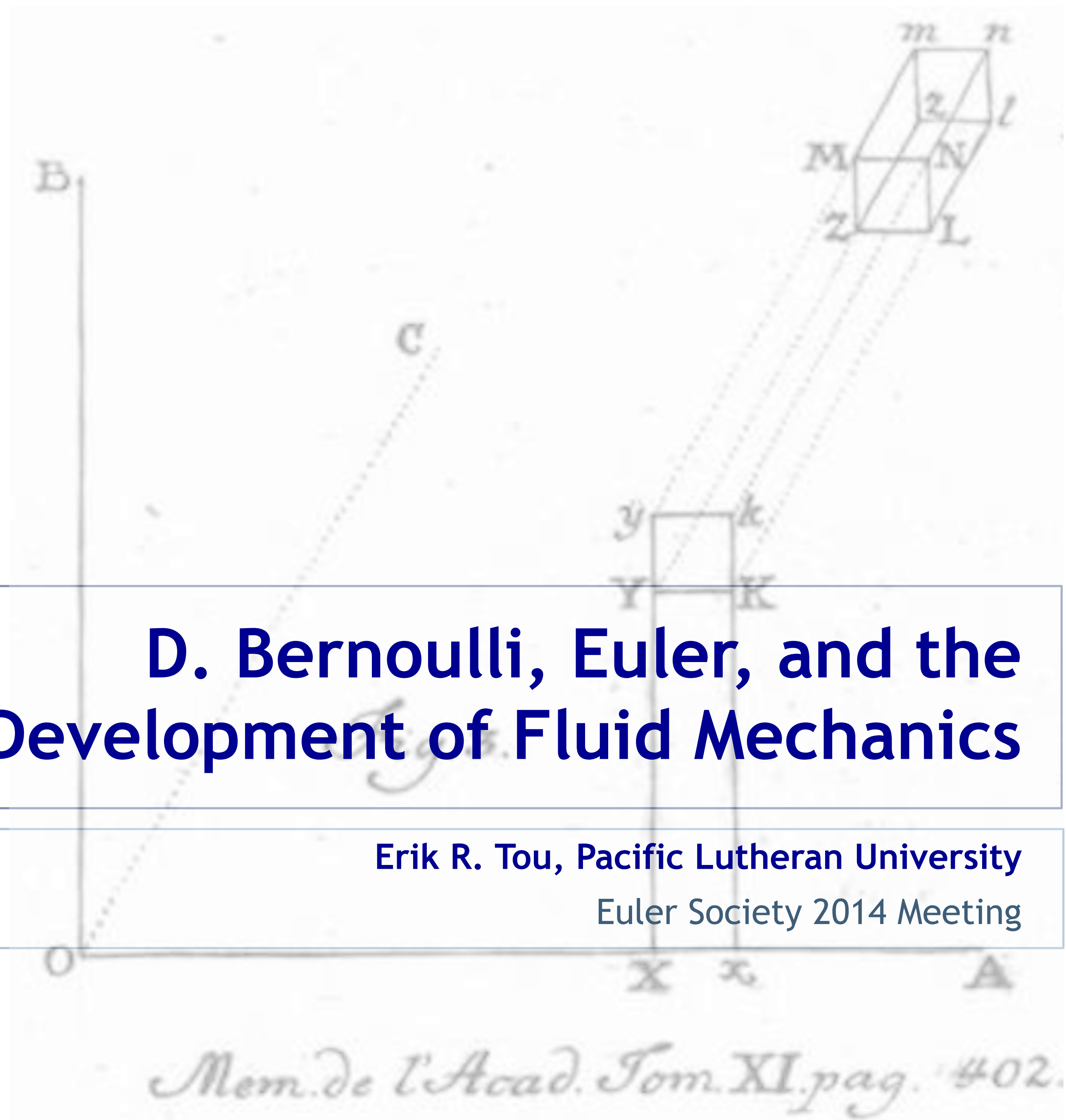
