Asymptotic Counting Theorems for Primitive Juggling Patterns

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1 Introduction

Juggling patterns are typically described using siteswap notation, which is based on the regular rhythm of the balls being thrown. Each throw is assigned a *height* which measures the number of beats, or units of juggling time, that the ball is in the air. For example, the sequence $(4, 2)$ indicates that the first ball should be thrown to height 4, and the second to height 2, at which point the pattern repeats. Mathematically, a *juggling sequence* or *siteswap* is a sequence $T = (t_1, t_2, \ldots, t_n)$ of length n for which the values $i+t_i \pmod{n}$ are all distinct. For jugglers, this condition is equivalent to the requirement that no two balls land in the same hand at the same time. We write $t_i = 0$ to indicate that no ball is thrown at time i. The progression of throws can be represented with an *arc diagram*, as shown in Figure 1 below. Readers are encouraged to consult Polster [6] on the use of siteswap notation in analyzing juggling patterns.

In addition to siteswaps, juggling patterns may be viewed as a sequence of

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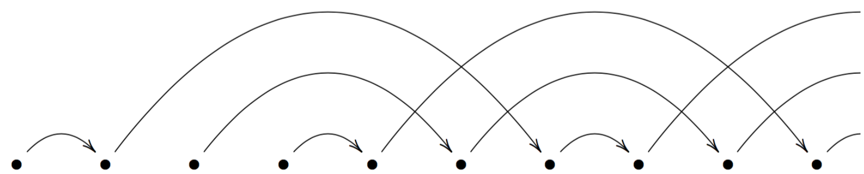

Figure 1: An arc diagram for the siteswap $(1, 5, 3)$.

juggling states or *landing schedules*. A juggling state σ is a vector of zeroes and ones, with a 1 in position i indicating that a ball is scheduled to land after i beats. The easiest way to think of a juggling state is to imagine that the juggler ceases juggling at a particular moment in time; the timing of "thuds" as the balls hit the ground will indicate the positions of the 1's in σ. Of course, the state will change after each throw, so that the full pattern is described by a sequence of n juggling states $\sigma^{(1)}$, $\sigma^{(2)}$, ..., $\sigma^{(n)}$, with $\sigma^{(m+n)} = \sigma^{(m)}$. The *height* of a juggling state is its length.[1] A juggling sequence with initial state σ is referred to as a σ-juggling sequence. The state $\sigma = \langle 1, 1, \ldots, 1\rangle$ with all 1's is called the *ground state*, and juggling sequences beginning with this state are referred to as *ground state sequences*. For clarity, we will always use angle brackets for juggling states and conventional parentheses for siteswaps. For example, the juggling sequence $(1, 5, 3)$ has initial state $\sigma = \sigma^{(1)} = \langle 1, 0, 1, 1\rangle$ and the throws 1, 5, and 3 produce the subsequent states $\sigma^{(2)} = \langle 1, 1, 1\rangle$, $\sigma^{(3)} = \langle 1, 1, 0, 0, 1\rangle$, $\sigma^{(4)} = \sigma^{(1)} = \langle 1, 0, 1, 1\rangle$. The arc diagram in Figure 1 ends at state $\sigma^{(2)} = \langle 1, 1, 1\rangle$ and it can be seen that, if the juggler were to stop, the balls would fall on each of the next three beats.

Given a fixed initial state σ, it is natural to consider the structure of the set of σ-juggling sequences. It is not too hard to see that a juggler can combine any two σ-juggling sequences by alternating between them. Mathematically, this

[1]Technically, we view σ as having infinite length, so that all but finitely many entries are 0's. I.e., if σ has height h, then $\sigma(h)$ is its last nonzero entry.

presents as concatenation; i.e., we can combine (or inversely, decompose) σ-juggling sequences, in analogy with multiplication and factorization of positive integers. Following in this vein, we define a σ-juggling sequence as *primitive* if it cannot be decomposed into shorter sequences. For example, the ground state sequence $(4, 2, 4, 4, 1, 3)$ can be decomposed into the primitive sequences $(4, 2)$, $(4, 4, 1)$, and (3).

The main goal of this paper is to prove an analogy of the prime number theorem for σ-juggling sequences. More specifically, we wish to answer the following question: given a b-ball juggling state σ and a positive integer n, how many primitive σ-juggling sequences are there with length $\leq n$? Ultimately, we will answer this question by applying analytic techniques to the generating functions described by Chung and Graham [2], to which we now turn.

