Introduction to Analysis of the Infinite, Book I, Chapter 7: Exponentials and Logarithms Expressed through Series

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(English translation by John D. Blanton)

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Leonhard Euler's *Introductio in analysin infinitorum* (1748) is an important work in the history of mathematics. In it, Euler provided the foundation for much of today's mathematical analysis, focusing in particular on functions and their development into infinite series as central objects of study. Below (in sans-serif font) is the text of John Blanton's English translation of Chapter VI of this work by Euler.[1]

114. Since $a^0 = 1$, when the exponent on a increases, the power itself increases, provided a is greater than 1. It follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small number. Let ω be an infinitely small number, or a fraction so small that, although not equal to zero, still $a^\omega = 1+\psi$, where ψ is also an infinitely small number. From the preceding chapter we know that unless ψ were infinitely small, then neither would ω be infinitely small. It follows that $\psi = \omega$, or $\psi > \omega$, or $\psi > \omega$. Which of these is true depends on the value of a, which is not now known, so we let $\psi = k\omega$. Then we have $a^\omega = 1 + k\omega$, and with a as the base for the logarithms, we have $\omega = \log(1 + k\omega)$.

EXAMPLE

In order that it may be clearer how the number k depends on a, let $a = 10$. From the table of common logarithms, we look for the logarithm of a number which exceeds 1 by the smallest possible amount, for instance, $1 + \frac{1}{1000000}$, so that $k\omega = \frac{1}{1000000}$. Then $\log\left(1 + \frac{1}{1000000}\right) = \log\frac{1000001}{1000000} = 0.00000043429 = \omega$. Since $k\omega = 0.00000100000$, it follows that $\frac{1}{k} = \frac{43429}{1000000}$ and $k = \frac{1000000}{43429} = 2.30258$. We see that k is a finite number which depends on the value of the base a. If a different base had been chosen, then the logarithm of the same number $1 + k\omega$ will differ from the logarithm of the number already given. It follows that a different value of k will result.

[1] from Leonhard Euler, *Introduction to Analysis of the Infinite.* Translated by John D. Blanton. New York: Springer-Verlag, 1988, 1990. pp. 92-100.

115. Since $a^\omega = 1 + k\omega$, we have $a^{j\omega} = (1 + k\omega)^j$, whatever value we assign to j. It follows that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{1\cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1\cdot 2\cdot 3}k^3\omega^3 + \cdots .$$

If now we let $j = \frac{z}{\omega}$, where z denotes any finite number, since ω is infinitely small, then j is infinitely large. Then we have $\omega = \frac{z}{j}$, where ω is represented by a fraction with an infinite denominator, so that ω is infinitely small, as it should be. When we substitute $\frac{z}{j}$ for ω then

$$a^z = (1+kz/j)^j = 1+\frac{1}{1}kz+\frac{1(j-1)}{1\cdot 2j}k^2z^2+\frac{1(j-1)(j-2)}{1\cdot 2j\cdot 3j}k^3z^3+\frac{1(j-1)(j-2)(j-3)}{1\cdot 2j\cdot 3j\cdot 4j}k^4z^4+\cdots .$$

This equation is true provided an infinitely large number is substituted for j, but then k is a finite number depending on a, as we have just seen.

116. Since j is infinitely large, $\frac{j-1}{j} = 1$, and the larger the number we substitute for j, the closer the value of the fraction $\frac{j-1}{j}$ comes to 1. Therefore, if j is a number larger than any assignable number, then $\frac{j-1}{j}$ is equal to 1. For the same reason, $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, and so forth. It follows that $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{j-2}{3j} = \frac{1}{3}$, $\frac{j-3}{4j} = \frac{1}{4}$, and so forth. When we substitute these values we obtain

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots .$$

This expresses a relationship between the numbers a and k, since when we let $z = 1$, we have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \cdots .$$

When $a = 10$, then k is necessarily approximately equal to 2.30258 as we have already seen.

117. Suppose $b = a^n$, and let a be the base for the logarithm, so that $\log b = n$. Since $b^z = a^{nz}$, we have the infinite series

$$b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1\cdot 2} + \frac{k^3n^3z^3}{1\cdot 2\cdot 3} + \frac{k^4n^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots .$$

Now we substitute $\log b$ for n, so that

$$b^z = 1 + \frac{kz}{1}\log b + \frac{k^2z^2}{1\cdot 2}(\log b)^2 + \frac{k^3z^3}{1\cdot 2\cdot 3}(\log b)^3 + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4}(\log b)^4 + \cdots .$$

Since we know the value of k from the given value of base a, the general exponential b^z can be expressed in an infinite series whose terms proceed with the powers of z. Having shown this fact, we now go on to show how logarithms can be expressed by infinite series.