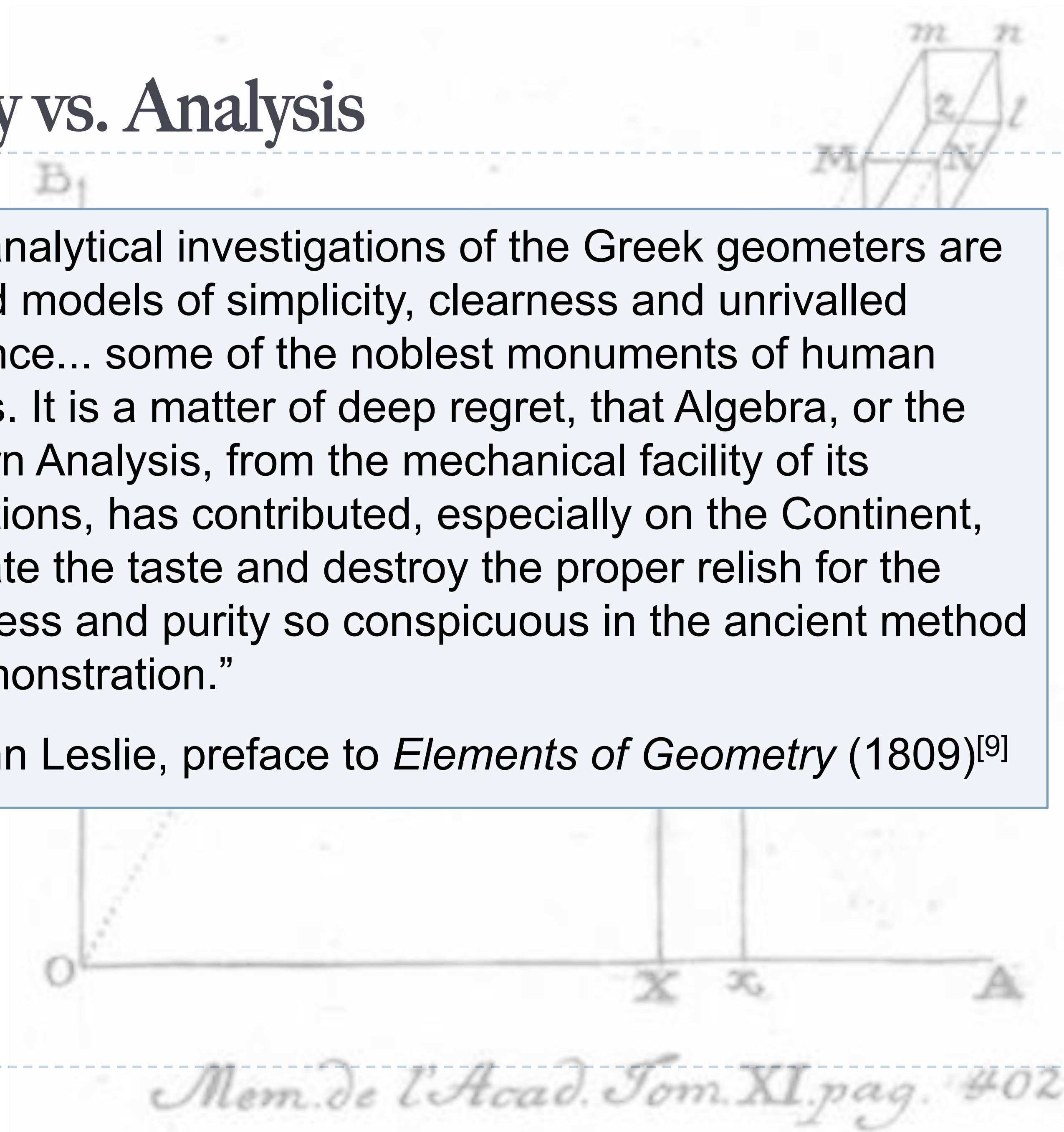


