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On Left: Gabriel Cramer (1704-1752), Swiss mathematician and correspondent of Euler's. On Right: the Marquis de Condorcet (1743-1794), an editor of Euler's *Letters to a German Princess*.

On Euler's Letter to Cramer of October 20, 1744

By Robert E. Bradley

The correspondence between Euler and Gabriel Cramer (1704-1752) will soon be readily available, because it will be included in the forthcoming Volume 7 of Series IVA of Euler's *Opera Omnia* [Euler, vol. IVA.7], scheduled to be published sometime soon. It consists of 19 letters in perfect alternation. The first one was a brief letter from Euler, written in 1743. Its contents and tone make it clear that there had previously been no direct contact between the two men. The final letter was written by Euler in late 1751, just a few weeks before Cramer's death.

However, the 1975 catalog of Euler's correspondence [Euler, vol. IVA.1] lists only 17 of these letters. One of the two missing documents was Cramer's final letter to Euler. Although its whereabouts remain a mystery, which is rather surprising, because Euler seems to have kept careful records of his correspondence, its contents are known and will be included in the *Opera Omnia*, because Cramer's draft survives in the archives of the public library in Geneva, where Cramer lived and taught. The other missing letter was Euler's third to Cramer. It was written at some point between Cramer's letters of September 30 and November 11, 1744, but was entirely unknown in 1975.

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The Omnipresent Savant

By Dominic Klyve

Seeking the Original Text of Euler's *Letters to a German Princess*

In my previous column, I documented some of the changes that various translators and editors of Leonhard Euler's *Lettres a une Princesse d'Allemagne* made to just his first letter. These changes, though a bit annoying to those of us who wish to see Euler's original work, could still perhaps be described as stylistic. As such, they seem minor in comparison to another set of changes made to Euler's text in the most popular French edition of the eighteenth century. These changes, made deliberately and with political purpose, were propagated into other editions and languages with such consistency that today we must work hard to uncover Euler's original text. In this column, we will try to examine these changes, with the goal of helping the English-speaking world read Euler's original work.

The most widely distributed and well-known edition of the *Lettres* (and the one from which Henry Hunter made the only English translation to date) is arguably the third (E343[4], published in 1787), which the Marquis de Condorcet played a significant role in publishing. This sounds like a wonderful match—a leading scientist helping to edit the works of a giant of the previous generation—and even reminds us of some of the great mathematicians (including Carathéodory, Lagrange, and Weber) who worked on the first volumes of the *Opera Omina*. I'm sad to say that Condorcet did not match the selfless and impressive work of the twentieth century editors; another thing that I've learned since writing my last column is that he systematically and deliberately edited Euler's work by excising passages that he didn't think belonged there.

Condorcet is a name perhaps not as well known in the mathematics or science communities as it is to historians. He was a competent but not a great mathematician. His greatest achievements were perhaps quasi-mathematical; he was one of the first people to apply mathematics to understand human behavior. (We still see his name while studying voting theory, for example.) He was an influential figure in the French Revolution and was, in the words of E. O. Wilson, "a complete revolutionary, both anticlerical and republican." (Wilson) He was committed to the idea of human progress, and to creating a "more perfect social order ruled by science and secular philosophy." (Wilson, p. 19)

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Condorcet's legacy in Euler's *Lettres* came to my attention via an unexpected source. Many *Opusculum* readers are aware that Andie Ho recently translated E92 (The Defense of Divine Revelation). In fact she did more; she translated an entire book published by Adrien Le Clere [anonymous] in 1805. The Le Clere book includes a brief discussion about how difficult it had been for the publishing house to locate a copy of E92, and then reprints the entire work (in a French translation of Euler's original German). The anonymous editor of the book then follows this with an essay that startled me, and which provided the final piece of what has now become a trilogy of *Opusculum* columns. The essay, "Comparison of the last edition of Euler's Letters published by de Condorcet, with the original edition" carefully demonstrates that Condorcet removed, wherever possible, Euler's references to God, salvation, and scripture.

