{P_{it}}{P_{it-1}}\right)$$

Note that

$$R_{it} = e^{r_{it}} - 1$$

Log returns are often preferred to simple returns for statistical modeling purposes.

1.2 Portfolios

Let $\mathbf{R}_t = (R_{1t}, \dots, R_{Nt})'$ and $\mathbf{r}_t = (r_{1t}, \dots, r_{Nt})'$. A portfolio of N assets is characterized by a $N \times 1$ vector of portfolio weights $\mathbf{w} = (w_1, \dots, w_N)'$ where w_i denote the share of wealth invested in asset i such that $\sum_{i=1}^n w_i = 1$. The one-period simple and log returns on the portfolio \mathbf{w} are

$$\begin{aligned} R_{w,t} &= \mathbf{x}' \mathbf{R}_t = \sum_{i=1}^N w_i R_{it} \\ r_{w,t} &= \ln(1 + R_{w,t}) \\ &= \ln(1 + \mathbf{w}' \mathbf{R}_t) \approx \mathbf{w}' \mathbf{r}_t = \sum_{i=1}^N w_i r_{it} \end{aligned}$$

Note that the log portfolio return is not exactly equal to the weighted average of the individual asset log returns.

1.3 Value-at-Risk Defined

Consider a one period investment in an asset with simple return R . Let $\$W_0$ denote the initial dollar amount invested. The value of the investment after one period in terms of the simple return is

$$\$W_1 = \$W_0(1 + R)$$

and the value of the investment in terms of the log return is

$$\$W_1 = \$W_0 e^r$$

1.3.1 VaR Based on Simple Returns

For $\alpha \in (0, 1)$, let q_α^R denote the $\alpha \times 100\%$ quantile of the probability distribution of the simple return R . Usually, q_α^R is a low quantile such that $\alpha = 0.01$ or $\alpha = 0.05$. As a result, q_α^R is typically a negative number. The $\alpha \times 100\%$ dollar Value-at-Risk ($\$VaR_\alpha$) is

$$\$VaR_\alpha = -\$W_0 \cdot q_\alpha^R$$

In words, $\$VaR_\alpha$ represents the dollar loss that could occur with probability α . By convention, it is reported as a positive number (hence the minus sign). The VaR as a percentage of the initial portfolio value is simply the (negative) low quantile of the simple return distribution:

$$VaR_\alpha = \frac{\$VaR_\alpha}{\$W_0} = -q_\alpha^R$$

1.3.2 VaR Based on Log Returns

Let q_α^r denote the $\alpha \times 100\%$ quantile of distribution of the log return $r = \ln(1+R)$. The simple return quantile q_α^R is related to the log return quantile using the relationship

$$q_\alpha^R = e^{q_\alpha^r} - 1$$

Therefore, $\$VaR_\alpha$ based on log-returns may be computed using

$$\$VaR_\alpha = -\$W_0 \cdot (e^{q_\alpha^r} - 1)$$

and

$$VaR_\alpha = \frac{\$VaR_\alpha}{\$W_0} = -(e^{q_\alpha^r} - 1)$$

1.4 Expected Tail Loss Defined

The $\alpha \times 100\%$ expected tail loss (ETL), in terms of the log return, is defined as

$$ETL_\alpha = -E[r|r < -VaR_\alpha]$$

In words, the ETL is the expected (negative) return conditional on the return being less than the $\alpha \cdot 100\%$ percentage VaR. If the initial investment is $\$W_0$, then the dollar ETL is

$$\$ETL = \$W_0 \times ETL_\alpha$$

2 Example Data

VaR and ETL calculations are illustrated using the daily log-returns on the 30 Dow Jones Industrial Average stocks over the period January 2, 1991 through January 2, 2001. The adjusted closing prices are in the S+FinMetrics “timeSeries” `DowJones30`. Log returns may be calculated using

```
> DowJones30.ret = getReturns(DowJones30)
```

These returns are illustrated in Figure 1.

Summary statistics on these returns are listed below

```
> summaryStats(DowJones30.ret)
```

Sample Moments:

	mean	std	skewness	kurtosis
AA	0.00066613	0.02027	0.47417262	6.256
AXP	0.00094778	0.02106	0.06744039	5.009
T	-0.00002873	0.02011	-1.36496467	27.055
BA	0.00041075	0.01919	-0.31983600	10.532