D. Bernoulli, Euler, and the Development of Fluid Mechanics

Erik R. Tou, Pacific Lutheran University

Euler Society 2014 Meeting

Geometry vs. Analysis



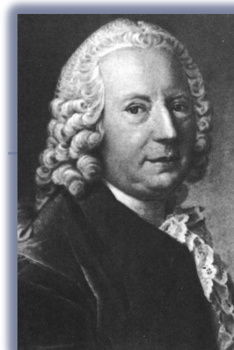
"The analytical investigations of the Greek geometers are indeed models of simplicity, clearness and unrivalled elegance... some of the noblest monuments of human genius. It is a matter of deep regret, that Algebra, or the Modern Analysis, from the mechanical facility of its operations, has contributed, especially on the Continent, to vitiate the taste and destroy the proper relish for the strictness and purity so conspicuous in the ancient method of demonstration."

— John Leslie, preface to *Elements of Geometry* (1809)[9]

Partial Timeline of Early Fluid Mechanics

- **1738 [Apr/May]** – Daniel Bernoulli's *Hydrodynamica* appears in print (first draft written *c*. 1733).
- **1738 [Oct]** – Johann Bernoulli writes to Leonhard Euler, first mentioning *Hydraulica*.
- **1739 [Mar/Apr]** – Euler receives copies of D. Bernoulli's *Hydrodynamica* and J. Bernoulli's *Hydraulica* (Part 1), nearly simultaneously.
- **1739 [May]** – Euler writes to each Bernoulli about their work on fluid mechanics.
- **1743** – *Hydraulica* appears in print (in J. Bernoulli's *Opera Omnia*), backdated to 1732.
- **1751** – Euler writes "Research on the movement of rivers" [E332], pub. 1767.
- **1752** – Euler writes "Principles of the motion of fluids" [E258], pub. 1761.
- **1755** – Euler writes "General principles of the motion of fluids" [E226], pub. 1757.

D. Bernoulli's *Hydrodynamica* (1738)



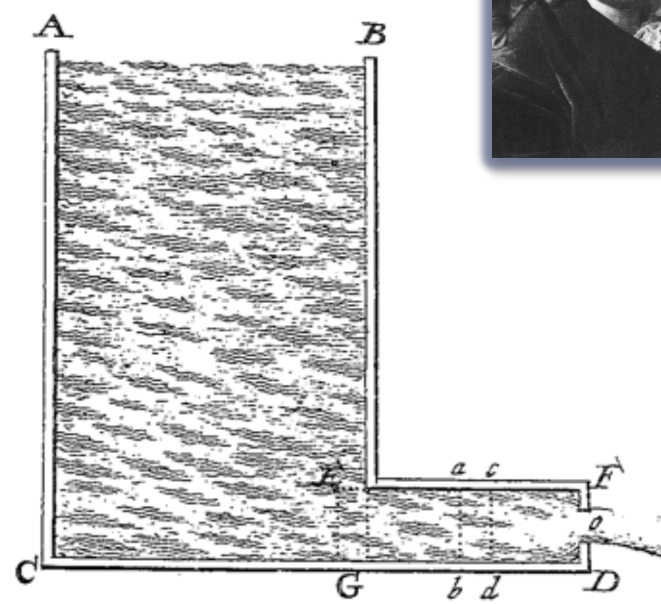
Problema.

Fig. 72. §. 5. Fuerit vas ampliſſimum ACEB (Fig. 72.) aqua conſtanter plenum conſervandum, tubo inſtructum cylindrico & horizontali ED; ſitque in extremitate tubi foramen o aquas velocitate uniformi emittens; quæritur preſſio aquæ in latera tubi ED.

Solutio.

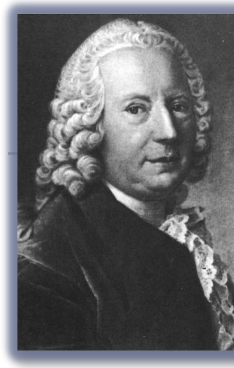
Sit altitudo ſuperficiei aqueæ AB ſupra orificium $o = a$; erit velocitas aquæ in o effluentis, ſi prima fluxus momenta excipias, uniformis cenſenda & $= \sqrt{a}$, quia vas conſtanter plenum conſervari aſſumimus; poſitaque ratione amplitudinum tubi ejusque foraminis $= \frac{n}{1}$, erit velocitas aquæ in tubo $= \frac{\sqrt{a}}{n}$: Si vero omne fundum FD abeſſet, foret velocitas ultima aquæ in eodem tubo $= \sqrt{a}$, quæ major eſt quam a; Igitur aqua in tubo tendit ad majorem motum, niſus autem ejus ab appoſito fundo FD impeditur: Ab hoc niſu & reniſu comprimitur aqua, quæ ipſa compreſſio coërcetur à lateribus tubi, hæcque proinde ſimilem preſſionem ſuſtinent. Apparet ſic preſſionem laterum proportionalem eſſe accelerationi ſeu incremento velocitatis, quod aqua ſit acceptura, ſi in inſtanti omne obſtaculum motus evaneſcat, ſic ut immediate in aërem ejiciatur.