That Euler was a devout man is well known, and he was content using theological ideas to buttress or explain his scientific ones. Condorcet found these ideas anathema to Euler's goal of teaching science through the *Lettres*, and to the larger Enlightenment goals of rationalism in all things. He removed Euler's theological references in every letter in which doing so wouldn't destroy the purpose of the letter. Hunter found and restored some of Condorcet's excisions, but not all of them. The first letter in which this occurs seems to be Letter 18. By the end of this letter, Euler believes that he has shown that Newton had erred in his theory of light emission from the sun (despite his obvious brilliance). He then concludes with a lengthy philosophical reflection:

> If we are prone to such sad mistakes in our research on the phenomena in this visible world, a world which we can sense, how unfortunate would we be if God had abandoned us to ourselves with regards to the invisible world and our eternal salvation. On this important point, a revelation is absolutely necessary to us. We should make the most of it with the greatest veneration; and when this revelation presents us with things that seem inconceivable, we have but to remember the weaknesses of our mind, which strays so easily, even for the visible things. Each time I hear these freethinkers criticize the truths of our religion and even mock it with the most impertinent self-importance, I think and say to myself, "Puny mortals, no matter how lightly you gloss over these things and how many you ignore, they are more sublime and elevated than those on which the great Newton was so grossly mistaken. I hope that Your Highness never forgets this thought; the times when you are in need of it come all too often. [Ho's translation]

This entire passage was cut by Condorcet. In his translation, Henry Hunter, a Scottish minister with no desire to hide Euler's piety, restores the passage in a footnote at the end of the letter. Hunter, however, didn't find every passage that Condorcet removed. The first example I can find that Hunter missed occurs in Letter 21. In an attempt to give some meaning to the vast distance to the stars, and the finite speed of light, Euler in his original letter invokes a Biblical reference:

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The Euler Line

New Publications and Some New Journals

Homer White's chapter, "An 'Impossible' Problem, Courtesy of Leonhard Euler", appears in a new MAA volume, *Mathematical Time Capsules*. White's work provides a guide to using Euler's results from "Reflections on a problem of geometry dealt with by certain geometers which nevertheless is impossible" [E220] as a project in a Calculus II class. **Robert Bradley**'s chapter in the same volume, "Cusps: Horns and Beaks", makes a similar use of Euler's "Sur le point de rebroussement de la seconde espece" [E169].

Several other historians of mathematics have contributed to this volume, including, Lawrence D'Antonio, Victor Katz, Kim Plofker, and Amy Shell-Gellasch. Each chapter presents a topic or a historical thread that can be used in an undergraduate mathematics course, along with further references and readings on the topic.

Eric Schliesser's article, "Newton's Challenge to Philosophy: A Programmatic Essay," appears in *HOPOS: The Journal of the International Society for the History of Philosophy of Science*, Vol. 1, No. 1, Spring 2011, pp. 101-128.

Schliesser's goal in this paper is to "identify a set of interlocking views that became (and still are) very influential within philosophy in the wake of Newton's success." Euler plays a small but prominent role in the demonstration of the pre-eminence that physical views held in philosophical discussion in the post-Newton era.

Karin Reich's paper, "Ein neues Blatt in Eulers Lorbeerkranz, durch Carl Friedrich Gauß eingeflochten" ("A new leaf twisted into Euler's wreath of laurel by Carl Friedrich Gauss"), has recently been published in *Abhandlungen der Akademie der Wissenschaften zu Göttingen*, new series Vol. 10 (2011), p. 223-273.

Reich's paper casts new light on a connection between Gauss to Leonhard Euler. This article is currently available (in part) via Google Books.

Some time ago, **Erik Koelink** and **Walter Van Assche** published their paper, "Leonhard Euler and a *q*-analogue of the logarithm" in the Proceedings of the American Mathematical Society (AMS), Vol. 137, No. 5 (2009).