2 Generating Functions

Suppose σ is a b-ball juggling state of height h and let $J_\sigma(n)$ denote the number of juggling sequences with initial state σ and length n. The juggling sequence generating function is the formal series

$$f_\sigma(x) \ = \ \sum_{k=0}^{\infty} J_\sigma(k) x^k,$$

where we define $J_\sigma(0) = 1$.[2] In the special case of a ground state $\sigma = \langle 1, 1, \ldots, 1 \rangle$, we write $J_b(n)$ instead. If $n < h$, it is possible that $J_\sigma(n)$ is zero. In fact, Chung and Graham [2] give a way to determine if $J_\sigma(n)$ is zero, and if not, count the number of σ-juggling sequences of that length. We improve on the first of these results by decomposing a juggling state into a set of smaller states.

Definition 1. *Given a juggling state $\sigma = \langle \sigma(1), \sigma(2), \sigma(3), \ldots \rangle$ and a positive*

[2]When generating functions are treated as formal objects, the variable x will be used. When they are treated as analytic objects, the variable z will be used instead.

integer n, define the ith thread *by $\sigma_i = \langle \sigma(i), \sigma(i+n), \sigma(i+2n), \ldots \rangle$. Each thread itself is a juggling state, though typically with fewer balls than σ itself.*

With this definition in hand, we may now give a necessary and sufficient condition for $J_\sigma(n)$ to be nonzero.

Proposition 1. *Let σ be a b-ball juggling state of height h. For $n < h$, $J_\sigma(n) \neq 0$ if and only if each σ_i $(1 \leq i \leq n)$ is a ground state.*

Proof. The necessary condition for $J_\sigma(n) \neq 0$ is Lemma 1 from [2], so here we need only prove that if σ_i is a ground state for all i, then $J_\sigma(n) \neq 0$. Given this condition, there are two cases for σ_1: either $\sigma(1) = 0$ or $\sigma(1) = 1$. In the first case, the thread σ_1 must consist of all zeroes, in which case the juggler must do nothing. In the second case, σ_1 consists of some nonzero number of 1's, say m many of them. Thus, the juggler may execute a throw of height mn, so that the ball's landing time places it at the end of σ_1. In either case, the result is a cyclic permutation of the threads: σ_1 moves to σ_{n-1}, while σ_i moves to σ_{i-1} for $i \geq 2$. This procedure can be repeated n times, at which point the threads will return to their original positions, and the overall juggling state will again be σ. Thus, we have constructed a σ-juggling sequence of length n, so $J_\sigma(n) \neq 0$. $\square$

Note further that if $n \geq h$, then each σ_i is necessarily a ground state (σ_i is either $\langle 0 \rangle$ or $\langle 1 \rangle$ depending on whether $\sigma(i) = 0$ or 1), so $J_\sigma(n) \neq 0$ for all $n \geq h$, as expected.

As an example, consider $\sigma = \langle 1, 0, 1, 1, 0, 1, 0, 0, 1 \rangle$ with $n = 3$. The corresponding threads are $\sigma_1 = \langle 1, 1 \rangle$, $\sigma_2 = \langle 0 \rangle$, and $\sigma_3 = \langle 1, 1, 1 \rangle$. To construct a juggling sequence of length 3, we begin with a throw of height $2 \cdot 3 = 6$ so that $\sigma = \sigma^{(1)}$ becomes $\sigma^{(2)} = \langle 0, 1, 1, 0, 1, 1, 0, 1 \rangle$, with threads $\sigma_1^{(2)} = \langle 0 \rangle$, $\sigma_2^{(2)} = \langle 1, 1, 1 \rangle$, and $\sigma_3^{(2)} = \langle 1, 1 \rangle$. Throws of height 0 and 9 give $\sigma^{(3)} = \langle 1, 1, 0, 1, 1, 0, 1 \rangle$ and $\sigma^{(4)} = \langle 1, 0, 1, 1, 0, 1, 0, 0, 1 \rangle = \sigma$, respectively. Thus, $(6, 0, 9)$ is a σ-juggling sequence of length 3.

Now that we know the values of n for which $J_\sigma(n)$ is nonzero, we turn to the combinatorial problem of finding the number of σ-juggling sequences of length n.

Definition 2. *Given a juggling state σ, define the* cumulative *juggling function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $\tau(j) = \sum_{i=0}^{j} \sigma(i)$. In the case of $j = 0$, take $\sigma(0) = 0$ so that $\tau(0) = 0$.*

When $J_\sigma(n)$ is nonzero, its value may be expressed in terms of a matrix permanent [2], though we see here that it can be given as a generalized factorial involving τ.

Proposition 2. *Let σ be a juggling state. If $J_\sigma(n) \neq 0$, then $J_\sigma(n) = [n]_\sigma!$, where*

$$[n]_\sigma! \;=\; \prod_{i=0}^{n-1} (\tau(i) + 1). \tag{1}$$

Proof. From [2], we know that if $J_\sigma(n) \neq 0$, then

$$J_\sigma(n) \;=\; \prod_{j=1}^{n} (v_j - j + 1), \tag{2}$$

where the v_j are a sequence of non-decreasing integers between 1 and n, inclusive. To properly define the v_j sequence we define a related integer sequence, denoted here as w_i, by taking

$$w_i \;=\; \begin{cases} i & \text{if } \sigma(i) = 0 \\ n & \text{if } \sigma(i) = 1. \end{cases}$$

The v_j are the same as the w_j, only rearranged to be non-decreasing.