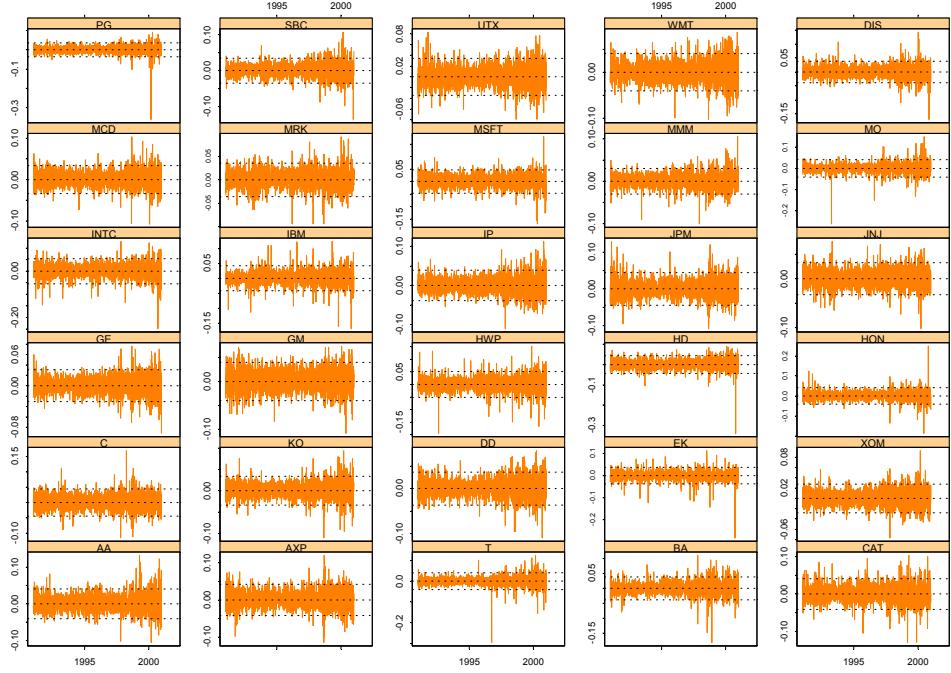


Figure 1: Daily log returns on Dow Jones 30 stocks.

CAT	0.00062407	0.02059	0.03768706	6.235
C	0.00130133	0.02228	0.20836742	5.813
KO	0.00071633	0.01690	0.01638419	6.310
DD	0.00050283	0.01821	0.00902287	5.423
EK	0.00019231	0.01856	-1.74461569	29.463
XOM	0.00050993	0.01394	0.16699197	4.986
GE	0.00088443	0.01537	-0.02254540	5.284
GM	0.00037496	0.01996	0.06570330	4.143
HWP	0.00080548	0.02589	-0.25593852	7.800
HD	0.00110341	0.02178	-1.47584163	27.498
HON	0.00080532	0.02058	0.37584235	17.145
INTC	0.00128791	0.02717	-0.54873050	8.072
IBM	0.00043914	0.02098	-0.14767612	9.317
IP	0.00025207	0.01941	0.29212499	5.685
JPM	0.00109663	0.02217	0.30356866	5.492
JNJ	0.00076479	0.01641	0.03577168	4.679
MCD	0.00061517	0.01701	0.11803896	6.200

MRK	0.00081026	0.01772	0.04197237	5.228
MSFT	0.00120271	0.02314	-0.14975931	7.430
MMM	0.00052147	0.01511	0.08429829	7.075
MO	0.00057575	0.02040	-0.82718665	20.006
PG	0.00051487	0.01798	-3.26960796	68.625
SBC	0.00064105	0.01737	-0.07638044	7.018
UTX	0.00082338	0.01709	0.08530716	5.019
WMT	0.00080274	0.02036	0.00005004	5.105
DIS	0.00050183	0.01910	0.02832071	9.202

The skewness and kurtosis values indicate non-normal distributions for the log-returns. Testing the null hypothesis that each return series is normally distributed using the Jarque-Bera statistic gives

```
> normalTest(DowJones30.ret, method="jb")
```

Test for Normality: Jarque-Bera

Null Hypothesis: data is normally distributed

Test Statistics:

	AA	AXP	T	BA	CAT
Test Stat	1211.2450	427.0028	61711.5570	6015.7899	1102.5776
p.value	0.0000	0.0000	0.0000	0.0000	0.0000
	C	KO	DD	EK	XOM
Test Stat	851.2940	1153.8869	618.1877	75018.2301	427.2389
p.value	0.0000	0.0000	0.0000	0.0000	0.0000
	GE	GM	HWP	HD	HON
Test Stat	549.5553	139.4029	2453.6285	64109.0577	21126.8931
p.value	0.0000	0.0000	0.0000	0.0000	0.0000
	INTC	IBM	IP	JPM	JNJ
Test Stat	2835.1289	4211.3738	795.0107	692.5375	297.3126
p.value	0.0000	0.0000	0.0000	0.0000	0.0000
	MCD	MRK	MSFT	MMM	MO
Test Stat	1084.0272	523.4963	2076.0047	1751.1850	30740.0087
p.value	0.0000	0.0000	0.0000	0.0000	0.0000
	PG	SBC	UTX	WMT	DIS
Test Stat	457961.0721	1702.1350	432.3921	466.7221	4050.9754