118. Since $a^\omega = 1 + k\omega$, where ω is an infinitely small fraction, and the relation between a and k is given by

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \cdots ,$$

if a is taken as the base of the logarithms, then $\omega = \log(1 + k\omega)$, and $j\omega = \log(1 + k\omega)^j$. It is clear that the larger the number chosen for j, the more $(1 + k\omega)^j$ will exceed 1. If we let j be an infinite number, the value of the power $(1 + k\omega)^j$ becomes greater than any number greater than 1. Now if we let $(1 + k\omega)^j = 1 + x$, then $\log(1 + x) = j\omega$. Since $j\omega$ is a finite number, namely the logarithm of $1 + x$, it is clear that j must be an infinitely large number; otherwise, $j\omega$ could not have a finite value.

119. Since we have let $(1+k\omega)^j = 1+x$, we have $1+k\omega = (1+x)^{\frac{1}{j}}$ and $k\omega = (1+x)^{\frac{1}{j}}-1$, so that $j\omega = \frac{j}{k}((1+x)^{\frac{1}{j}}-1)$. Since $j\omega = \log(1+x)$, it follows that $\log(1+x) = \frac{j}{k}(1+x)^{\frac{1}{j}} - \frac{j}{k}$ where j is a number infinitely large. But we have

$$(1+x)^{\frac{1}{j}} = 1 + \frac{1}{jx} - \frac{1(j-1)}{j \cdot 2j}x^2 + \frac{1(j-1)(2j-1)}{j \cdot 2j \cdot 3j}x^3 - \frac{1(j-1)(2j-1)(3j-1)}{j \cdot 2j \cdot 3j \cdot 4j}x^4 + \cdots .$$

Since j is an infinite number, $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{2j-1}{3j} = \frac{2}{3}$, $\frac{3j-1}{4j} = \frac{3}{4}$, etc. now it follows that

$$j(1+x)^{\frac{1}{j}} = j + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

As a result we have

$$\log(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right),$$

where a is the base of the logarithm and

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \cdots .$$

120. Since we have a series for the logarithm of $1 + x$, we can use this to define the number k when a is the base. If we let $1 + x = a$, since $\log a = 1$, we have

$$1 = \frac{1}{k}\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \cdots\right).$$

It follows that

$$k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \cdots .$$

If we let $a = 10$, the value of this infinite series must be approximately equal to 2.30258. We have

$$2.30258 = \frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \cdots ,$$

but it is difficult to see how this can be since the terms of this series continually grow larger and the sum of several terms does not seem to approach any limit. We will soon have an answer to this paradox.

121. Since

$$\log(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right),$$

when we substitute $-x$ for x, we obtain

$$\log(1-x) = -\frac{1}{k}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right).$$

If we subtract the second series from the first we obtain

$$\log(1+x) - \log(1-x) = \log\left(\frac{1+x}{1-x}\right) = \frac{2}{k}\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right).$$

Now if we let $\frac{1+x}{1-x} = a$, so that $x = \frac{a-1}{a+1}$, and because $\log a = 1$, we have

$$k = 2\left(\frac{a-1}{a+1} + \frac{(a-1)^3}{3(a+1)^3} + \frac{(a-1)^5}{5(a+1)^5} + \cdots\right).$$

From this equation we can find the value of k when a is given. For example, if $a = 10$, then $k = 2(\frac{9}{11} + \frac{9^3}{3\cdot 11^3} + \frac{9^5}{5\cdot 11^5} + \frac{9^7}{7\cdot 11^7} + \cdots)$ and the terms of this series decrease in a reasonable way so that soon a satisfactory approximation for k can be obtained.

122. Since we are free to choose the base a for the system of logarithms, we now choose a in such a way that $k = 1$. Suppose now that $k = 1$; then the series found above in section 116, $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$, is equal to a. If the terms are represented as decimal fractions and summed, we obtain the value for

$$a = 2.71828182845904523536028\cdots.$$

When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms. For the sake of brevity for this number $2.718281828459\cdots$ we will use the symbol e, which will denote the base of the natural or hyperbolic logarithms, which corresponds to the value $k = 1$, and e represents the sum of the infinite series $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$.