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The lost letter [Euler 1744b] became known to Euler scholars at the meeting of the Euler Society in August 2003. At some point in the 20th century, it found its way into the private collection of Bern Dibner (1897-1988). Dibner was an engineer, entrepreneur and philanthropist, as well as a historian of science. Over the course of his long life, he amassed an impressive private collection of rare books, manuscripts and letters. He donated about a quarter of this collection to the Smithsonian in 1974 and Euler's missing letter of October 20, 1744, was part of that gift. Mary Lynn Doan, professor of mathematics at Victor Valley Community College, had contacted the Dibner Library of the Smithsonian Institution in the summer of 2003 and had learned that they have a small collection of documents by Leonhard Euler [Euler Mss]. She visited the Library on her way to the Euler Society's meeting that summer and brought a photocopy of the letter with her. I was able to identify the addressee as Cramer and shortly afterwards I brought the letter to the attention of Andreas Kleinert, co-editor of the forthcoming [Euler, vol. IVA.7]. Thanks to Mary Lynn and the excellent archivists at the Smithsonian, Euler's *Opera Omnia* will now include the complete correspondence with Cramer.

What follows is my English translation of this letter, now catalogued as R.461a. For more about the contents of the letter, see Ed Sandifer's How Euler Did It column for Nov-ember 2009 at maa.org and the article "When Nine Points are Worth But Eight: Euler's Resolution of Cramer's Paradox" by Lee Stemkoski and me in *Convergence*.

— R. Bradley

Sir,

As I have not yet seen my work, which has just come off the press,[1] I am infinitely obliged to you for the particular trouble you have taken with the corrections. But great though my obligation to you may be, so much greater should be my sympathy for the precious time you have expended, and the scientific community [*les Scavans*] should be very displeased with me for causing you to have turned away from your usual occupations, so highly esteemed by all. It is because of this consideration that I completely approve of the reply you have made to Mr. Bousquet, in refusing your assistance with respect to proofreading my work,[2] which he wishes to publish, not doubting for a moment that he would never find a man as qualified for the task as you in Lausanne. I have learned with great pleasure that you have composed a piece on the same material[3] and, as I am extremely curious to see it, I add my wishes to those of Mr. Bousquet to encourage you to publish it.

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> If, at the beginning of the world, the stars had been created at about the same time as Adam, he would not have been able to see even the closest ones for six years; he would have had to wait even longer before discovering the others, since they are even farther from the Earth. [Ho's translation]

Condorcet cut this sentence from Euler's letter, presumably on the belief that reference to figures such as Adam did not befit a scientific work. This sentence does not appear in the first English edition. Nor, as far as I have yet been able to determine, is there any mention of Adam in any of the English printings of the Letters (although I must confess to being limited in this claim; my talented and helpful interlibrary loan librarian only secured for me the front matter and first letter of most of the editions that I have).

The book published by Le Clere goes on to list dozens of instances in which Condorcet and the other French editors removed passages from Euler's work. They range from the trivial to the significant, and the reader who only sees Euler in a later French edition, or in English translation, will end up with a distorted sense of Euler's thought.

Condorcet was unapologetic about rewriting Euler's work. It is interesting and instructive to read his *Avert-issement* at the beginning of the book. "Without failing in the respect due to Euler," Condorcet writes, "I thought myself at liberty to omit some passages altogether, and to correct the style of others" [this and following trans-lations taken from the 2nd English edition, presumably done by Hunter]. He then spends more than a page justifying this, claiming that although it may be unreasonable to expect a non-native speaker to write "a foreign language with classical purity", some readers who didn't already know Euler's greatness might judge him harshly for this. Wanting to save Euler from such judgment, Condorect apparently edited his French for style. Having devoted considerable attention to this, he then almost glosses over the changes we discussed above:

> As to other retrenchments, they affect, almost all of them, reflections which relate less to the science and philosophy, than to theology, and frequently even to the peculiar doctrines of that ecclesiastical commun-ion in which Euler lived. It is unnecessary to assign a reason for omissions of this description.

