Lectures on Structural Change

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1 Overview of Testing for and Estimating Structural Change in Econometric Models

1. Day 1: Tests of Parameter Constancy
2. Day 2: Estimation of Models with Structural Change
3. Day 3: Time Varying Parameter Models

2 Some Preliminary Asymptotic Theory

Reference: Stock, J.H. (1994) "Unit Roots, Structural Breaks and Trends," in *Handbook of Econometrics, Vol. IV.*

3 Tests of Parameter Constancy in Linear Models

3.1 Motivation

- Diagnostics for model adequacy
- Provide information about out-of-sample forecasting accuracy
- Within-sample parameter constancy is a necessary condition for super-exogeneity

3.2 Example Data Sets

3.2.1 Simulated Data

Consider the linear regression model

$$\begin{aligned} y_t &= \alpha + \beta x_t + \varepsilon_t,\ t = 1, \ldots, T = 200 \\ & x_t \sim iid\ N(0,1) \\ & \varepsilon_t \sim iid\ N(0, \sigma^2) \end{aligned}$$

No structural change parameterization: $\alpha = 0, \beta = 1, \sigma = 0.5$

Structural change cases

- Break in intercept: $\alpha = 1$ for $t > 100$
- Break in slope: $\beta = 3$ for $t > 100$
- Break in error variance: $\sigma = 0.25$ for $t > 100$
- Random walk in slope: $\beta = \beta_t = \beta_{t-1} + \eta_t$, $\eta_t \sim iid\ N(0, 0.1)$ and $\beta_0 = 1$.

(show simulated data)

3.2.2 US/DM Monthly Exchange rate data

Let

$$\begin{aligned} s_t &= \text{log of spot exchange rate in month } t \\ f_t &= \text{log of forward exchange rate in month } t \end{aligned}$$

The forward rate unbiased hypothesis is typically investigated using the so-called differences regression

$$\begin{aligned} \Delta s_{t+1} &= \alpha + \beta(f_t - s_t) + \varepsilon_{t+1} \\ f_t - s_t &= i_t^{US} - i_t^{DM} = \text{ forward discount} \end{aligned}$$

If the forward rate f_t is an unbiased forecast of the future spot rate s_{t+1} then we should find

$$\alpha = 0 \text{ and } \beta = 1$$

The forward discount is often modeled as an AR(1) model

$$f_t - s_t = \delta + \phi(f_{t-1} - s_{t-1}) + u_t$$

Statistical Issues

- Δs_{t+1} is close to random walk with large variance
- $f_t - s_t$ behaves like highly persistent AR(1) with small variance
- $f_t - s_t$ appears to be unstable over time

3.3 Chow Forecast Test

Reference: Chow, G.C. (1960). "Tests of Equality between Sets of Coefficients in Two Linear Regressions," *Econometrica*, 52, 211-22.

Consider the linear regression model with k variables

$$\begin{aligned} y_t &= \mathbf{x}_t'\boldsymbol{\beta} + u_t,\ u_t \sim (0, \sigma^2),\ t = 1, \ldots, n \\ \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \end{aligned}$$

Parameter constancy hypothesis

$$H_0 : \boldsymbol{\beta} \text{ is constant}$$

Intuition

- If parameters are constant then out-of-sample forecasts should be unbiased (forecast errors have mean zero)

Test construction:

- Split sample into $n_1 > k$ and $n_2 = n - n_1$ observations

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \begin{matrix} n_1 \\ n_2 \end{matrix}, \mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \begin{matrix} n_1 \\ n_2 \end{matrix}$$

- Fit model using first n_1 observations

$$\begin{aligned} \hat{\boldsymbol{\beta}}_1 &= (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y}_1 \\ \hat{\mathbf{u}}_1 &= \mathbf{y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1 \\ \hat{\sigma}_1^2 &= \hat{\mathbf{u}}_1'\hat{\mathbf{u}}_1/(n_1 - k) \end{aligned}$$