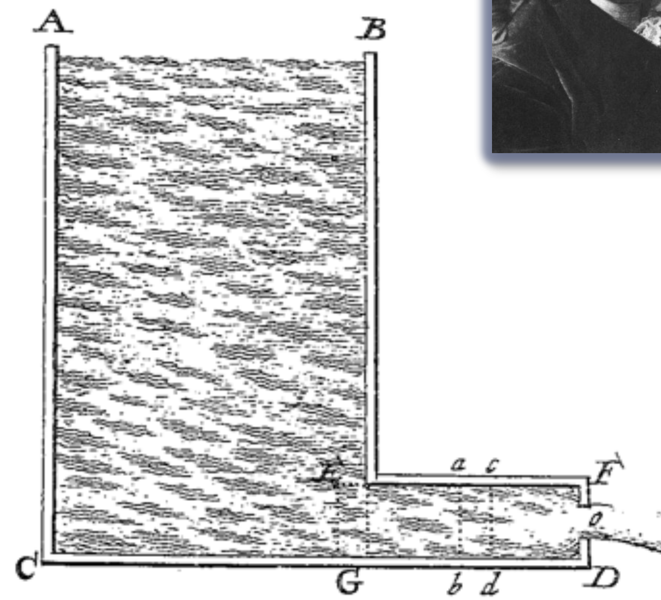
Res igitur jam eo perducta eſt, ut ſi durante fluxu aquæ per o, tubus ED in temporis puncto abrumpatur in cd, quæratur quantam accelerationem guttula $acbd$ inde ſit perceptura: tantam enim preſſionem ſentiet particula ac in lateribus tubi ſumta à præterfluente aqua: Hunc in finem conſiderandum eſt vas ABEcdC, atque pro eo invenienda acceleratio particulæ aqueæ effluxui proximæ, ſi hæc habuerit velocitatem $\frac{\sqrt{a}}{n}$: Iſtud negotium fecimus generaliſſime in *paragrapho tertio ſect. V.* Attamen quia in hoc caſu particulari brevis eſt calculus, motum in vaſe decurtato ABEcdC hic iterum calculo ſubducemus.



Sit velocitas aquæ in tubo Ed, quæ nunc ut variabilis conſideranda eſt, $= v$: amplitudo tubi ut antea $= n$, longitudo E$c = c$: indicetur longitudo ac particulæ aqueæ infinite parvæ & effluxui proxime per dx: Erit guttula æqualis in E tubum ingreſſura eodem temporis puncto quo altera $acdb$ ejicitur: dum autem guttula in E, cujus maſſa $= ndx$, tubum ingreditur acquirit velocitatem v, atque *vim vivam* $nvvdx$, quæ *vis viva* tota fuit de novo generata; nullum enim, ob amplitudinem vaſis AE infinitam, motum guttula in E habuit tubum nondum ingreſſa: huic *vi vivæ* $nvvdx$ addendum eſt incrementum *vis vivæ*, quod aqua in Eb accipit, dum guttula ad effluit, nempe $2ncvdv$: aggregatum debetur *deſcenſui actuali* guttulæ ndx per altitudinem BE ſeu a: habetur igitur $nvvdx + 2ncvdv = nadx$ ſive $\frac{vdv}{dx} = \frac{a - vv}{2c}$.

Bernoulli's Geometry





Let the height of the surface of the water AB above the hole $o = a$; the velocity of the water flowing out at o, if the first moment of flow be excepted, is to be regarded as uniform and $= \sqrt{a}$, since we assume the vessel to be kept constantly full. Let the ratio of the width of the tube to that of the hole be $n/1$; then the velocity of the water in the tube $= \sqrt{a}/n$ …

… the problem is now changed into this: if during the flow of the water through o the tube ED were broken at cd at an instant, one seeks the magnitude of the acceleration the volume element $acbd$ would thence be about to obtain…