At the end of this, the third of my trilogy of essays on Euler's *Lettres a une Princesse d'Allemagne*, what have we found? First, we are pleased that 250 years after they

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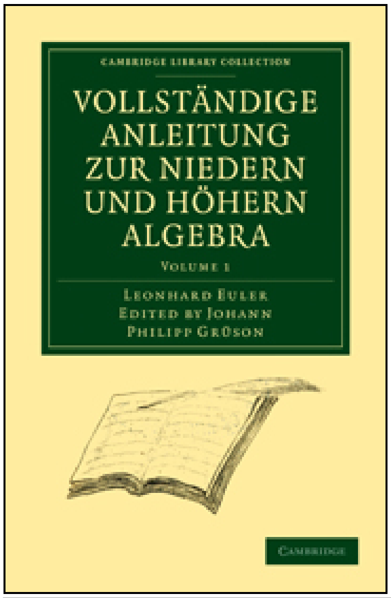
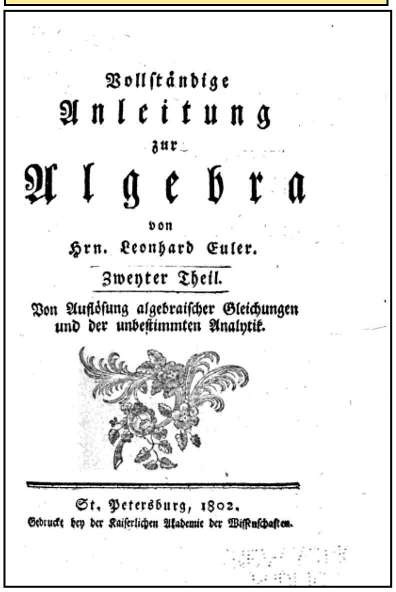
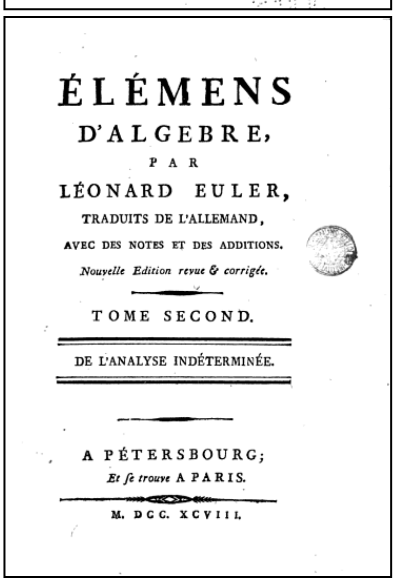
1: Here Euler is referring to *Methodus inveniendi lineas curvas* [Euler 1744a], published in Lausanne by Bousquet in 1744. 2: Here Euler is referring to *Introductio in analysin infnitorum* [Euler 1748], published in Lausanne by Bousquet in 1748. 3: Cramer's *Introduction à l'analyse des lignes courbes algébraique* [Cramer 1750], published by Bousquet in 1750, and volume 2 of Euler's *Introductio* [Euler 1748] both dealt with the theory of equations.

Review: Johann Philipp Grüson's edition of Euler's *Algebra*







By Rüdiger Thiele

This paperback edition in three volumes is a useful reprint of Grüson's edition of Euler's famous algebra textbook for students "Vollständige Anleitung zur Nieder-en und Höheren Algebra" (Complete instruction to elementary and advanced algebra) in German. It appeared in 1796/97, including also a German translation of Lagrange's French additions by Kaußler. The bibliography of Euler's "Vollständige Anleitung zur Algebra" (Elements of Algebra) [E387, E388] is a little bit confusing. The "Elements" appeared first in 1768/69, published by the St. Petersburg Academy, but in a Russian translation by P. Inokhodtsov and I. Iudin, then the same Academy published a German edition in 1770. In 1774 Euler's successor in Berlin, Joseph-Louis Lagrange, initiated a French edition of the "Elements" with his own additions (about 100 pages); so Euler's German text was translated into French by Johann III Bernoulli who also lived in Berlin. This French edition was the starting point for the English edition, which was begun by Francis Horner. He died before he completed the work, and left it to John Hewlett who finally edited an English translation in 1797. There were further translations into other European languages, for example into Dutch in 1773, into Russian in a second translation by V. Viskovatov in 1812, and even Latin in 1790; also selections of the "Elements" appeared (in German by J. Ebert in 1789, in English by J. Farrar in 1818). In 1972 C. Truesdell and in 2007 C. Sangwin edited the English issue, the former with an introduction "Euler, supreme geometer", the latter with a modernized text and shortened annotations of Part I of Hewlett's edition. Finally, the *Cambridge Library Collection* contains not only the books under review but also a reprint of Hewlett's translation (2009).