- Use $\hat{\boldsymbol{\beta}}_1$ and $\mathbf{X}_2$ to predict $\mathbf{y}_2$ using next n_2 observations

$$\hat{\mathbf{y}}_2 = \mathbf{X}_2\hat{\boldsymbol{\beta}}_1$$

- Compute out-of-sample prediction errors

$$\hat{\mathbf{u}}_2 = \mathbf{y}_2 - \hat{\mathbf{y}}_2 = \mathbf{y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}_1$$

Under $H_0 : \boldsymbol{\beta}$ is constant

$$\hat{\mathbf{u}}_2 = \mathbf{u}_2 - \mathbf{X}_2(\hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta})$$

and

$$\begin{aligned} E[\hat{\mathbf{u}}_2] &= \mathbf{0} \\ var(\hat{\mathbf{u}}_2) &= \sigma^2\left(\mathbf{I}_{n_2} + \mathbf{X}_2(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2'\right) \end{aligned}$$

Further, If the errors u are Gaussian then

$$\begin{aligned} &\hat{\mathbf{u}}_2 \sim N(\mathbf{0}, var(\hat{\mathbf{u}}_2)) \\ &\hat{\mathbf{u}}_2' var(\hat{\mathbf{u}}_2)^{-1}\hat{\mathbf{u}}_2 \sim \chi^2(n_2) \\ &(n_1 - k)\hat{\sigma}_1^2/\sigma^2 \sim \chi^2(n_1 - k) \end{aligned}$$

This motivates the Chow forecast test statistic

$$Chow_{FCST}(n_2) = \frac{\hat{\mathbf{u}}_2'\left(\mathbf{I}_{n_2} + \mathbf{X}_2(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2'\right)\hat{\mathbf{u}}_2}{n_2\hat{\sigma}_1^2} \sim F(n_2, n_1 - k)$$

Decision: Reject H_0 at 5% level if

$$Chow_{FCST}(n_2) > cv_{0.05}$$

Remarks:

- Test is a general specification test for unbiased forecasts
- Popular with LSE methodology
- Implementation requires *a priori* splitting of data into fit and forecast samples

3.3.1 Application: Simulated Data

Chow Forecast Test			
		n_2	
Model	100	50	25
No SC	1.121	1.189	1.331
Mean shift	9.130***	1.329*	1.061
Slope shift	9.055***	2.067***	1.545*
Var shift	0.568	0.726	0.864
RW slope	2.183***	1.302	0.550

3.4 CUSUM and CUSUMSQ Tests

Reference: Brown, R.L., J. Durbin and J.M. Evans (1975). "Techniques for Testing the Constancy of Regression Relationships over Time," *Journal of the Royal Statistical Society*, Series B, 35, 149-192.

3.4.1 Recursive least squares estimation

The *recursive least squares* (RLS) estimates of $\boldsymbol{\beta}$ are based on estimating

$$y_t = \boldsymbol{\beta}_t' \mathbf{x}_t + \xi_t, \ t = 1, \ldots, n$$

by least squares recursively for $t = k+1, \ldots, n$ giving $n-k$ least squares (RLS) estimates $(\hat{\boldsymbol{\beta}}_{k+1}, \ldots, \hat{\boldsymbol{\beta}}_T)$.

- RLS estimates may be efficiently computed using the Kalman Filter
- If $\boldsymbol{\beta}$ is constant over time then $\hat{\boldsymbol{\beta}}_t$ should quickly settle down near a common value.