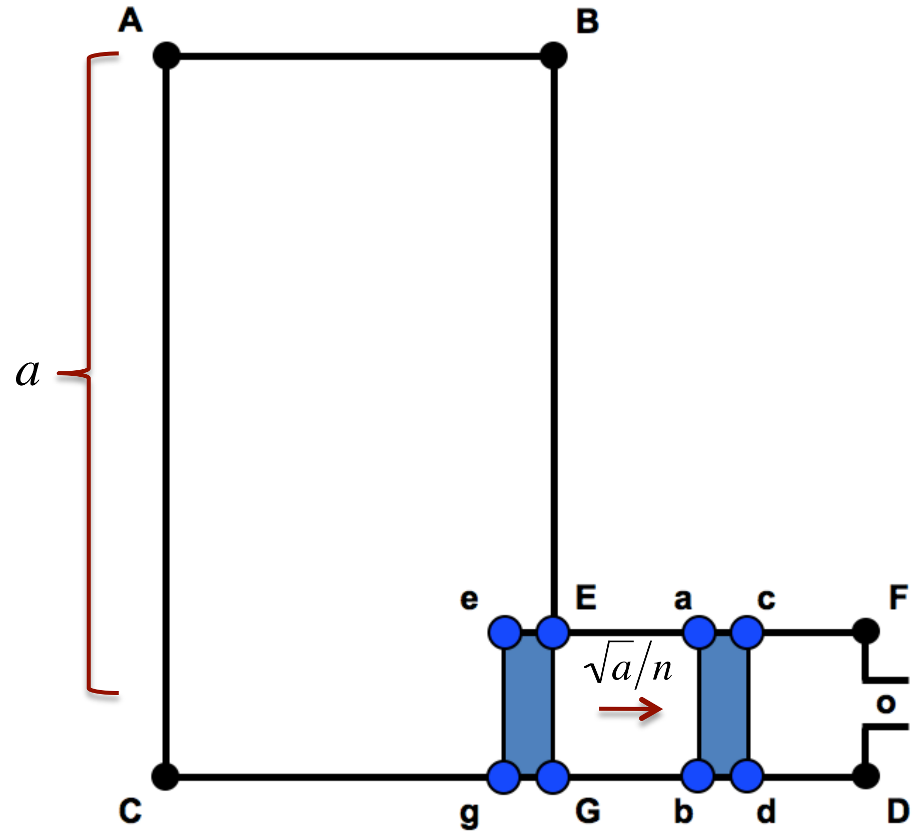
Bernoulli's Geometry

a = distance from surface AB to hole o

n = ratio of FD to o (FD is n times the width of the hole)

Aqueous particles at $acbd$ (midway through tube) and $eEGg$ (about to enter tube).

Velocity of water in tube is $\sqrt{a}/n$.



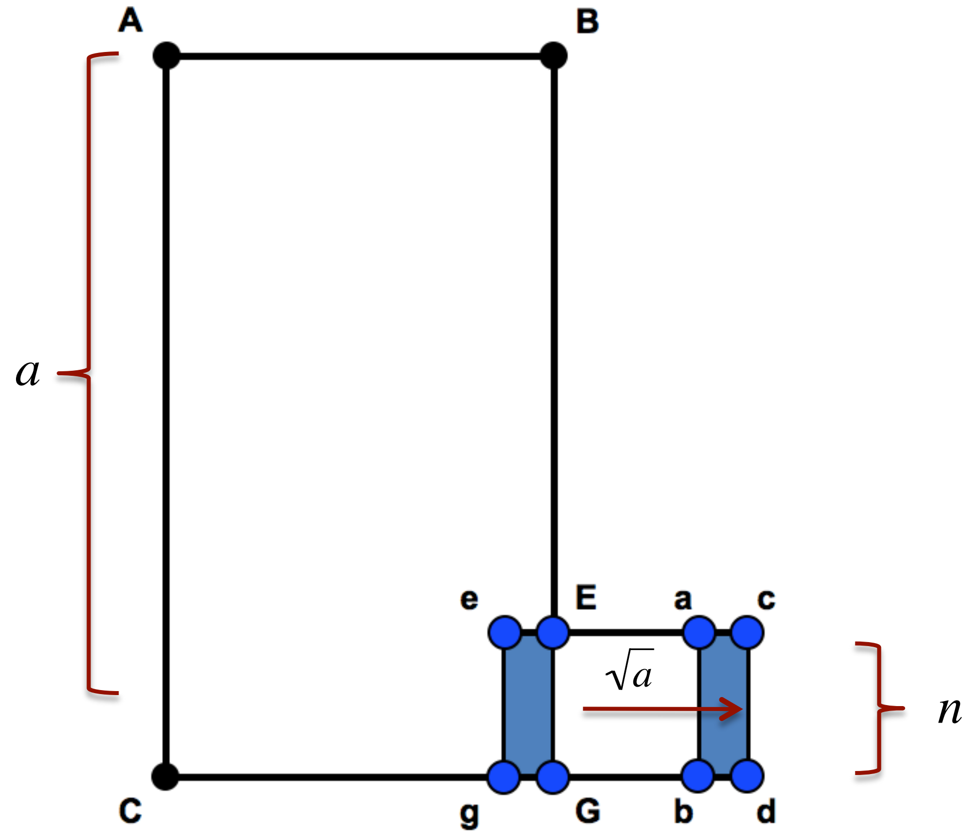
Bernoulli's Geometry

a = distance from surface AB to hole o

n = ratio of FD to o (FD is n times the width of the hole)

Aqueous particles at $acbd$ (midway through tube) and $eEGg$ (about to enter tube).