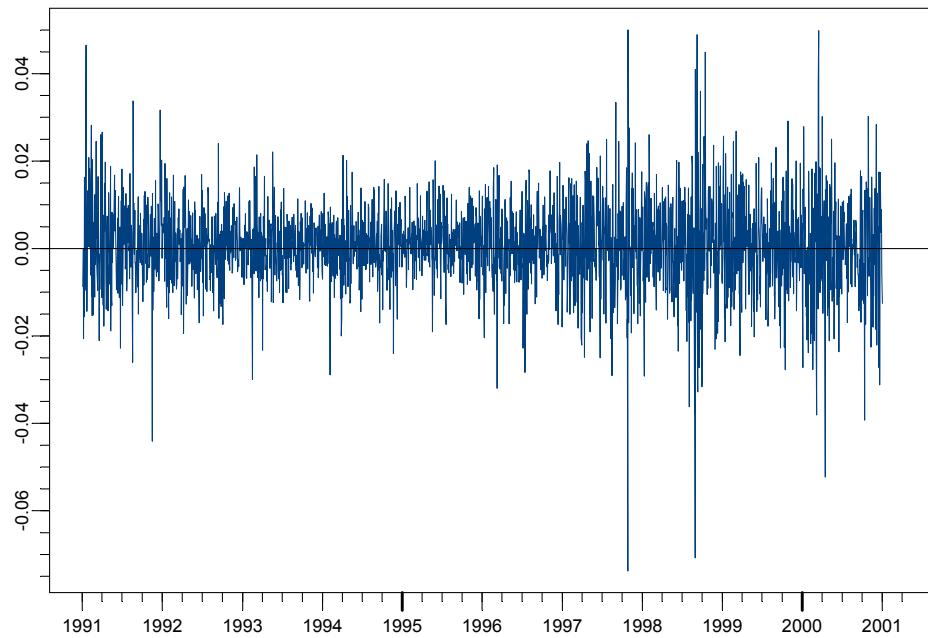


Figure 2: Daily log returns on equally weighted portfolio of Dow Jones 30 stocks.

```
p.value      0.0000      0.0000      0.0000      0.0000      0.0000
```

```
Dist. under Null: chi-square with 2 degrees of freedom
Total Observ.: 2527
```

An equally weighted portfolio of the Dow Jones 30 stocks may be computed using

```
> port.ret.ts = rowSums(DowJones30.ret, weights=rep(1,30)/30)
```

The daily log-returns are illustrated in Figure 2.

The summary statistics for the portfolio are

```
> summaryStats(port.ret.ts)
```

Sample Quantiles:

min	1Q	median	3Q	max
-0.0738	-0.00441	0.0007142	0.005973	0.05001

Sample Moments:

```
mean      std skewness kurtosis
0.0006889 0.009707 -0.3138    7.663
```

Number of Observations: 2527

and the Jarque-Bera test for normality is

```
> normalTest(port.ret.ts, method="jb")
```

Test for Normality: Jarque-Bera

Null Hypothesis: data is normally distributed

Test Statistics:

Portfolio

Test Stat 2330.622

p.value 0.000

Dist. under Null: chi-square with 2 degrees of freedom

Total Observ.: 2527

3 Unconditional Models for VaR and ETL

This section reviews common models for computing unconditional estimates of VaR and ETL. In unconditional models, the multivariate return distribution is assumed to be covariance stationary and ergodic and, hence, have time invariant moments. Usually, the moments of the return distribution are estimated from historical data and are assumed to be fixed over the period in which estimates of VaR and ETL are required. The models considered are historical simulation, normal approximation and extreme value theory.

3.1 Historical Simulation

Historical simulation (HS) simply refers to the empirical distribution of the observed returns. As a result, the $\alpha \times 100\%$ VaR based on HS is just the $\alpha \times 100\%$ empirical quantile of the return distribution. Similarly, the HS estimate of ETL_α is simply the average of the returns below the HS VaR_α estimate.

Example 1 $VaR_{.01}$ for Dow Jones 30 stocks based on historical simulation

In S-PLUS, empirical quantiles may be computed using the `quantile` or `colQuantiles` function. To compute $VaR_{.01}$ for the equally weighted portfolio of Dow Jones 30 stocks based on HS use

```

> VaR.hs.01 = quantile(port.ret.ts, probs=0.01)
> VaR.hs.01
  1%
-0.0247706

```

With 1% probability the loss is about 2.5% or higher. If there is \$1M initially invested in the portfolio, the 1% dollar VaR is

```

> 1000000*( exp(VaR.hs.01) - 1 )
  1%
-24466.33