3.4.2 Recursive residuals

Formal tests for structural stability of the regression coefficients may be computed from the standardized $1 - \textit{step}$ ahead recursive residuals

$$\begin{aligned} w_t &= \frac{v_t}{\sqrt{f_t}} = \frac{y_t - \hat{\boldsymbol{\beta}}'_{t-1}\mathbf{x}_t}{\sqrt{f_t}} \\ f_t &= \hat{\sigma}^2 \left[1 + \mathbf{x}'_t(\mathbf{X}'_t\mathbf{X}_t)^{-1}\mathbf{x}_t\right] \end{aligned}$$

Intuition:

- If β_i changes in the next period then the forecast error will not have mean zero
- w_t are recursive Chow Forecast "t-statistics" with $n_2 = 1$

3.4.3 CUSUM statistic

The CUSUM statistic of Brown, Durbin and Evans (1975) is

$$\begin{aligned} CUSUM_t &= \sum_{j=k+1}^{t} \frac{\hat{w}_j}{\hat{\sigma}_w} \\ \hat{\sigma}_w^2 &= \frac{1}{n-k}\sum_{t=1}^{n}(w_t - \bar{w})^2 \end{aligned}$$

Under the null hypothesis that $\boldsymbol{\beta}$ is constant, $CUSUM_t$ has mean zero and variance that is proportional to $t - k - 1$.

3.4.4 CUSUMSQ statistic

THE CUSUMSQ statistic is

$$CUSUMSQ_t = \frac{\sum_{j=k+1}^{t} \hat{w}_j^2}{\sum_{j=k+1}^{n} \hat{w}_j^2}$$

Under the null that $\boldsymbol{\beta}$ is constant, $CUSUMSQ_t$ behaves like a $\chi^2(t)$ and confidence bounds can be easily derived.

3.4.5 Application: Simulated Data

(insert graphs here)

Remarks

- Ploberger and Kramer (1990) show the CUSUM test can be constructed with OLS residuals instead of recursive residuals

- CUSUM Test is essentially a test to detect instability in intercept alone
- CUSUM Test has power only in direction of the mean regressors
- CUSUMSQ has power for changing variance
- There are tests with better power

3.4.6 Application: Exchange Rate Regression

(insert graphs here)

3.5 Nyblom's Parameter Stability Test

Reference:Nyblom, J. (1989). "Testing for the Constancy of Parameters Over Time," *Journal of the American Statistical Association*, 84 (405), 223-230.

Consider the linear regression model with k variables

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t,\ t = 1, \ldots, n$$

The time varying parameter (TVP) alternative model assumes

$$\boldsymbol{\beta} = \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t,\ \eta_{it} \sim (0, \sigma^2_{\eta_i}),\ i = 1, \ldots, k$$

The hypotheses of interest are

$$\begin{aligned} H_0 &: \boldsymbol{\beta} \text{ is constant } \Leftrightarrow \sigma^2_{\eta_i} = 0 \text{ for all } i \\ H_1 &: \sigma^2_{\eta_i} > 0 \text{ for some } i \end{aligned}$$

Nyblom (1989) derives the locally best invariant test as the Lagrange multiplier test. The score assuming Gaussian errors is

$$\sum_{t=1}^{n} \mathbf{x}_t\hat{\varepsilon}_t = \mathbf{0}$$

$$\hat{\varepsilon}_t = y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}}$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

Define

$$\begin{aligned} \mathbf{f}_t &= \mathbf{x}_t\hat{\varepsilon}_t \\ \mathbf{S}_t &= \sum_{j=1}^{t} \mathbf{f}_t = \text{ cumulative sums} \\ \mathbf{V} &= n^{-1}\mathbf{X}'\mathbf{X} \end{aligned}$$

Note that

$$\sum_{j=1}^{n} \mathbf{f}_t = \mathbf{0}$$

Nyblom derives the LM statistic

$$\begin{aligned} L &= \frac{1}{n\hat{\sigma}^2}\sum_{t=1}^{n}\mathbf{S}_t\mathbf{V}^{-1}\mathbf{S}_t \\ &= \frac{1}{n\hat{\sigma}^2} tr\left[\mathbf{V}^{-1}\sum_{t=1}^{n}\mathbf{S}_t\mathbf{S}_t'\right] \end{aligned}$$