The Russian Academy published a second and third German edition in 1771 and 1802; later German editions are due to the Reclam publisher, starting in 1883 (ed. L. Natani) up to 1942, with more than 100.000 copies sold. In 1959 a revised edition by J.E. Hofmann and J. Niessner was published, finally in 1911 the "Opera Omnia Euleri" opened with Euler's "Elements" edited by H. Weber (series I, volume 1). The edition under review is a revised one. Grüson (1768-1857), also Gruson, was a German mathematician, from 1794 in Berlin and from 1798 a member of Berlin's Academy. He wrote in his Preface that he dressed the wordy and lengthy presentation (wortreichen und weitläufigen Vortrag) by Euler in a more attractive robe (in ein den Geschmack weniger beleidigendes Gewand) but finally he himself also added some long-winded notes (at least from our viewpoint). So did the translator of Lagrange's additions (vol. 3) by supplying an appendix of 79 pages. Grüson announced in his Preface (Vorbericht) that the third volume would contain also the elements of the calculus; but the actual volume does not. The editors of the German issue of the "Elements" (1770) inserted a Preface in which they report that the almost blind Euler dictated the "Elements" to an uneducated German servant. The translator of the French edition Johann III Bernoulli remarked: "I have endeavored to translate this algebra in the style best suited to works of the kind. My chief anxiety was to enter into the sense of the original, and to render it with the greatest perspicuity. Perhaps I may presume to give my translation some superiority over the original, because that work having been dictated, and admitting of no revision from the [almost blind] author himself, it is easy to conceive that in many passages it would stand in need of correction. If I have not submitted to translate literally, I have not failed to follow my author step by step. ... Nor shall I take any more notice of the notes which I have added to the first part. They are not so numerous as to make me fear the reproach of having unnecessarily increased the volume." Grüson improved some numerical calculations made by Euler (for example in vol. 2: part II, §§ 109, 111 (table), 140) but did not notice further mis-

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Frontispieces and covers from the various publications of Euler's *Algebra*. Top: Grüson's publication, which has recently become available in paperback. Center: The original German *Vollständige Anleitung zur Algebra*. Bottom: the French *Élémens d'Algebre*, volume 2.

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takes (for example vol. 2: part I, §198, x = -3 and x = -4 are not solutions; §227, case III 1126819/894348 is correct; part II, §175 negative q omitted). Moreover, he made also some confusing remarks such as this one: "Positive quantities are less than nothing." (I, § 18, Zusätze 3 and 4)

Euler left Berlin on June 1, 1766 and arrived in July 1766 in St. Petersburg. Already in 1769 the first part of the Russian translation was printed. Probably Euler began the writing of the "Elements" in Berlin, at least he made a draft for the book in Berlin which must have been finished in 1768 at latest. Four small matters support this conjecture: in two examples Euler embedded the numbers 1765 and 1766 (§§ 243, 421) also in his "Letters" (part II, E344), he liked to play with dates by speaking of an 1761-gon (Letter written February 7, 1761) or asking for the invention of gunpowder (1734 – 354 = 1380) in his "Rechenkunst", ch. 3 (E17).

We have about 120 examples in the "Elements." Euler was familiar with the typical and attractive examples of algebra already taught to him by his father who used Stifel's Coss (1553) "with beautiful examples", and later Euler selected such examples for his "Elements"; in Part I are about 40 examples which directly refer to Stifel's book (A. Heefer). For example, the problem of the nails for a horseshoe (I, §419) is due to Rudolff's "Coss" (1525) which Euler knew in Stifel's version. Finally, Euler's examples are an essential part of the very clear and pleasing style of the composition, and surely they do their bit for its success. In his Preface Grüson repeats the nice story already told by the editors of the 1770-issue and repeated by many others later (for example N. Fuss in his "Éloge", 1783), that the young man who wrote it down was by profession a tailor; he received no instructions but in a short time was able to perform the most difficult algebraic calculations, and to resolve with readiness whatever analytical questions were proposed to him. So we can recommend the book to teachers and of course to "lovers of Algebra" (eds. of 1760 issue) and, moreover, all Eulerians.