Velocity of water in tube is $\sqrt{a}/n$.



If tube is broken at cd, velocity of water in tube increases to $\sqrt{a}$.

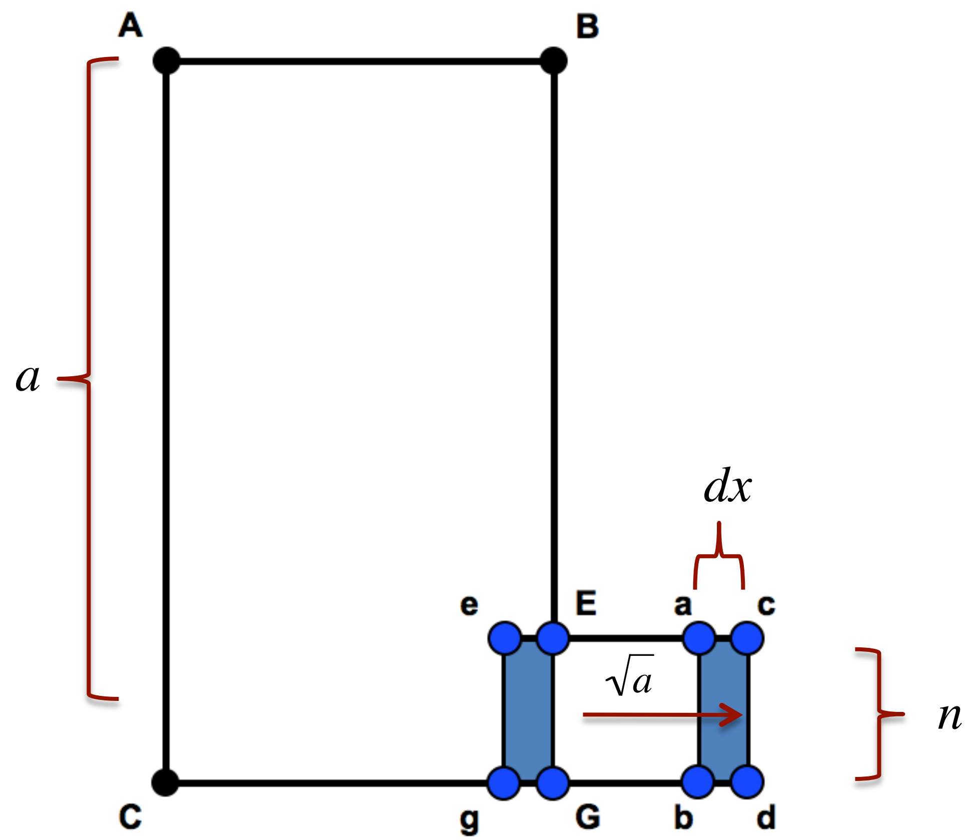
From Geometry to Calculus

v = velocity of water in tube (viewed as a variable)

c = Ec = length of tube

dx = width of aqueous particle, “infinitely small and about to flow out”

$n\ dx$ = area of particle = mass of particle (density = 1).



From Geometry to Calculus

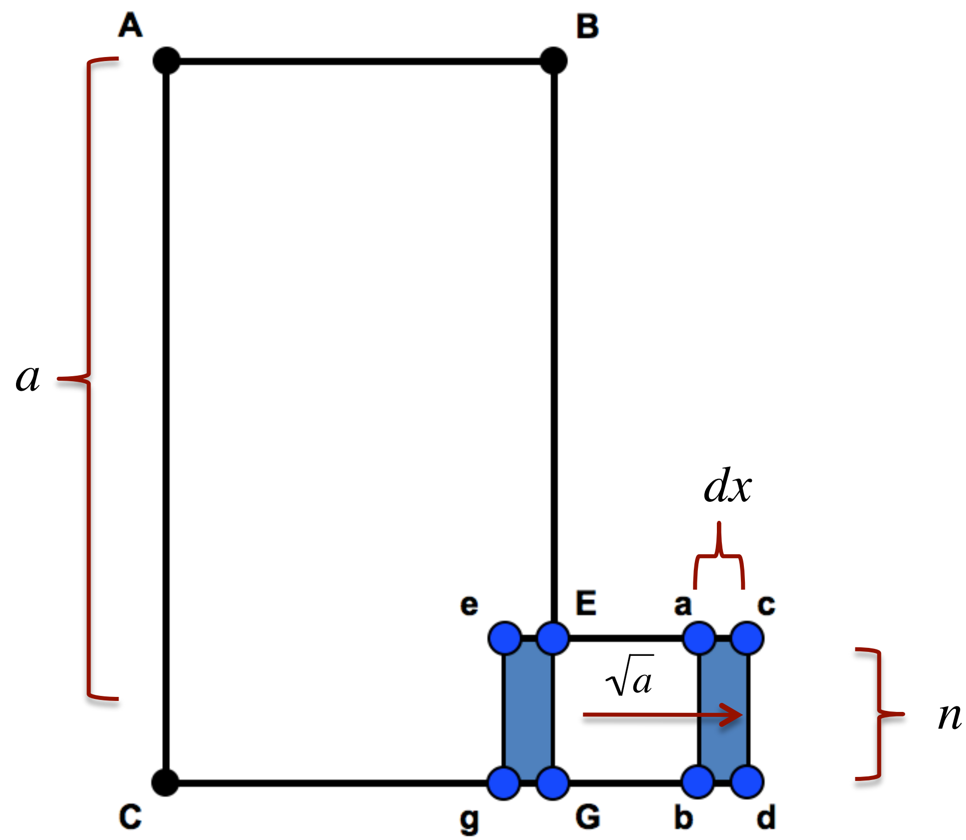
v = velocity of water in tube (viewed as a variable)