Under mild assumptions regarding the behavior of the regressors, the limiting distribution of L under the null is a Camer-von Mises distribution:

$$\begin{aligned} L &\Rightarrow \int_0^1 \mathbf{B}_k^{\mu}(\lambda)\mathbf{B}_k^{\mu}(\lambda)' d\lambda \\ \mathbf{B}_k^{\mu}(\lambda) &= \mathbf{W}_k(\lambda) - \lambda\mathbf{W}_k(1) \\ \mathbf{W}_k(\lambda) &= \text{k dimensional Brownian motion} \end{aligned}$$

Decision: Reject H_0 at 5% level if

$$L > cv_{0.05}$$

Remarks:

- Distribution of L is non-standard and depends on k.
- Critical values are computed by simulation and are given in Nyblom, Hansen (1992) and Hansen (1997)
- Test is for constancy of all parameters
- Test is not informative about the date or type of structural change
- Test is applicable for models estimated by methods other than OLS
- Distribution of L is different if $\mathbf{x}_t$ is non-stationary (unit root, deterministic trend). See Hansen (1992).

3.5.1 Application: Simulated Data

Nyblom Test	
Model	L_c
No SC	.332
Mean shift	13.14***
Slope shift	14.13***
var shift	.351
RW slope	9.77***

3.5.2 Application: Exchange rate regression

Nyblom Test	
Model	L_c
AR(1)	1.27***
Diff reg	.413

3.6 Hansen's Parameter Stability Tests

References

1. Hansen, B.E. (1992). "Testing for Parameter Instability in Linear Models "*Journal of Policy Modeling*, 14(4), 517-533.

2. Hansen, B.E. (1992). "Tests for Parameter Instability in Regressions with I(1) Processes," *Journal of Business and Economic Statistics*, 10, 321-336.

Idea: Extension of Nyblom's LM test to individual coefficients.

Under the null of constant parameters, the score vector from the linear model with Gaussian errors is

$$\begin{aligned} \sum_{t=1}^{n} x_{it}\hat{\varepsilon}_t &= 0, \ i = 1, \ldots, k \\ \sum(\hat{\varepsilon}_t^2 - \hat{\sigma}^2) &= 0 \\ \hat{\varepsilon}_t &= y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}} \\ \hat{\sigma}^2 &= n^{-1}\sum_{t=1}^{n}\hat{\varepsilon}_t^2 \end{aligned}$$

Define

$$\begin{aligned} f_{it} &= \begin{cases} x_{it}\hat{\varepsilon}_t & i = 1, \ldots k \\ \hat{\varepsilon}_t^2 - \hat{\sigma}^2 & i = k+1 \end{cases} \\ S_{it} &= \sum_{j=1}^{t} f_{ij}, \ i = 1, \ldots, k+1 \end{aligned}$$

Note that

$$\sum_{i=1}^{n} f_{it} = 0, \ i = 1, \ldots, k+1$$

3.6.1 Individual Coefficient Tests

Hansen's LM test for

$$H_0 : \beta_i \text{ is constant}, \ i = 1, \ldots, k$$

and for

$$H_0 : \sigma^2 \text{ is constant}$$

is

$$\begin{aligned} L_i &= \frac{1}{nV_i} \sum_{t=1}^{n} S_{it}^2, \ i = 1, \ldots, k \\ V_i &= \sum_{t=1}^{n} f_{it}^2 \end{aligned}$$

Under $H_0 : \beta_i$ is constant or $H_0 : \sigma^2$ is constant

$$L_i \Rightarrow \int_0^1 B_1^{\mu}(\lambda) B_1^{\mu}(\lambda) d\lambda$$

Decision: Reject H_0 at 5% level if

$$L_i > cv_{0.05} = 0.470$$

3.6.2 Joint Test for All Coefficients

For testing the joint hypothesis

$$H_0 : \boldsymbol{\beta} \text{ and } \sigma^2 \text{ are constant}$$

define the $(k+1) \times 1$ vectors

$$\begin{aligned} \mathbf{f}_t &= (f_{1t}, \ldots, f_{k+1,t})' \\ \mathbf{S}_t &= (S_{1t}, \ldots, S_{k+1,t})' \end{aligned}$$