```

To compute the VaR._{.01} for all of the Dow Jones 30 stocks based on HS use

```

> unlist(colQuantiles(DowJones30.ret, probs=0.01))
    AA.1%      AXP.1%      T.1%      BA.1%      CAT.1%      C.1%
-0.04692524 -0.05137398 -0.05050954 -0.04699595 -0.04966563 -0.05252321

    KO.1%      DD.1%      EK.1%      XOM.1%      GE.1%      GM.1%
-0.04102243 -0.04533577 -0.04449885 -0.03381674 -0.04063354 -0.04914696

    HWP.1%      HD.1%      HON.1%      INTC.1%      IBM.1%      IP.1%
-0.06547625 -0.05123644 -0.05445059 -0.07176775 -0.05232081 -0.04953661

    JPM.1%      JNJ.1%      MCD.1%      MRK.1%      MSFT.1%      MMM.1%
-0.05121868 -0.03776024 -0.03966271 -0.04359857 -0.05891276 -0.04029453

    MO.1%      PG.1%      SBC.1%      UTX.1%      WMT.1%      DIS.1%
-0.05290309 -0.03873091 -0.04528876 -0.04237849 -0.05424686 -0.04840624

```

The bootstrap may be used to assess the sampling uncertainty associated with the empirical quantile. The S-PLUS function `bootstrap` may be used for this purpose. Alternatively, the `bootstrap` function from the library S+Resample may be used¹. For example, to compute a bootstrap standard error for the HS estimate of VaR._{.05} for the first Dow Jones 30 stock use

```

> VaR.05.boot = bootstrap(port.ret.ts, statistic=quantile,
+                         args.stat=list(probs=0.05))
> summary(VaR.05.boot)
Call:
bootstrap(data = port.ret.ts, statistic = quantile, args.stat = list(probs
= 0.05))

```

¹The library S+Resample may be downloaded from the Insightful research page. This library greatly extends the bootstrapping and resampling functionality in S-PLUS.

Number of Replications: 1000

Summary Statistics:

Observed	Bias	Mean	SE
5% -0.01466	-0.00009937	-0.01476	0.0005054

Empirical Percentiles:

2.5%	5%	95%	97.5%
5% -0.01588	-0.01561	-0.01403	-0.01388

BCa Confidence Limits:

2.5%	5%	95%	97.5%
5% -0.01588	-0.01561	-0.01403	-0.01388

The bootstrap standard error estimate is 0.0005 and a 95% bootstrap confidence interval is $[-0.0159, -0.0139]$. The bootstrap distribution may be visualized using

```
> plot(VaR.05.boot)
```

Example 2 $ETL_{.01}$ for Dow Jones 30 stocks based on historical simulation

The HS estimate of $ETL_{.01}$ is simply the average of the returns below the HS VaR $_{.01}$ estimate. For the equally weighted portfolio, the HS estimate of $ETL_{.01}$ is

```
> mean(port.ret.ts[port.ret.ts < VaR.hs.01])
[1] -0.03496077
```

The average loss in portfolio value when the return is less than the 1% VaR is about 3.5%.

To compute the HS estimate of $ETL_{.01}$ for all of the Dow Jones 30 stocks use

```
> ETL.hs.01 = function(x) { mean(x[x < quantile(x, probs=0.01)]) }
> apply(DowJones30.ret, 2, ETL.hs.01)
          AA        AXP         T        BA        CAT         C
-0.06058743 -0.06734013 -0.08510645 -0.07228072 -0.0703473 -0.06861809

          KO        DD        EK        XOM        GE        GM
-0.05804336 -0.05913512 -0.07994174 -0.04177331 -0.05104979 -0.06092251

          HWP        HD        HON       INTC        IBM        IP
-0.09570858 -0.07645151 -0.07704453 -0.104592 -0.07876983 -0.06014644

          JPM        JNJ        MCD        MRK        MSFT        MMM
```

-0.06735523	-0.04954164	-0.05503747	-0.05697714	-0.08095875	-0.05277178
MO	PG	SBC	UTX	WMT	DIS

-0.08194002	-0.07168527	-0.05853913	-0.05536674	-0.06888778	-0.06691007
-------------	-------------	-------------	-------------	-------------	-------------

3.2 Normal Distribution

Assume the $N \times 1$ vector of log-returns \mathbf{r} has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu}$ has elements μ_i ($i = 1, \dots, N$) and $\boldsymbol{\Sigma}$ has elements σ_{ij} ($i, j = 1, \dots, N$). For an individual asset,

$$r_i \sim N(\mu_i, \sigma_{ii})$$

The $\alpha \times 100\%$ quantile of the normal distribution for r_i is

$$q_\alpha^i = \mu_i + \sigma_i q_\alpha^z$$

where q_α^z is the $\alpha \times 100\%$ quantile of the standard normal distribution. The distribution of r_i given that $r_i \leq q_\alpha^i$ is truncated normal. The mean of this distribution is the normal ETL $_\alpha$. Greene (2004) shows that ²

$$E[r_i | r_i \leq q_\alpha^i] = \mu_i + \sigma_i \times \frac{\phi(z_\alpha^i)}{\Phi(z_\alpha^i)}$$

where $z_\alpha^i = (\mu_i - VaR_\alpha)/\sigma_i$, $\phi(z)$ is the standard normal pdf and $\Phi(z)$ is the standard normal CDF.