Now suppose that the juggling state σ has k 0's in it. It follows that for

$1 \leq j \leq k$, each v_j equals the index i_j of the jth 0 in σ. Also, for $j > k$, we have $v_j = n$. In other words, the v_j can be written in order as $i_1 \leq i_2 \leq \cdots \leq i_k \leq n = n = \cdots = n$.

For the first k terms it is easy to see that $i_j = \tau(i_j - 1) + (j-1) + 1$, i.e., the number of 1's up to position i_j plus the number of 0's up to (but not including) position i_j equals $i_j - 1$. Thus, $v_j - j = i_j - j = \tau(i_j - 1)$ for the first k terms. The remaining $n-k$ terms may be written as a factorial:

$$\prod_{j=k+1}^{n} (n - j + 1) \;=\; \prod_{j=1}^{n-k} j \;=\; (n-k)!.$$

Now let I_j denote the index of the jth 1 in σ. Since the cumulative function satisfies $\tau(0) = 0$ and only increases when there is a 1 in σ, we must have $\tau(I_j - 1) = j - 1$ for the last $n-k$ terms.

With all of this in mind, we may rearrange the terms in product (2) to correspond to their respective indices in σ:

$$J_\sigma(n) \;=\; \prod_{i=1}^{n} (\tau(i-1) + 1).$$

In other words, we have a non-decreasing product which begins with a factor of 1, and increases only when $\sigma(i) = 1$. A simple reindexing gives the product formula we set out to show. $\square$

Returning to our previous example, let $\sigma = \langle 1, 0, 1, 1, 0, 1, 0, 0, 1 \rangle$ and take $n = 3$. We already know that $J_\sigma(3)$ is nonzero, so we may apply the formula from Proposition 2 to get

$$J_\sigma(3) \;=\; [3]_\sigma! \;=\; \prod_{i=0}^{2} (\tau(i) + 1) \;=\; 1 \cdot 2 \cdot 2 \;=\; 4.$$

This tells us that there are three additional juggling sequences of length 3, other than $(6,0,9)$. Inspection shows that they are $(1,5,9)$, $(1,10,4)$, and $(11,0,4)$.

Also, for $n \geq h$, notice that since $\tau(h) = b$ and $\tau(i)$ is constant for all $i \geq h$, we must have $J_\sigma(n) = J_\sigma(h) \cdot (b+1)^{n-h}$. We now give two simple corollaries that will be key to proving the main result of this paper.

Corollary 1. *For any b-ball juggling state σ with height h, $J_\sigma(n)$ is nonzero for at most b values of $n = 0, 1, 2, \ldots, h-1$.*

Proof. Let k be the number of zeroes in σ, and let i_1, i_2, $\ldots$, i_k denote the indices of these zeroes. Then for each $j = 1, 2, \ldots, k$, we may take $n = h - i_j$ to get $\sigma_{i_j}(1) = \sigma(i_j) = 0$ and $\sigma_{i_j}(2) = \sigma(i_j + h - i_j) = \sigma(h) = 1$. In other words, the thread σ_{i_j} fails to be a ground state for $n = h - i_j$ and so by Proposition 1, $J_\sigma(h - i_j) = 0$ for all $j = 1, 2, \ldots, k$. Since $h - k = b$, there are no more than b nonzero values of $J_\sigma(n)$ for $n = 0, 1, 2, \ldots, h-1$. □

Corollary 2. *For any b-ball juggling state σ, $J_\sigma(n) \leq J_b(n)$. Specifically, $[n]_\sigma!$ is bounded above by $n!$ when $n < b$ and $b! \cdot (b+1)^{n-b}$ when $n \geq b$.*

Proof. Since σ is a b-ball state, it is obvious that $\tau(i) \leq i$ whenever $i < b$ and, more generally, $\tau(i) \leq b$ for all $i \geq 0$. Applying this observation to equation (1) from Proposition 2 gives the result. □

Notice also that when σ is the ground state, $h = b$ and $\tau(i) = i$ whenever $i \leq b$. This means that $J_b(n) = n!$ for $n \leq b$ and $b! \cdot (b+1)^{n-b}$ for $n > b$.[3]

With all of this in mind, we refine our analysis of juggling sequences by considering their factorability. Recall that a primitive juggling sequence is one that cannot be decomposed into shorter sequences, and let $P_\sigma(n)$ denote the number of primitive juggling sequences of length n with initial state σ.