— R. Thiele

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L. Euler: *An introduction to the elements of algebra*. Selected by J. Farrar. Cambridge 1818, 1821, 1828, 1836, 2010.

R. Thiele: *Der Weg zur modernen Algebra*, in: K. Reich und A. Kreuzer (eds.), *Emil Artin*. Augsburg 2007.

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were written, the letters remain rich in scientific, philosophical, and didactic content. However, we seem to find ourselves in an uncomfortable position; this brilliant work, one of Euler's most popular books, exists in English only in Bowdlerized form. Going back to the French original is helpful only if we first the very first original—too many changes have snuck in (and back out again) during later printings. The original is hard to find, but happily the *Opera Omnia* has reprinted and retypeset the original faithfully. To Eulerians interested in reading or using the *Letters to a German Princess*, I cannot recommend too strongly the importance of checking assertions and quotations against their edition (or the first French printing, if it can be found).

It's now clear to me how very vital it is to have a new translation of the *Lettres*—one written in modern English, and one faithful to Euler's work. Andie Ho is already considering such a project, and I couldn't be happier. Until she (or someone else) actually completes this work, Euler's legacy to the English-speaking world will not be complete.

— D. Klyve

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Anonymous, *Défense de le Révélation contre les objections des esprits-forts, by M. Euler, suivie des penseées de cet auteur sur la religion, suprimeées dans la dernière edition de les letters a une princesse d'Allemagne*. Chez Adrien Le Clere, Imprimeur de N.S.P. le Pape. Original scan and English translation by Andie Ho; available on the Euler Archive.

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In my opinion, these matters have, by and large, not yet been properly explained and I do not doubt that you have clarified a great number of situations that have eluded me, as well as others who have written on the subject. One finds there questions so thorny, that one must apply to them all possible attention so as not to fall into error, as happened to me in my explanation of the cuspidal point of the second kind.[4] Mr. the Marquis de l'Hopital showed that there are in fact curves endowed with such a point, but Mr. Gua de Malves holds that the two branches of a curve which form the point always extend to the other side so that, according to him, this point is nothing but the intersection of two branches, which cross in an infinitely small angle. These arguments convinced me that he was right, as you no doubt noted in looking over my work. But since then, I have recognized quite clearly that I was mistaken on this and that there actually are curves that have such a cusp point by itself, one that cannot be regarded as the infinitely close intersection of two branches. Even in the fourth order there is a curved line of this kind, whose equation is[5]

$$y^4 - 2xy^2 + xx = x^3 + 4yxx,$$

which simplifies to

$$y = \sqrt{x} \pm \sqrt[4]{x^3}.$$

This reason I was mistaken is that I believed that this curve ought to have a diameter[6], since $\sqrt{x}$ may take a negative value as well as positive, but since the other term $\sqrt[4]{x^3}$ is equal to the first one, $\sqrt{x}$, multiplied by its own square root $\sqrt[4]{x^3}$, one sees clearly we may not take the first $\sqrt{x}$ to be negative, without the other $\sqrt[4]{x^3}$ becoming imaginary. And in fact, if we give the $\sqrt{x}$ term the $-$ sign, then the equation

$$y = -\sqrt{x} \pm \sqrt[4]{x^3}$$

is resolved as

$$y^4 - 2xy^2 + xx = x^3 - 4yxx,$$

which is not the same curve in the same position. As I do not have a copy of my manuscript here, I beg you to add a little note at this location, if you have not already returned home.

I have seen that Mr. Maclaurin already had the same doubt concerning the number of points which determine curves of a given order: he says that to determine a line of the third order, the number of nine points may be too small, yet still the number of ten is too great, which in my opinion is an overt contradiction.