Given a random sample of size T of observed returns on N assets from the multivariate normal distribution, the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ may be estimated using the sample statistics

$$\hat{\boldsymbol{\mu}} = T^{-1} \sum_{t=1}^T \mathbf{r}_t, \quad \hat{\boldsymbol{\Sigma}} = T^{-1} \sum_{t=1}^T (\mathbf{r}_t - \hat{\boldsymbol{\mu}})(\mathbf{r}_t - \hat{\boldsymbol{\mu}})'$$

The normal quantile may then be estimated using the plug-in method

$$\hat{q}_\alpha^i = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^z$$

where $\hat{\mu}_i$ is the i th element of $\hat{\boldsymbol{\mu}}$, and $\hat{\sigma}_i$ is the square root of the i th diagonal element of $\hat{\boldsymbol{\Sigma}}$. Similarly, the estimate of normal ETL $_\alpha$ is

$$\hat{E}[r_i | r_i \leq q_\alpha^i] = \hat{\mu}_i + \hat{\sigma}_i \times \frac{\phi(\hat{z}_\alpha^i)}{\Phi(\hat{z}_\alpha^i)}$$

where $\hat{z}_\alpha^i = (\hat{\mu}_i - \widehat{VaR}_\alpha)/\hat{\sigma}_i$, and $\widehat{VaR}_\alpha = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^z$. Standard errors for these estimates may be conveniently computed using the bootstrap.

²See Greene (2004) page xxx for properties of the truncated normal distribution.

3.2.1 Active versus Passive Portfolio Risk Measurement

In a passive portfolio, the portfolio weight vector \mathbf{w} stays fixed over time. The portfolio return is normally distributed

$$r_w = \mathbf{w}' \mathbf{r} \sim N(\mu_w, \sigma_w^2)$$

with time invariant mean and variance

$$\mu_w = \mathbf{w}' \boldsymbol{\mu}, \sigma_w^2 = \mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}$$

The $\alpha \times 100\%$ quantile of the normal distribution for r_w is

$$q_\alpha^w = \mu_w + \sigma_w q_\alpha^z$$

and the $\alpha \times 100\%$ ETL is

$$E[r_w | r_w \leq q_\alpha^w] = \mu_w + \sigma_w \times \frac{\phi(z_\alpha^w)}{\Phi(z_\alpha^w)}$$

In a passive portfolio, the portfolio mean and variance may be estimated from observed returns by first computing the portfolio return $r_{w,t} = \mathbf{w}' \mathbf{r}_t$, and then computing the sample mean and variance from the portfolio returns

$$\hat{\mu}_w = T^{-1} \sum_{t=1}^T \mathbf{w}' \mathbf{r}_t, \hat{\sigma}_w^2 = T^{-1} \sum_{t=1}^T (r_{w,t} - \hat{\mu}_w)^2$$

In an active portfolio, the weights on the individual assets may change over time as the portfolio manager rebalances the portfolio. Let \mathbf{w}_t denote the $N \times 1$ vector of portfolio weights at time t . Then the portfolio return is normally distributed

$$r_{w,t} = \mathbf{w}_t' \mathbf{r} \sim N(\mu_{w,t}, \sigma_{w,t}^2)$$

with period specific mean and variance

$$\mu_{w,t} = \mathbf{w}_t' \boldsymbol{\mu}, \sigma_{w,t}^2 = \mathbf{w}_t' \boldsymbol{\Sigma} \mathbf{w}_t$$

In an active portfolio, the portfolio mean and variance at time t are usually estimated with

$$\hat{\mu}_{w,t} = \mathbf{w}_t' \hat{\boldsymbol{\mu}}, \hat{\sigma}_{w,t}^2 = \mathbf{w}_t' \hat{\boldsymbol{\Sigma}} \mathbf{w}_t$$

3.2.2 S-PLUS Functions

Simple S-PLUS functions for computing VaR and ETL based on the normal distribution are

```

norm.quantile = function(x, p=0.01, mu=NULL) {
  ## required arguments:
  ## x      timeSeries of returns (simple or continuous)
  ## optional arguments:
  ## p      scalar probability level
  ## mu    vector of mean values.
  ## value:
  ## numeric value giving quantile estimate from estimated normal
  ## distribution
  if(is.null(mu))
    q = colMeans(x) + colStdevs(x)*qnorm(p)
  else
    q = mu + colStdevs(x)*qnorm(p)
  q
}