Hansen's LM statistic for testing the constancy of all parameters is

$$\begin{aligned} L_c &= \frac{1}{n} \sum_{t=1}^{n} \mathbf{S}_t' \mathbf{V}^{-1} \mathbf{S}_t = \frac{1}{n} tr \left(\mathbf{V}^{-1} \sum_{t=1}^{n} \mathbf{S}_t \mathbf{S}_t' \right) \\ \mathbf{V} &= \sum_{t=1}^{n} \mathbf{f}_t \mathbf{f}_t' \end{aligned}$$

Under the null of no-structural change

$$L_c \Rightarrow \int_0^1 \mathbf{B}_{k+1}^{\mu}(\lambda) \mathbf{B}_{k+1}^{\mu}(\lambda) d\lambda$$

Decision: Reject H_0 at 5% level if

$$L_c > cv_{0.05}$$

Remarks

- Tests are very easy to compute and are robust to heteroskedasticity
- Null distribution is non-standard and depends upon number of parameters tested for stability
- Individual tests are informative about the type of structural change
- Tests are not informative about the date of structural change
- Hansen's L_1 test for constancy of intercept is analogous to the CUSUM test
- Hansen's L_{k+1} test for constancy of variance is analogous to CUSUMSQ test
- Hansen's L_c test for constancy of all parameters is similar to Nybolom's test
- Distribution of tests is different if data are nonstationary (unit root, deterministic trend) - see Hansen (1992), JBES.

3.6.3 Application: Simulated Data

Hansen Tests				
Model	α	β	σ^2	Joint
No SC	.179	.134	.248	.503
Mean shift	13.19***	.234	.064	13.3***
Slope shift	.588	5.11***	.067	5.25***
var shift	.226	.119	.376*	.736
RW slope	.253	4.08***	.196	4.4***

3.6.4 Application: Exchange rate regression cont'd

Hansen Tests				
Model	intercept	slope	variance	Joint
AR(1)	.382	.147	2.94***	3.90***
Diff reg	.104	.153	.186	.520

4 Tests for Single Structural Change

Consider the linear regression model with k variables

$$y_t = \mathbf{x}_t' \boldsymbol{\beta}_t + \varepsilon_t,\ t = 1\ldots, n$$

No structural change null hypothesis

$$H_0 : \boldsymbol{\beta}_t = \boldsymbol{\beta}$$

Single break date alternative hypothesis

$$
\begin{aligned}
H_1 &: \begin{cases} \boldsymbol{\beta}_t = \boldsymbol{\beta},\ t \leq m = \text{break date} \\ \boldsymbol{\beta}_t = \boldsymbol{\beta} + \boldsymbol{\gamma},\ t > m \text{ and } \boldsymbol{\gamma} \neq \mathbf{0} \end{cases} \\
k &< m < n - k \\
\lambda &= \frac{m}{n} = \text{ break fraction}
\end{aligned}
$$

Remarks:

- Under no break null $\boldsymbol{\gamma} = \mathbf{0}$.
- Pure structural change model: all coefficients change ($\gamma_i \neq 0$ for $i = 1, \ldots, k$)
- Partial structural change model: some coefficients change ($\gamma_i \neq 0$ for some i)
- $m = [\lambda \cdot n]$, $[\cdot]$ = integer part

4.1 Chow's Test with Known Break Date

Assume: m or λ is known
For a data interval $[r, \ldots, s]$ such that $s - r > k$ define

- $\hat{\boldsymbol{\beta}}_{r,s}$ = OLS estimate of $\boldsymbol{\beta}$
- $\hat{\boldsymbol{\varepsilon}}_{r,s}$ = OLS residual vector
- $SSR_{r,s} = \hat{\boldsymbol{\varepsilon}}'_{r,s}\hat{\boldsymbol{\varepsilon}}_{r,s}$ = sum of squared residuals

