Euler's Letter to Cramer of October 20, 1744

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The correspondence between Euler and Gabriel Cramer (1704-1752) will soon be readily available, because it will be included in the forthcoming Volume 7 of Series IVA of Euler's *Opera Omnia* [Euler, vol. IVA.7], scheduled to be published sometime soon. It consists of 19 letters in perfect alternation. The first one was a brief letter from Euler, written in 1743. Its contents and tone make it clear that there had previously been no direct contact between the two men. The final letter was written by Euler in late 1751, just a few weeks before Cramer's death.

However, the 1975 catalog of Euler's correspondence [Euler, vol. IVA.1] lists only 17 of these letters. One of the two missing documents was Cramer's final letter to Euler. Although its whereabouts remain a mystery, which is rather surprising, because Euler seems to have kept careful records of his correspondence, its contents are known and will be included in the *Opera Omnia*, because Cramer's draft survives in the archives of the public library in Geneva, where Cramer lived and taught. The other missing letter was Euler's third to Cramer. It was written at some point between Cramer's letters of September 30 and November 11, 1744, but was entirely unknown in 1975.

The lost letter [Euler 1744b] became known to Euler scholars at the meeting of the Euler Society in August 2003. At some point in the 20th century, it found its way into the private collection of Bern Dibner (1897-1988). Dibner was an engineer, entrepreneur and philanthropist, as well as a historian of science. Over the course of his long life, he amassed an impressive private collection of rare books, manuscripts and letters. He donated about a quarter of this collection to the Smithsonian in 1974 and Euler's missing letter of October 20, 1744, was part of that gift. Mary Lynn Doan, professor of mathematics at Victor Valley Community College, had contacted the Dibner Library of the Smithsonian Institution in the summer of 2003 and had learned that they have a small collection of documents by Leonhard Euler [?]. She visited the Library on her way to the Euler Society's meeting that summer and brought a photocopy of the letter with her. I was able to identify the addressee as Cramer and shortly afterwards I brought the letter to the attention of Andreas Kleinert, co-editor of the forthcoming [Euler, vol. IVA.7]. Thanks to Mary Lynn and the excellent archivists at the Smithsonian, Euler's *Opera Omnia* will now include the complete correspondence with Cramer.

What follows is my English translation of this letter, now catalogued as R.461a. For more about the contents of the letter, see Ed Sandifer's *How Euler Did It* column for November 2009 at maa.org and the article "When Nine Points are Worth But Eight: Euler's Resolution of Cramer's Paradox" by Lee Stemkoski and me in *Convergence* at mathdl.maa.org.

Sir,

As I have not yet seen my work, which has just come off the press,[1] I am infinitely obliged to you for the particular trouble you have taken with the corrections. But great though my obligation to you may be, so much greater should be my sympathy for the precious time you have expended, and the scientific community [*les Scavans*] should be very displeased with me for causing you to have turned away from your usual occupations, so highly esteemed by all. It is because of this consideration that I completely approve of the reply you have made to Mr. Bousquet, in refusing your assistance with respect to proofreading my work,[2] which he wishes to publish, not doubting for a moment that he would never find a man as qualified for the task as you in Lausanne. I have learned with great pleasure that you have composed a piece on the same material[3] and, as I am extremely curious to see it, I add my wishes to those of Mr. Bousquet to encourage you to publish it. In my opinion, these matters have, by and large, not yet been properly explained and I do not doubt that you have clarified a great

[1] Here Euler is referring to *Methodus inveniendi lineas curvas* [Euler 1744a], published in Lausanne by Bousquet in 1744.

[2] Here Euler is referring to *Introductio in analysin infinitorum* [Euler 1748], published in Lausanne by Bousquet in 1748.

[3] Cramer's *Introduction à l'analyse des lignes courbes algébraique* [Cramer 1750], published by Bousquet in 1750, and volume 2 of Euler's *Introductio* [Euler 1748] both dealt with the theory of equations.

number of situations that have eluded me, as well as others who have written on the subject. One finds there questions so thorny, that one must apply to them all possible attention so as not to fall into error, as happened to me in my explanation of the cuspidal point of the second kind.[4] Mr. the Marquis de l'Hopital showed that there are in fact curves endowed with such a point, but Mr. Gua de Malves holds that the two branches of a curve which form the point always extend to the other side so that, according to him, this point is nothing but the intersection of two branches, which cross in an infinitely small angle. These arguments convinced me that he was right, as you no doubt noted in looking over my work. But since then, I have recognized quite clearly that I was mistaken on this and that there actually are curves that have such a cusp point by itself, one that cannot be regarded as the infinitely close intersection of two branches. Even in the fourth order there is a curved line of this kind, whose equation is[5]

$$y^4 - 2xy^2 + xx = x^3 + 4yxx,$$

which simplifies to

$$y = \sqrt{x} \pm \sqrt[4]{x^3}.$$

This reason I was mistaken is that I believed that this curve ought to have a diameter[6], since $\sqrt{x}$ may take a negative value as well as positive, but since the other term $\sqrt[4]{x^3}$ is equal to the first one, $\sqrt{x}$, multiplied by its own square root $\sqrt{\sqrt{x}}$, one sees clearly we may not take the first $\sqrt{x}$ to be negative, without the other $\sqrt[4]{x^3}$ becoming imaginary. And in fact, if we give the $\sqrt{x}$ term the $-$ sign, then the equation

$$y = -\sqrt{x} \pm \sqrt[4]{x^3}$$

is resolved as

$$y^4 - 2xy^2 + xx = x^3 - 4yxx,$$

which is not the same curve in the same position. As I do not have a copy of my manuscript here, I beg you to add a little note at this location, if you have not already returned home.