```

Example 3 *VaR_{.01} for Dow Jones 30 stocks based on normal distribution*

The normal estimate of VaR_{.01} for the (passive) equally weighted portfolio is

```

> norm.quantile(port.ret.ts)
  Portfolio
-0.02189391

```

A 95% confidence interval for VaR_{.01} based on the bootstrap may be computed using

```

> VaR.01.boot = bootstrap(port.ret.ts, statistic=norm.quantile)
> limits.emp(VaR.01.boot)
      2.5%          5%         95%       97.5%
Portfolio -0.02320047 -0.02299066 -0.02088363 -0.0207183

```

For an active portfolio, the 1% quantile may be estimated using

```

> mu.hat = colMeans(DowJones30.ret)
> Sigma.hat = var(DowJones30.ret)
> w = rep(1,30)/30
> t(w)%*%mu.hat + sqrt( t(w)%*%Sigma.hat%*%w )*qnorm(.01)
[1]
[1,] -0.02189391

```

The normal estimates of VaR_{.01} for all of the Dow Jones 30 stocks are

```

> norm.quantile(DowJones30.ret)
      AA        AXP        T        BA        CAT        C
-0.04649377 -0.04803599 -0.04681135 -0.04423398 -0.04728698 -0.05053085

```

KO	DD	EK	XOM	GE	GM
-0.03861008	-0.04186347	-0.04298664	-0.03193052	-0.0348732	-0.04604977
HWP	HD	HON	INTC	IBM	IP
-0.05941707	-0.04955317	-0.04707859	-0.06192249	-0.04836213	-0.04489961
JPM	JNJ	MCD	MRK	MSFT	MMM
-0.05046832	-0.03740214	-0.03896121	-0.04041834	-0.05262862	-0.03462914
MO	PG	SBC	UTX	WMT	DIS
-0.04688573	-0.04131749	-0.0397562	-0.0389281	-0.04655835	-0.04393198

Example 4 $ETL_{.01}$ for Dow Jones 30 stocks based on the normal distribution

3.3 Extreme Value Theory

Extreme value theory (EVT) is concerned with modeling the tails of a probability distribution F . To illustrate, let X_1, X_2, \dots be a sequence of *iid* random variables representing risks or losses with an unknown CDF F . A natural measure of extreme events are values of the X_i that exceed a high threshold u . Define the *excess distribution* above the threshold u as the conditional probability:

$$F_u(y) = \Pr\{X - u \leq y | X > u\} = \frac{F(y + u) - F(u)}{1 - F(u)}, \quad y > 0 \quad (1)$$

it can be shown that for large enough u there exists a positive function $\beta(u)$ such that the excess distribution (1) is well approximated by the *generalized Pareto distribution* (GPD)

$$G_{\xi, \beta(u)}(y) = \begin{cases} 1 - \left(1 + (\xi y / \beta(u))^{-1/\xi}\right) & \text{for } \xi \neq 0 \\ 1 - \exp(-y / \beta(u)) & \text{for } \xi = 0 \end{cases}, \quad \beta(u) > 0 \quad (2)$$

defined for $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\beta(u)/\xi$ when $\xi < 0$. The parameter ξ is determines the tail shape of the distribution of exceedences over the threshold u . If $\xi < 0$ then F is in the thin-tailed Weibull family and $G_{\xi, \beta(u)}$ is a *Pareto type II* distribution; if $\xi = 0$ then F is in the Gumbell family and $G_{\xi, \beta(u)}$ is an *exponential* distribution; and if $\xi > 0$ then F is in the fat-tailed Fréchet family and $G_{\xi, \beta(u)}$ is a *Pareto* distribution. For $\xi > 0$, the most relevant case for risk management purposes, it can be shown that $E[X^k] = \infty$ for $k \geq \alpha = 1/\xi$. For example, if $\xi = 0.5$ then $E[X^2] = \infty$ and the distribution of losses, X , does not have finite variance. If $\xi = 1$ then $E[X] = \infty$.

The threshold parameter $u = u_0$ may be determined by computing the *empirical mean excess function*

$$e_n(u) = \frac{1}{n_u} \sum_{i=1}^{n_u} (x_{(i)} - u) \quad (3)$$

where $x_{(i)}$ ($i = 1, \dots, n_u$) are the values of x_i such that $x_i > u$, and then plotting $e_n(u)$ against u . The plot should be linear in u for $u > u_0$. An upward sloping plot indicates heavy-tailed behavior. In particular, a straight line with positive slope above u_0 is a sign of Pareto behavior in tail. A downward trend shows thin-tailed behavior, whereas a line with zero slope shows an exponential tail.