Chow's breakpoint test for testing H_0 vs. H_1 with m known is

$$
F_n\left(\frac{m}{n}\right) = F_n(\lambda) = \frac{(SSR_{1,n} - (SSR_{1,m} + SSR_{m+1,n}))/k}{(SSR_{1,m} + SSR_{m+1,n})/(n-2k)}
$$

The Chow test may also be computed as the F-statistic for testing $\boldsymbol{\gamma} = \mathbf{0}$ from the dummy variable regression

$$
\begin{aligned}
y_t &= \mathbf{x}'_t\boldsymbol{\beta} + D_t(m)\mathbf{x}'_t\boldsymbol{\gamma} + \varepsilon_t \\
D_t(m) &= 1 \text{ if } t > m; 0 \text{ otherwise}
\end{aligned}
$$

Under $H_0 : \boldsymbol{\gamma} = \mathbf{0}$ with m known

$$
\begin{aligned}
&F_n(\lambda) \sim F(k, n-2k) \\
&k \cdot F_n(\lambda) \xrightarrow{d} \chi^2(k)
\end{aligned}
$$

Decision: Reject H_0 at 5% level if

$$
\begin{aligned}
F_n(\lambda) &> F_{0.95}(k, n-k) \\
k \cdot F_n(\lambda) &> \chi^2_{0.95}(k)
\end{aligned}
$$

4.1.1 Application: Simulated Data

	Chow Breakpoint Test		
	$F_{200}(0.5)$	$F_{200}(0.25)$	$F_{200}(0.75)$
No SC	0.808	0.081	1.55
Mean shift	377***	13.03***	11.21***
Slope shift	374***	10.97***	17.57***
Var shift	1.071	0.117	1.204
RW slope	80.14***	4.058**	2.218

4.2 Quandt's LR Test with Unknown Break Date

References:

1. Quandt, R.E. (1960). "Tests of Hypotheses that a Linear System Obeys Two Separate Regimes," *Journal of the American Statistical Association*, 55, 324-330.

2. Davies, R.A. (1977). "Hypothesis Testing When a Nuisance Parameter is Present only Under the Alternative,“ *Biometrika*, 64, 247-254.

3. Kim, H.-J., and D. Siegmund (1989). "The Likelihood Ratio Test for a Change-Point in Simple Linear Regression," *Biometrika*, 76, 3, 409-23.

4. Andrews, D.W.K. (1993). "Tests for Parameter Instability and Structural Change with Unknown Change Point," *Econometrica*, 59, 817-858.

5. Hansen, B.E. (1997). "Approximate Asymptotic P Values for Structural-Change Tests," *Journal of Business and Economic Statistics*, 15, 60-67.

Assume: m or λ is unknown.

Quandt considered the LR statistic for testing $H_0 : \boldsymbol{\gamma} = \mathbf{0}$ vs. $H_1 : \boldsymbol{\gamma} \neq \mathbf{0}$ when m is unknown. This turns out to be the maximal $F_n(\lambda)$ statistic over a range of break dates $m_0, \ldots, m_1$:

$$\begin{aligned} QLR &= \max_{m \in [m_0, m_1]} F_n\left(\frac{m}{n}\right) = \max_{\lambda \in [\lambda_0, \lambda_1]} F_n(\lambda) \\ \lambda_i &= \frac{m_i}{n} = \text{ trimming parameters, } i = 0, 1 \end{aligned}$$

Remarks

- QLR is also know as Andrews' sup $-F$ statistic
- Trimming parameters λ_0 and λ_1 must be set
  - Cannot have $\lambda_0 = 1$ and $\lambda_1 = 1$ because breaks are hard to identify near beginning and end of sample
  - Information about location of break can be used to specify λ_0 and λ_1

- Andrews recommends $\lambda_0 = 0.15$ and $\lambda_1 = 0.85$ if there is no knowledge of break date

- Implicitly, the break data m and break fraction λ are estimated using

$$\begin{aligned} \hat{m} &= \arg\max_m F_n\left(\frac{m}{n}\right) \\ \hat{\lambda} &= \hat{m}/n \end{aligned}$$

- Under the null, m defined under the alternative is not identified. This is an example of the "Davies problem".