$c = Ec$ = length of tube

dx = width of aqueous particle, "infinitely small and about to flow out"

$n\ dx$ = area of particle = mass of particle (density = 1).

"There will be an equal volume element [$eEGg$] at E entering the tube at the same instant that $acdb$ flows out … it acquires the velocity v and the live force [*vis vivam*] $nvv\ dx$…"



From Geometry to Calculus

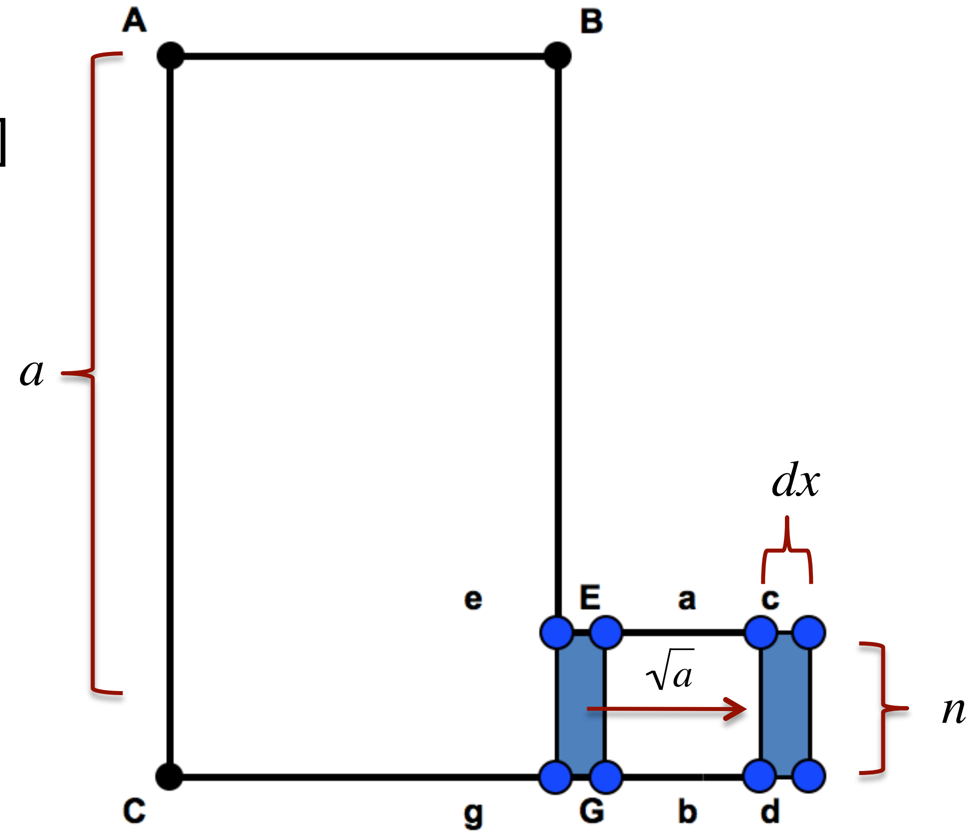
"…to this … is to be added the increment of live force [$2ncv\ dv$] which the water at Eb receives while the volume element at ad flows out…"