123. Natural logarithms have the property that the logarithm of $1 + \omega$ is equal to ω, where ω is an infinitely small quantity. From this it follows that $k = 1$, and the natural logarithms of all numbers can be found. Let e stand for the number found above, then

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots,$$

and the natural logarithms themselves can be found from these series where $\log(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$, and $\log\left(\frac{1+x}{1-x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \cdots$. This last series is strongly convergent if we substitute an extremely small fraction for x. For instance, if $x = \frac{1}{5}$, then

$$\log\frac{6}{4} = \log\frac{3}{2} = \frac{2}{1\cdot 5} + \frac{2}{3\cdot 5^3} + \frac{2}{5\cdot 5^5} + \frac{2}{7\cdot 5^7} + \cdots.$$

If $x = \frac{1}{7}$, then

$$\log\frac{4}{3} = \frac{2}{1\cdot 7} + \frac{2}{3\cdot 7^3} + \frac{2}{5\cdot 7^5} + \frac{2}{7\cdot 7^7} + \cdots,$$

and if $x = \frac{1}{9}$, then

$$\log\frac{5}{4} = \frac{2}{1\cdot 9} + \frac{2}{3\cdot 9^3} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} + \cdots.$$

From the logarithms of these fractions we can find the logarithms of integers. From the nature of logarithms we have $\log\frac{3}{2} + \log\frac{4}{3} = \log 2$, and $\log\frac{3}{2} + \log 2 = \log 3$, and $2\log 2 = \log 4$. Further we have $\log\frac{5}{4} + \log 4 = \log 5$, $\log 2 + \log 3 = \log 6$, $3\log 2 = \log 8$, $2\log 3 = \log 9$, $\log 2 + \log 5 = \log 10$.

EXAMPLE

We can now state the values of the natural logarithms of integers from 1 to 10.

$\log 1 = 0.00000\ 00000\ 00000\ 00000\ 00000$	$\log 2 = 0.69314\ 71805\ 59945\ 30941\ 72321$
$\log 3 = 1.09861\ 22886\ 68109\ 69139\ 52452$	$\log 4 = 1.38629\ 43611\ 19890\ 61883\ 44642$
$\log 5 = 1.60943\ 79124\ 34100\ 37460\ 07593$	$\log 6 = 1.79175\ 94692\ 28055\ 00081\ 24773$
$\log 7 = 1.94591\ 01490\ 55313\ 30510\ 54639$	$\log 8 = 2.07944\ 15416\ 79835\ 92825\ 16964$
$\log 9 = 2.19722\ 45773\ 36219\ 38279\ 04905$	$\log 10 = 2.30258\ 50929\ 94045\ 68401\ 79914$

All of these logarithms are computed from the above three series, with the exception of $\log 7$, which can be found as follows. When in the last series given above, we let $x = \frac{1}{99}$, we obtain $\log\frac{100}{98} = \log\frac{50}{49} = 0.02020\ 27073\ 17519\ 44840\ 78230$. When this is subtracted from $\log 50 = 2\log 5 + \log 2 = 3.91202\ 30054\ 28146\ 05861\ 87508$ we obtain $\log 49$. But $\log 7 = \frac{1}{2}\log 49$.

124. Let the natural logarithm of $1+x$ be equal to y, then

$$y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Now let a be the base of a system of logarithms and let v be the logarithm of $1+x$ in this system. Then as we have seen,

$$v = \frac{1}{k}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) = \frac{y}{k}.$$

It follows that $k = \frac{y}{v}$, and this is the most convenient method of calculating the value of k corresponding to the base a; it is given by the quotient of the natural logarithm of any number divided by the logarithms of that same number with the base a. Suppose the number is a, then $v = 1$ and k is equal to the natural logarithm of a. In the system of common logarithms, where the base is $a = 10$, then k is the natural logarithm of 10. It follows that $k = 2.30258\ 50929\ 94045\ 68401\ 79914$, which is the value calculated not far above. If each natural logarithm is divided by this number k, or, which comes to the same thing, multiplied by the decimal fraction $0.43429\ 44819\ 03251\ 82765\ 11289$, then the results are the common logarithms, with base $a = 10$.

125. Since

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \cdots,$$

if we let $a^y = e^z$, then after taking natural logarithms, we have $y \log a = z$, since $\log e = 1$. We now substitute this value in the series to obtain

$$a^y = 1 + \frac{y \log a}{1} + \frac{y^2 (\log a)^2}{1 \cdot 2} + \frac{y^3 (\log a)^3}{1 \cdot 2 \cdot 3} + \cdots.$$

In this way, any exponential, with the aid of natural logarithms, can be expressed as an infinite series. Now let j be an infinitely large number, then both exponentials and logarithms can be expressed as powers. That is, $e^z = \left(1 + \frac{z}{j}\right)^j$ and so $a^y = \left(1 + \frac{y \log a}{j}\right)^j$. For natural logarithms, we have $\log(1 + x) = j((1 + x)^{\frac{1}{j}} - 1)$. Other uses of natural logarithms are discussed in integral calculus.