Once the threshold parameter $u = u_0$ is chosen, the remaining parameters ξ and $\beta(u)$ may be estimated by maximum likelihood. Given the ML estimates $\hat{\xi}$ and $\hat{\beta}(u)$, the tails of the loss distribution F may be estimated using

$$\hat{F}(x) = 1 - \frac{k}{n} \left(1 + \hat{\xi} \cdot \frac{x - u}{\hat{\beta}(u)} \right) \quad (4)$$

where k denotes the number of exceedences over the threshold u . Then, an estimate of VaR_α based on inverting $\hat{F}(x)$ is

$$\widehat{\text{VaR}}_\alpha = u + \frac{\hat{\beta}(u)}{\hat{\xi}} \left(\left(\frac{n}{k} (1 - q) \right)^{-\hat{\xi}} - 1 \right) \quad (5)$$

An estimate of ETL_α is

$$\widehat{\text{ETL}}_\alpha = \frac{\widehat{\text{VaR}}_\alpha}{1 - \hat{\xi}} + \frac{\hat{\beta}(u) - \hat{\xi}u}{1 - \hat{\xi}}. \quad (6)$$

Remarks:

1. EVT methods are easily applied to passive portfolios.
2. EVT methods are difficult to apply to active portfolios. This usually requires building an EVT model for the full multivariate distribution of returns. The most promising methodology for doing this is based on copulas.

Example 5 $\text{VaR}_{.01}$ and $\text{ETL}_{.01}$ for equally weighted portfolio based on EVT

Consider estimating $\text{VaR}_{.01}$ and $\text{ETL}_{.01}$ based on EVT for the equally weighted portfolio of Dow Jones 30 stocks. The first step is to determine the threshold parameter u using empirical mean excess function. This may be computed using the S+FinMetrics function `meplot`

```
> tmp = meplot(-port.ret.ts)
```

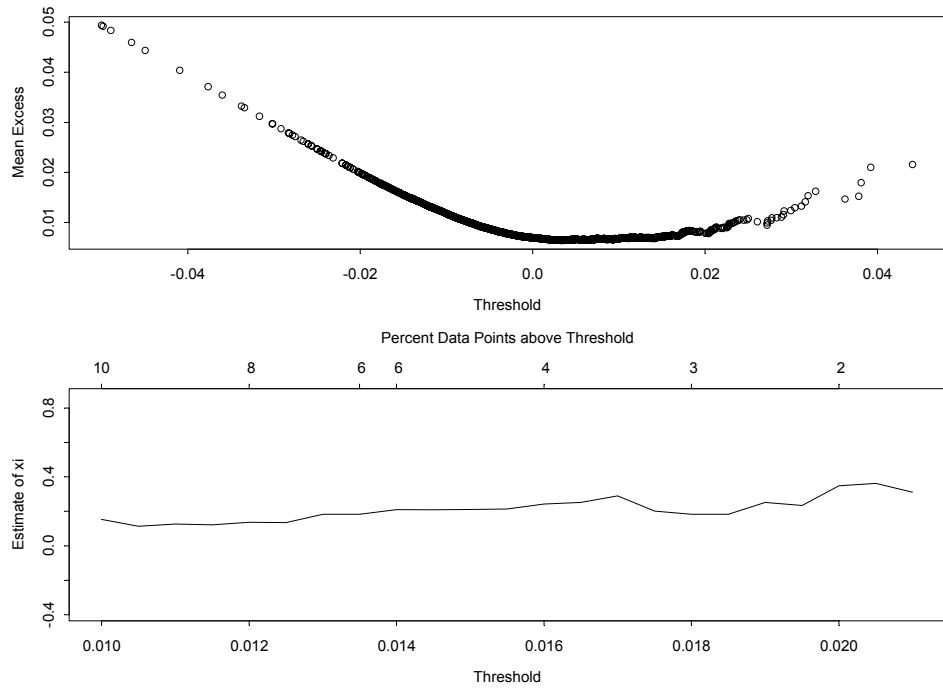


Figure 3: Mean excess plot (top panel) and MLEs of GPD shape parameter for equally weighted portfolio.

which produces the graph in the top panel of Figure 3. The negative returns are used as an input to `meplot` to investigate the low tail of the distribution. The plot becomes linear with positive slope near $u = 0.015$ suggesting fat tailed behavior in the lower tail. To better pin down the threshold, the S+FinMetrics `shape.plot` may be used to see how the MLEs of the GPD shape parameter varies with the selected threshold u . For example

```
> shape.plot(-port.ret.ts, from=0.9, to=0.98)
```

computes the MLEs of ξ using threshold values from the 90th quantile through the 98th quantile. These values are illustrated in the lower panel of Figure 3. The two plots suggest a threshold value around $u = 0.015$.