- Davies (1977) showed that if estimated parameters are unidentified under the null, standard χ^2 inference does not obtain.

Under $H_0 : \boldsymbol{\gamma} = \mathbf{0}$, Kim and Siegmund (1989) showed

$$\begin{aligned} k \cdot QLR &\Rightarrow \sup_{\lambda \in [\lambda_0, \lambda_1]} \frac{\mathbf{B}_k^{\mu}(\lambda)' \mathbf{B}_k^{\mu}(\lambda)}{\lambda(1-\lambda)} \\ \mathbf{B}_k^{\mu}(\lambda) &= \mathbf{W}_k(\lambda) - \lambda \mathbf{W}_k(1) = \text{ Brownian Bridge} \end{aligned}$$

Decision: Reject H_0 at 5% level if

$$k \cdot QLR > cv_{0.05}$$

Remarks

- Distribution of QLR is non-standard and depends on the number of variables k and the trimming parameters λ_0 and λ_1

- Critical values for various values of λ_0 and λ_1 computed by simulation are given in Andrews (1993), and are larger than $\chi^2(k)$ critical values. For $\lambda_0 = 0.15$ and $\lambda_1 = 0.85$

5% critical values			
k	$\chi^2(k)$	QLR	$k \cdot QLR$
1	3.84	8.85	
10	18.3	27.03	

- P-values can be computed using techniques from Hansen (1997)

- Graphical plot of $F_n(\lambda)$ statistics is informative to locate the break date

4.2.1 Application: Simulated data

QLR or sup-F Test			
	QLR	$\widehat{m}$	$\widehat{\lambda}$
No SC	2.87	142	0.71
Mean shift	377***	101	0.51
Slope shift	374***	101	0.51
Var shift	2.36	142	0.71
RW slope	113***	77	0.39

(insert graphs here)

4.2.2 Application: Exchange rate data

QLR or sup-F Test			
	QLR	$\widehat{m}$	$\widehat{\lambda}$
AR(1)	12.13***	1989:05	0.65
Diff reg	4.08	1991:03	0.74

4.3 Optimal Tests with Unknown Break Date

References:

1. ANDREWS, D.W.K. AND W. PLOBERGER (1994). "Optimal Tests When a Nuisance Parameter Is Present Only Under the Alternative," *Econometrica*, 62, 1383-1414.

Andrews and Ploberger (1994) derive tests for structural change with an unknown break date with optimal power. These tests turn out to be weighted averages of the Chow breakpoint statistics $F_n(\frac{m}{n})$ used to compute the QLR statistic:

$$\begin{aligned} ExpF_n &= \ln\left(\frac{1}{m_2 - m_1 + 1}\sum_{t=m_1}^{m_2} \exp\left(\frac{1}{2}k \cdot F_n\left(\frac{t}{n}\right)\right)\right) \\ AveF_n &= \frac{1}{m_2 - m_1 + 1}\sum_{t=m_1}^{m_2} k \cdot F_n\left(\frac{t}{n}\right) \\ k &= \text{number of regressors being tested} \end{aligned}$$

Remarks

- Asymptotic null distributions are non-standard and depend on k, λ_0 and λ_1
- Critical values are given in Andrews and Ploberger; P-values can be computed using techniques of Hansen (1997)
- Tests can have higher power than QLR statistic
- Tests are not informative about location of break date

4.4 Empirical Application

Reference: Stock and Watson (199?), " " *Journal of Business and Economic Statistics*.