For example, the state $\sigma = \langle 1, 1, 0, 1, 0, 1 \rangle$ has $J_\sigma(4) = 18$ juggling sequences of length 4. If a sequence of length 4 is not primitive, it must be decomposable

[3]In fact, this is Theorem 1 from [2].

into two sequences of length 2, two sequences of length 1 and 3, or four sequences of length 1. However, $J_\sigma(1) = 0$ since σ is not the ground state, so the second and third cases are impossible. We also know that $J_\sigma(2) = 2$, and inspection reveals these sequences to be $(2, 6)$ and $(7, 1)$. Thus, the non-primitive σ-juggling sequences of length 4 are $(2, 6, 2, 6)$, $(2, 6, 7, 1)$, $(7, 1, 2, 6)$, and $(7, 1, 7, 1)$. This means that the remaining 14 sequences of length 4 must be primitive, so $P_\sigma(4) = 14$.

As with the rational integers, the task of counting primitive juggling sequences of a given size requires analytic techniques, though we begin with a combinatorial approach here. The generating function for primitive juggling sequences is

$$g_\sigma(x) \;=\; \sum_{k=0}^{\infty} P_\sigma(k)x^k.$$

In [2], we see that the generating functions f_σ and g_σ satisfy

$$\begin{aligned} f_\sigma(x) &= \frac{r_\sigma(x)(1-(b+1)x) + J_\sigma(h)x^h}{1-(b+1)x} = \frac{s_\sigma(x)}{1-(b+1)x} \\ g_\sigma(x) &= 1 - \frac{1}{f_\sigma(x)} = \frac{f_\sigma(x)-1}{f_\sigma(x)}, \end{aligned}$$

where $r_\sigma(x) = \sum_{k=0}^{h-1} J_\sigma(k)x^k$ and $s_\sigma(x) = r_\sigma(x)(1-(b+1)x) + J_\sigma(h)x^h$. For convenience, we will instead examine the natural pairs $\bar{r}_\sigma(x)$ and $\bar{s}_\sigma(x)$ of these polynomials, defined as $\bar{r}_\sigma(x) = x^{h-1} \cdot r_\sigma(x^{-1}) = \sum_{k=0}^{h-1} J_\sigma(k)x^{h-1-k}$ and $\bar{s}_\sigma(x) = x^h \cdot s_\sigma(x^{-1}) = J_\sigma(h) + (x-(b+1)) \cdot \bar{r}_\sigma(x)$. As with the counting function $J_b(n)$, we let $r_b(x)$ and $s_b(x)$ (and their pairs $\bar{r}_b(x)$, $\bar{s}_b(x)$) indicate the ground state case.

3 Primitive Ground State Sequences

First, we restrict to ground states $\sigma = \langle 1, 1, \ldots, 1 \rangle$, so that $h = b$ and $J_b(n) = n!$ for all positive $n < b$. Viewing the polynomials $\bar{r}_b$ and $\bar{s}_b$ as analytic objects, we have

$$\begin{aligned} \bar{r}_b(z) &= \sum_{k=0}^{b-1} k! \cdot z^{b-1-k}, \\ \bar{s}_b(z) &= b! + (z - (b+1)) \sum_{k=0}^{b-1} k! \cdot z^{b-1-k}. \end{aligned}$$

In this section, we decompose $\bar{s}_b(z)$ as the sum of two functions of a complex variable:

$$u(z) = (z - (b+1))(z^{b-1} - (b-1)!), \quad v(z) = b! + (z - (b+1)) \sum_{k=1}^{b-2} k! z^{b-1-k}.$$

Our goal is to show that, given certain conditions on $|z|$, we necessarily have $|u(z)| < |v(z)|$. Then, Rouché's theorem may be used to determine bounds for the roots of $\bar{s}_b(z)$. This technique is a variation on the one used by Klyve, Elsner, and the author in [3], albeit with a different objective in mind.

Lemma 1. *For $b \geq 4$ and $|z| \geq b$,* $\left| \sum_{k=1}^{b-2} k! z^{b-1-k} \right| \leq \frac{1}{2} \cdot (|z|^{b-1} - (b-1)!)$.

Proof. Since $\sqrt[k]{(k+1)!}$ is an increasing function of k, it follows that $(k+1)! \leq (\sqrt[b-3]{(b-2)!}\,)^k$ for all $k \leq b-3$. (Note also that this inequality is strict for all

$k < b-3$.) Letting $\mu_b = \sqrt[b-3]{(b-2)!}$, it then follows that for $b \geq 4$,

$$\begin{aligned}\left|\sum_{k=1}^{b-2} k! z^{b-1-k}\right| &\leq \sum_{k=1}^{b-2} k! |z|^{b-1-k} \\ &< |z| \sum_{k=0}^{b-3} \mu_b^k |z|^{b-3-k} \\ &= |z| \cdot \frac{|z|^{b-2} - \mu_b^{b-2}}{|z| - \mu_b} \\ &\leq \frac{|z|^{b-1} - |z|\mu_b^{b-2}}{|z| - \mu_b}.\end{aligned}$$