Given the threshold $u = 0.015$, the MLEs of the GPD parameters ξ and $\beta(0.015)$ may be computed using the S+FinMetrics function `gpd`

```
> gpd.fit = gpd(-port.ret.ts, threshold=0.015)
> gpd.fit
Generalized Pareto Distribution Fit --
```

```

Total of 2527 observations

Upper Tail Estimated with ml --
Upper Threshold at 0.015 or 4.788 % of the data
ML estimation converged.
Log-likelihood value: 480.9

```

Parameter Estimates, Standard Errors and t-ratios:

	Value	Std.Error	t value
xi	0.2109	0.1066	1.9792
beta	0.0056	0.0008	7.2066

The MLE for ξ is 0.211, with an estimated standard error of 0.107, indicating fat tailed behavior. Since $\hat{\alpha} = 1/\hat{\xi} = 4.74$ the fitted GPD suggests that the equally portfolio return distribution has about four moments finite. The fit of the GPD to the tails of negative returns may be visualized using the S+FinMetrics function `tailplot`

```
> tailplot(gpd.fit)
```

which is illustrated in Figure 4. The GPD appears to be a good fit.

From the fitted GPD, estimates of $\text{VaR}_{.01}$ and $\text{ETL}_{.01}$ may be computed using the S+FinMetrics function `riskmeasures`

```
> riskmeasures(gpd.fit, p=0.99)
      p   quantile     sfall
beta 0.99 0.02539038 0.03526328
```

The estimate of $\text{VaR}_{.01}$ is -0.0254 and the estimate of $\text{ETL}_{.01}$ is -0.0353. These values are quite close to the HS estimates. 95% confidence intervals for these estimates may be computed using the S+FinMetrics functions `gpd.q` and `gpd.sfall`

```
> gpd.q(pp=0.99, ci.p = 0.95, plot=F)
      Lower CI   Estimate   Upper CI
0.02345692 0.02539038 0.02792725
> gpd.sfall(pp=0.99, ci.p = 0.95, plot=F)
      Lower CI   Estimate   Upper CI
0.03084265 0.03526328 0.04554279
```

4 Conditional Models for VaR and ETL

In conditional models, it is recognized that the moments of the multivariate return distribution may change over time

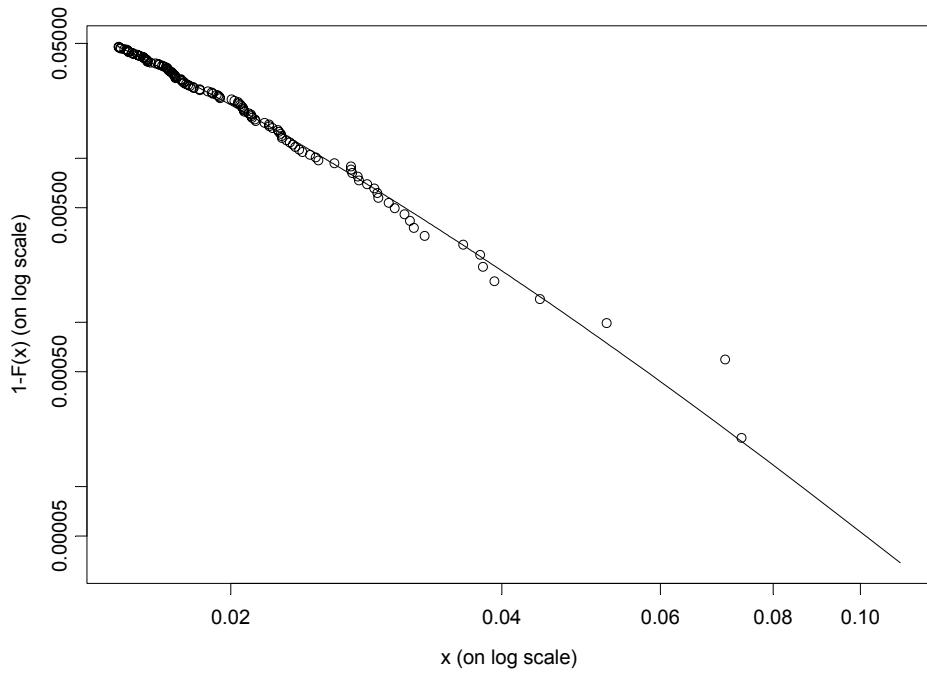


Figure 4: Fit of GDP to the lower tail of the equally weighted portfolio return distribution.

4.1 EWMA Estimates

JP Morgan's RiskMetrics system for market risk management utilizes the following exponentially weighted moving average (EWMA) model for time-varying return variances

$$\sigma_{t+1}^2 = (1 - \lambda) \sum_{\tau=1}^{\infty} \lambda^{\tau-1} r_{t+1-\tau}^2, \quad 0 < \lambda < 1$$

which may be re-written as

$$\sigma_{t+1}^2 = \lambda \sigma_t^2 + (1 - \lambda) r_t^2$$

In the RiskMetrics system λ is set at 0.94 for forecasting daily variances.

4.2 GARCH Models

to be completed

4.3 GARCH+EVT

to be completed

5 Backtesting VaR

to be completed

References

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