The aforementioned Braikenridge is also absolutely mistaken in holding that a line of order n may be described by n^2+1 points and it is a disputed truth, as you have very well remarked, that this number is but $(nn+3n)/2$. Further-more, one may not doubt that two curved lines, one of which is of order m and the other of order n, may intersect in mn points, though you will be the first to have given a perfect proof of this truth, for I freely admit that my proof is all but complete. At first, all of these reflections only served to bring to my attention the difficulties of the case, which you were so good as to propose to me. However, I finally found the solution to this doubt, with which I hope you will be satisfied. I say, then, that although it is indeed true that a line of order n be determined by $(nn+3n)/2$ points, this rule is nevertheless subject to certain exceptions. For although the general equation of lines of order n has $(nn+3n)/2$ coefficients to be determined, it may happen that such a number of equations, which we draw from the same number of given points, are not sufficient for this effect: this is evident, when two or several of these equations become identical. In such a case, one finds after having reduced the matter to the determination of the final coefficient, the value of this is expressed by a fraction, whose numerator and denominator both become = 0. I conceive therefore, that this inconvenience will take place when the nine points, which ought to determine a line of the 3rd order, are disposed such that two curved lines of this order may be drawn through them. In this case, the nine given points, since they contain two identical equations, are worth but 8, and we may then add the tenth point in order to render the problem determined. We may clarify this article to our further satisfaction by considering lines of the second order, for the determination of which 5 points may not always be sufficient. For when all the five points are arranged on a straight line so that they give, for example, these equations[7]

$$\begin{array}{lllll} x=0\ ; & x=1\ ; & x=2\ ; & x=3\ ; & x=4\ ; \\ y=0\ ; & y=1\ ; & y=2\ ; & y=3\ ; & y=4\ ; \end{array}$$

all of the coefficients of the general equation[5]

$$\alpha yy + \beta xy + \gamma xx + \delta y + \varepsilon x + \varsigma = 0$$

will not be determined, for after having introduced all of the given determinations, we are brought to this equation

$$\alpha yy - (\alpha + \gamma)xy + \gamma xx + \delta y - \delta x = 0,$$

so that there still remain two coefficients to be determined. If from the five given points there had been but 4 arranged in a straight line, then there would remain but one coefficient to be determined. From this, one easily understands that if the nine points, from which one ought to draw a line of the third order, are at the same time the intersections of two curved lines of this order, then, after having completed all of the calculations, there must remain in the general equation for this order an undetermined coefficient, and beginning from

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4: For more on the cuspidal point of the second kind and the importance of the 4th degree equation that follows, see [Bradley 2006]. 5: The equations in these paragraphs were actually written by Euler as in-line equations. We have set them as displayed equations for greater clarity. 6: That is, Euler thought the curve was symmetric about the x-axis. 7: In modern notation, Euler is considering the points (0, 0), (1, 1), (2, 2), (3, 3), and (4, 4).

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this case not only two, but an infinity of lines of the 3rd order may be drawn from the same nine points.

The use, which you have made of continued[8] fractions in dioptrics is admirably beautiful and I am extremely obliged to you for the theorem, which you have communicated to me. I am charmed that you recognize, along with me, that this material is of great use in mathematics and that it is quite worthy of attention. It is not only arithmetic that can draw much profit from it, but also the integral calculus, as I made known in several pieces on this subject that I left in Petersburg, one of which[9] has already been published in the ninth volume of the *Comm.*

It is already a long time since Mr. Bousquet wrote to tell me that you had the kindness to send me a copy of the *Works* of Mr. James Bernoulli, which was shipped here along with a quantity of books for Mr. Neaulme. But since this latter was not willing to accept the package, I have received nothing. Had it not been for this, I would not have failed to thank you infinitely. I am therefore embarrassed that I do not find myself in a position to show my gratitude except in words, but rest assured, that should an opportunity present itself for me to render you service, I will employ all of my energy to discharge my obligation. I have the honor of being, with the most perfect esteem,

Sir,

Your very humble and very obedient servant L. Euler

Berlin this 20 October 1744

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8: For some reason, Euler had underlined the word *continues* (translated *continued* here) in this letter. 9: Euler is referring to E71, "*De fractionibus continuis dissertatio*," which was presented to the St. Petersburg Academy on March 7, 1737. However, volume 9 of the *Commentarii academiae scientiarum imperialis Petropolitanae*, for the year 1737, did not actually appear until 1744.