We know $|z| \geq b > \mu_b > \sqrt[b-1]{(b-1)!}$, so $|z|^{b-1} - |z|\mu_b^{b-2} \leq |z|^{b-1} - (b-1)!$. Moreover, since $|z| \geq b$ and $\mu_b \leq \frac{1}{2}b$ for all $b \geq 4$, $\frac{1}{|z|-\mu_b} \leq \frac{1}{b-b/2} = \frac{2}{b} \leq \frac{1}{2}$. These two observations complete the proof. ☐

We can now use this inequality to compare the functions $u(z)$ and $v(z)$.

Theorem 1. *Given any $b \geq 4$, let z be any complex number satisfying $b \leq |z| \leq b+1-c\cdot\frac{b^{3/2}}{e^b}$, where $c = \frac{2e^4}{e^3-2} \approx 6.0378$. Then $|v(z)| < |u(z)|$.*

Proof. First, note that since $|z|^{b-1} \geq b^{b-1} > (b-1)!$, we have $|u(z)| \geq (b+1-|z|)\cdot(|z|^{b-1}-(b-1)!)$. Also, from Lemma 1 we know that for $b \geq 4$ and $|z| \geq b$, $|v(z)| < b! + \frac{1}{2}\cdot(b+1-|z|)\cdot(|z|^{b-1}-(b-1)!)$. Accordingly, if we can show that

$$b! \quad \leq \quad \frac{1}{2}\cdot(b+1-|z|)\cdot(|z|^{b-1}-(b-1)!),$$

then we will have proven the theorem. Taking the bounds on $|z|$ into account, we see that it suffices to show that

$$b! \quad \leq \quad \frac{c}{2}\cdot\frac{b^{3/2}}{e^b}\cdot(b^{b-1}-(b-1)!),$$

which easily simplifies to

$$\frac{2}{c} \quad \leq \quad \frac{\sqrt{b}}{e^b} \cdot \left(\frac{b^b}{b!} - 1 \right).$$

After using the definition of c and Stirling's inequality $b! \leq e\sqrt{b} \cdot \left(\frac{b}{e}\right)^b$, we need only show

$$\frac{1}{e} - \frac{2}{e^4} \quad \leq \quad \frac{\sqrt{b}}{e^b} \cdot \left(\frac{e^{b-1}}{\sqrt{b}} - 1 \right) \quad = \quad \frac{1}{e} - \frac{\sqrt{b}}{e^b}. \tag{3}$$

It is easy to see that the right hand side of (3) is an increasing function of b, and has a value of $\frac{1}{e} - \frac{2}{e^4}$ when $b = 4$. This concludes the proof. □

As a consequence, Rouché's theorem implies that, if a real number R satisfies $b \leq R \leq b + 1 - c \cdot \frac{b^{3/2}}{e^b}$, $\bar{s}_b(z)$ has the same number of roots within a circle of radius R as $u(z) = (z - (b+1))(z^{b-1} - (b-1)!)$. Since the first root of $u(z)$ is $b+1$ itself, and the other roots all lie on a circle of radius $\sqrt[b-1]{(b-1)!} < b$, we then know that $\bar{s}_b(z)$ has a single root (of multiplicity 1) with modulus greater than $b + 1 - c \cdot \frac{b^{3/2}}{e^b}$ and $b - 1$ roots of modulus less than b. Moreover, when $b < 4$, a computer can easily determine the roots of $\bar{s}_b(z)$ as $z = 1 \pm \sqrt{2}$ when $b = 2$ and $z \approx 3.6891, 0.34455 \pm 0.65071i$ when $b = 3$. In both cases, $\bar{s}_b(z)$ has a real root of multiplicity 1 between b and $b+1$, with the other root(s) lying within a circle of radius $\sqrt[b-1]{(b-1)!}$.

Having determined the approximate locations of the roots of $\bar{s}_b(z)$, we now refine this result and establish the asymptotic growth of $P_b(n)$. The primary analytic technique comes from Flajolet and Sedgewick [4], which allows us to determine the asymptotic growth of $P_b(n)$ from the roots of its generating function.