I have seen that Mr. Maclaurin already had the same doubt concerning the number of points which determine curves of a given order: he says that to determine a line of the third order, the number of nine points may be too small, yet

[4]For more on the cuspidal point of the second kind and the importance of the 4th degree equation that follows, see [Bradley 2006]

[5]The four equations in this paragraph were actually written by Euler as in-line equations. We have set them as displayed equations for greater clarity.

[6]That is, Euler thought the curve was symmetric about the x-axis.

still the number of ten is too great, which in my opinion is an overt contradiction. The afore-mentioned Braikenridgeis also absolutely mistaken in holding that a line of order n may be described by $n^2 + 1$ points and it is a disputed truth, as you have very well remarked, that this number is but $\frac{nn+3n}{2}$. Furthermore, one may not doubt that two curved lines, one of which is of order m and the other of order n, may intersect in mn points, though you will be the first to have given a perfect proof of this truth, for I freely admit that my proof is all but complete. At first, all of these reflections only served to bring to my attention the difficulties of the case, which you were so good as to propose to me. However, I finally found the solution to this doubt, with which I hope you will be satisfied. I say, then, that although it is indeed true that a line of order n be determined by $\frac{nn+3n}{2}$ points, this rule is nevertheless subject to certain exceptions. For although the general equation of lines of order n has $\frac{nn+3n}{2}$ coefficients to be determined, it may happen that such a number of equations, which we draw from the same number of given points, are not sufficient for this effect: this is evident, when two or several of these equations become identical. In such a case, one finds after having reduced the matter to the determination of the final coefficient, the value of this is expressed by a fraction, whose numerator and denominator both become $= 0$. I conceive therefore, that this inconvenience will take place when the nine points, which ought to determine a line of the 3rd order, are disposed such that two curved lines of this order may be drawn through them. In this case, the nine given points, since they contain two identical equations, are worth but 8, and we may then add the tenth point in order to render the problem determined. We may clarify this article to our further satisfaction by considering lines of the second order, for the determination of which 5 points may not always be sufficient. For when all the five points are arranged on a straight line so that they give, for example, these equations[7]

$x = 0 \quad ; \quad x = 1 \quad ; \quad x = 2 \quad ; \quad x = 3 \quad ; \quad x = 4 \quad ;$

$y = 0 \quad ; \quad y = 1 \quad ; \quad y = 2 \quad ; \quad y = 3 \quad ; \quad y = 4 \quad ;$

all of the coefficients of the general equation $\alpha yy + \beta xy + \gamma xx + \delta y + \epsilon x + \zeta = 0$ will not be determined, for after having introduced all of the given determinations, we are brought to this equation $\alpha yy - (\alpha + \gamma)xy + \gamma xx + \delta y - \delta x = 0$, so that there still remain two coefficents to be determined. If from the five given points there had been but 4 arranged in a straight line, then there would remain but one coefficient to be determined. From this, one easily understands that if

[7] In modern notation, Euler is considering the points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.

the nine points, from which one ought to draw a line of the third order, are at the same time the intersections of two curved lines of this order, then, after having completed all of the calculations, there must remain in the general equation for this order an undetermined coefficient, and beginning from this case not only two, but an infinity of lines of the 3rd order may be drawn from the same nine points.

The use, which you have made of continued[8] fractions in dioptrics is admirably beautiful and I am extremely obliged to you for the theorem, which you have communicated to me. I am charmed that you recognize, along with me, that this material is of great use in mathematics and that it is quite worthy of attention. It is not only arithmetic that can draw much profit from it, but also the integral calculus, as I made known in several pieces on this subject that I left in Petersburg, one of which[9] has already been published in the ninth volume of the *Comm.*

It is already a long time since Mr. Bousquet wrote to tell me that you had the kindness to send me a copy of the *Works* of Mr. James Bernoulli, which was shipped here along with a quantity of books for Mr. Neaulme. But since this latter was not willing to acept the package, I have received nothing. Had it not been for this, I would not have failed to thank you infinitely. I am therefore embarassed that I do not find myself in a position to show my gratitude except in words, but rest assured, that should an opportunity present itself for me to render you service, I will employ all of my energy to discharge my obligation. I have the honor of being, with the most perfect esteem,

Sir,

Your very humble and very obedient servant L. Euler

Berlin this 20 October 1744

[8]For some reasaon, Euler had underlined the word *continues* in this letter.

[9]Euler is refering to E71, "*De fractionibus continuis dissertatio*," which was presented to the St. Petersburg Academy on March 7, 1737. However, volume 9 of the *Commentarii academiae scientiarum imperialis Petropolitanae*, for the year 1737, did not actually appear until 1744.

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