Theorem 2. *Suppose $b \geq 4$, let $P_b(n)$ denote the number of primitive, ground state juggling patterns of length n, and let ρ be the largest root (by modulus) of*

$\bar{s}_b(z)$. Then, ρ is a real number satisfying $b+1-c\cdot\frac{b^{3/2}}{e^b} \le \rho < b+1$, where c is the same constant defined in Theorem 1. Moreover,

$$P_b(n) \;=\; \frac{b+1-\rho}{|s_b'(1/\rho)|}\cdot\left(\rho^n + O\left(\frac{\rho^n}{n}\right)\right).$$

Proof. Theorem 1 ensures that ρ has multiplicity 1. To show that it is real, note that $\bar{s}_b(b+1) = b! > 0$ and, for $b \ge 3$,

$$\bar{s}_b(b) \;=\; b! - \sum_{k=0}^{b-1} k!\cdot b^{b-1-k} \;\le\; b! - b^{b-1} \;<\; 0.$$

Since ρ is the largest root (by modulus) of $\bar{s}_b(z)$, it follows that ρ^{-1} is the smallest root (by modulus) of $s_b(z)$. Since the generating function $g_b(z) = \frac{s_b(z)-(1-(b+1)z)}{s_b(z)}$ is a rational function in z that is analytic at 0, we know from [4, p. 256] that

$$P_b(n) \;=\; |C|(\rho^{-1})^{-n-1}\left(1+O\left(\frac{1}{n}\right)\right),$$

where $C = \lim_{z\to\rho^{-1}}(z-\rho^{-1})g_b(z)$. Using the definition of $g_b(z)$ given above, along with the fact that $s_b(\rho^{-1}) = 0$, we compute C as

$$\begin{aligned}
\lim_{z\to\rho^{-1}}(z-\rho^{-1})g_b(z) &= \lim_{z\to\rho^{-1}}(z-\rho^{-1})\cdot\frac{s_b(z)-(1-(b+1)z)}{s_b(z)} \\
&= ((b+1)\rho^{-1}-1)\lim_{z\to\rho^{-1}}\frac{z-\rho^{-1}}{s_b(z)} \\
&= \left(\frac{b+1}{\rho}-1\right)\lim_{z\to\rho^{-1}}\frac{z-\rho^{-1}}{s_b(z)-s_b(\rho^{-1})} \;=\; \frac{b+1-\rho}{\rho\cdot s_b'(1/\rho)}.
\end{aligned}$$

Putting it all together, we find that

$$P_b(n) \;=\; \frac{b+1-\rho}{|s_b'(1/\rho)|}\cdot\rho^n\left(1+O\left(\frac{1}{n}\right)\right) \;=\; \frac{b+1-\rho}{|s_b'(1/\rho)|}\cdot\left(\rho^n+O\left(\frac{\rho^n}{n}\right)\right),$$

which is what we set out to prove. □

For an example, take $b = 5$, so that $\bar{r}_5(z) = z^4 + z^3 + 2z^2 + 6z + 24$ and $\bar{s}_5(z) = x^5 - 5x^4 - 4x^3 - 6x^2 - 12x - 24$. Then $\rho \approx 5.9235$ and $|s'_b(1/\rho)| \approx 7.1920$, so $P_\sigma(n) \sim 0.010636 \cdot 5.9235^n$. As implied by the error term in Theorem 2, this estimate is extremely accurate. For example, the actual value of $P_5(12)$ is 19,848,757 while Theorem 2 gives a first-order estimate of 19,848,735.021, with a relative error of about -0.00011073%.

More generally, while the results in this section provide only a lower bound on ρ, it would be not too difficult to provide an effective upper bound by using techniques from elementary Calculus. If $\bar{s}_b(z)$ is convex (as a real-valued function) on $[\rho, b+1]$, as seems likely, then Newton's method could be applied beginning at $z = b+1$ to obtain ever-tighter upper bounds on ρ. At any rate, computational evidence suggests that this lower bound is probably sharp—e.g., for $b \geq 6$, $b + 1 - \rho$ appears to be bounded below by $\frac{b^{1.45}}{e^b}$. It is reasonable to expect that $\frac{b^\lambda}{e^b}$ is $o(b+1-\rho)$ for all positive $\lambda < \frac{3}{2}$.

4 Primitive σ-Juggling Sequences

It is not difficult to extend this result to general primitive juggling patterns.

Theorem 3. *Let σ be a b-ball juggling state with $b \geq 4$, let $P_\sigma(n)$ denote the number of primitive, σ-juggling sequences of length n, and let ρ be the largest root (by modulus) of the polynomial $\bar{s}_\sigma(z)$. Then, ρ is a real number satisfying $b + 1 - c \cdot \frac{b^{3/2}}{e^b} \leq \rho < b + 1$, where c is the same constant defined in Theorem 1. Moreover,*

$$P_\sigma(n) \;=\; \frac{b+1-\rho}{|s'_\sigma(1/\rho)|}\left(\rho^n + O\left(\frac{\rho^n}{n}\right)\right).$$

Proof. Our strategy is to decompose $\bar{s}_\sigma(z)$ into two function $u(z)$ and $v(z)$, and then reduce the problem to the ground state case. First, recall that

$$\bar{s}_\sigma(z) \;=\; J_\sigma(h) + (z-(b+1)) \cdot \sum_{k=0}^{h-1} J_\sigma(k) z^{h-1-k}.$$

We know from Corollary 1 that there are no more than b nonzero terms in the sum. Also, we know that $J_\sigma(0) = 1$ for all σ. Moreover, since $|z| \geq b$ and each factor appearing in $J_\sigma(k)$ is $\leq b$, the largest terms in the sum are those with small index k. So, letting ℓ be the largest index for which $J_\sigma(\ell)$ is nonzero, we have

$$\sum_{k=0}^{h-1} J_\sigma(k)|z|^{h-1-k} \;\leq\; |z|^{h-1} + J_\sigma(\ell)|z|^{h-1-\ell} + \sum_{k=1}^{b-2} J_\sigma(k)|z|^{h-1-k}$$

Additionally, if $u(z) = (z-(b+1))\cdot(z^{h-1}+J_\sigma(\ell)z^{h-1-\ell})$, then since $J_\sigma(\ell)|z|^{h-1-\ell} \leq J_\sigma(b-1)|z|^{h-b}$, we know from the triangle inequality that

$$\begin{aligned} |u(z)| \;&\geq\; |z-(b+1)| \cdot |z|^{h-b}(|z|^{b-1} - J_\sigma(b-1)) \\ &\geq\; |z-(b+1)| \cdot |z|^{h-b}(|z|^{b-1} - (b-1)!), \end{aligned}$$

with the final inequality following from Corollary 2. Finally, since Corollary 2 also implies $J_\sigma(h) = [h]_\sigma! \leq b! \cdot b^{h-b}$, it suffices to show that

$$b!\cdot|z|^{h-b} + |z-(b+1)|\cdot|z|^{h-b}\cdot\sum_{k=1}^{b-2} J_\sigma(k)|z|^{b-1-k} \;<\; |z-(b+1)|\cdot|z|^{h-b}(|z|^{b-1}-(b-1)!).$$

However, we can see that canceling the common factor $|z|^{h-b}$ from all terms reduces it the ground state case, and the proof follows Theorem 2 identically from this point onward. □

Crucially, one must note that the ground state polynomial $s_b(z)$ provides a bound on the size of ρ, even though $1/\rho$ is a root of the σ state's polynomial

$s_\sigma(z)$. As an illustration, consider the case where $b = 5$ and $h = 7$, for which there are 15 possible juggling states. For each possible σ, Theorem 3 dictates that the largest root ρ_σ of $\bar{s}_\sigma$ satisfies $5.5451 \le \rho_5 \le \rho_\sigma < 6$, where ρ_5 is the largest root of $\bar{s}_5$. A graph of all the $\bar{s}_\sigma(z)$ functions alongside $\bar{s}_5(z)$ is shown in Figure 2.

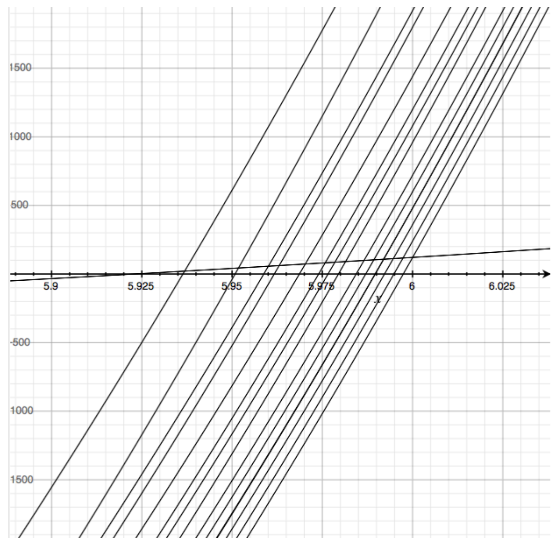

Figure 2: The graphs of $\bar{s}_\sigma(z)$ for each of the 5-ball juggling states of height 7, alongside the graph of the ground state polynomial $\bar{s}_5(z)$.

Finally, suppose we wish to estimate the number of primitive, b-ball juggling sequences of length *up to* n. This involves the *cumulative* counting function $\Pi_\sigma(n) = \sum_{k=1}^{n} P_\sigma(n)$, and a simple estimate of its growth can be determined from Theorem 3.

Corollary 3. *Suppose* $b \ge 4$, *let* $\Pi_\sigma(n)$ *denote the number of primitive,* σ*-juggling sequences of length* $\le n$, *and let* ρ *be the largest root (by modulus) of* $\bar{s}_\sigma(z)$. *Then,* ρ *is a real number satisfying* $b + 1 - c \cdot \frac{b^{3/2}}{e^b} \le \rho < b + 1$, *where* c

is the same constant defined in Theorem 1. Moreover,

$$\Pi_\sigma(n) \sim \frac{\rho(b+1-\rho)}{(\rho-1)|s'_\sigma(1/\rho)|} \cdot \rho^n.$$

Proof. By the Stolz-Cesàro theorem (cf. [5, p. 85]), $\Pi_\sigma(n)$ is asymptotic to $\sum_{k=1}^{n} \frac{b+1-\rho}{|s'_\sigma(1/\rho)|} \cdot \rho^k$. Since this is a geometric sum, we easily see that

$$\Pi_\sigma(n) \sim \frac{b+1-\rho}{|s'_\sigma(1/\rho)|} \cdot \sum_{k=1}^{n} \rho^k = \frac{b+1-\rho}{|s'_\sigma(1/\rho)|} \cdot \frac{\rho(\rho^n-1)}{\rho-1}$$

and so

$$\Pi_\sigma(n) \sim \frac{\rho(b+1-\rho)}{(\rho-1)|s'_\sigma(1/\rho)|} \cdot \rho^n,$$

as desired. □

5 Conclusion

In the end, a straightforward application of analytic techniques to the generating functions from [2] was sufficient to determine the asymptotic growth of $P_\sigma(n)$ and its cumulative cousin $\Pi_\sigma(n)$, though the process is laborious at points. However, it should not be understated that we now have an analogue to the prime number theorem for juggling patterns. This analysis comes by viewing each state in isolation, and viewing its juggling sequences as a set (technically, a non-commutative semigroup) that shares some properties with the positive integers.

Alternatively, one could view a juggling sequence more holistically as a collection of juggling states. This requires the construction of a directed graph, called a *state diagram*, in which each b-ball juggling state is a node and a throw corresponds to a directed edge connecting two states. In this context, a σ-juggling sequence is a path in the state diagram that begins and ends at σ.

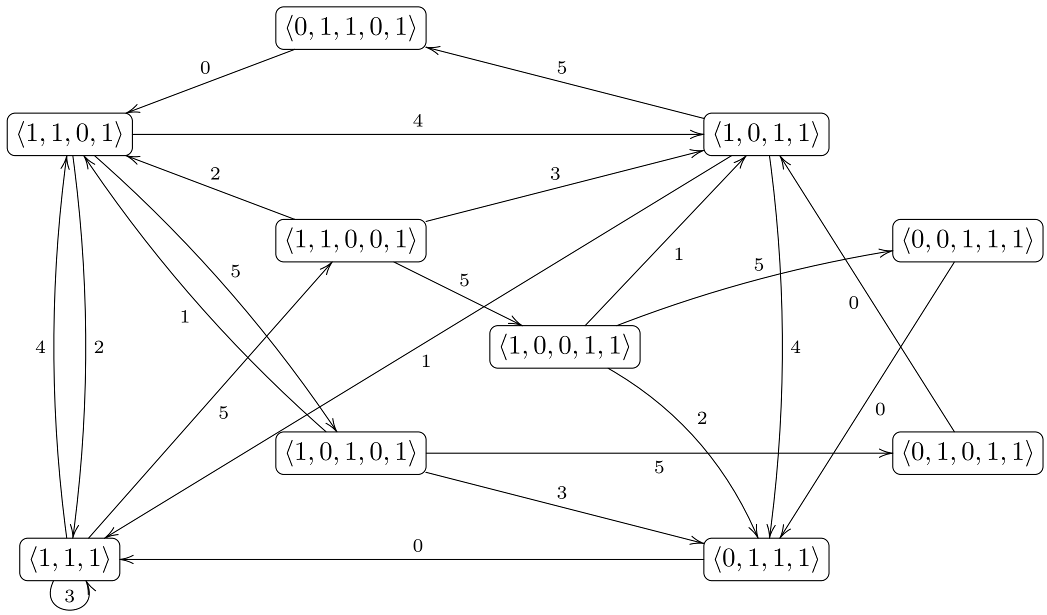

Figure 3: The 3-ball state diagram, restricted to a maximum height of 5.

Since such a path is necessarily a cycle, we may untether the juggling sequence from the chosen state σ, which leads to a new analogue for primality. Specifically, we may define a *prime* juggling sequence to be one that corresponds to a simple cycle in the state diagram (i.e., there are no repeated states). Following the 3-ball state diagram restricted to height 5 as shown in Figure 5, we see that the $\langle 1, 1, 0, 1\rangle$-juggling sequence $(5, 5, 0, 5, 0)$ is prime, while the $\langle 1, 1, 0, 1\rangle$-juggling sequence $(2, 4, 4, 5, 0)$ is not. Prime juggling sequences are necessarily primitive for each juggling state through which they pass. (Note that the converse is false, since a non-simple cycle may be viewed as primitive if one chooses a state σ that it passes through only once.) Some interesting combinatorial work has been done in this area, notably by [1] in the case $b = 2$. In this venue, it is more appropriate to examine prime juggling sequences instead of primitive juggling sequences. An Ihara-type zeta function could serve as a foundation for an analytic approach.

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