Live forces (*vis viva*):

Potential ascent:

- *Inflow*: $(n\ dx)v^2$
- *Outflow*: $2ncv\ dv$

Actual descent: $(n\ dx)a$



From Geometry to Calculus

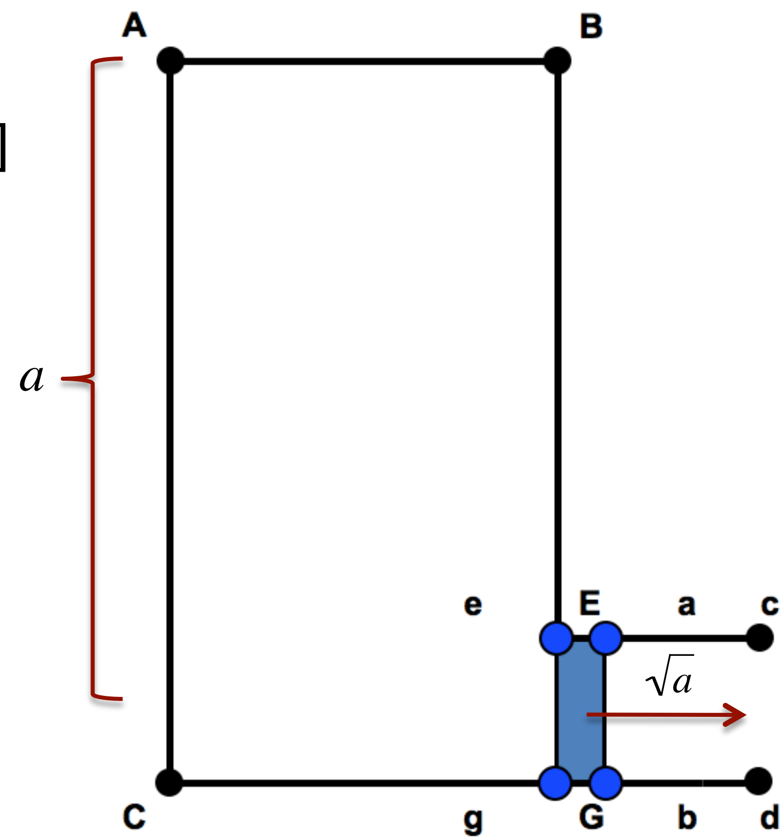
"…to this … is to be added the increment of live force [$2ncv\ dv$] which the water at Eb receives while the volume element at ad flows out…"

Live forces (*vis viva*):

Potential ascent:

- *Inflow*: $(n\ dx)v^2$
- *Outflow*: $2ncv\ dv$

Actual descent: $(n\ dx)a$



Equate potential ascent & actual descent:

$$nv^2\ dx + 2ncv\ dv = na\ dx$$

Modernizing Bernoulli's Conclusion

$nv^2\, dx + 2ncv\, dv = na\, dx$

This is (almost) the Bernoulli equation for fluid flow:

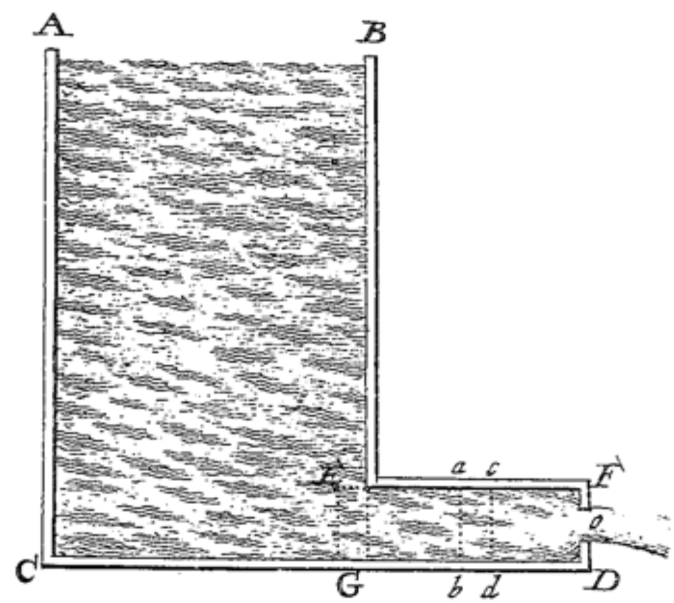
$$nv^2\, dx + 2ncv\, dv = na\, dx$$

$$v^2 + 2cv\frac{dv}{dx} = a$$

$$[pressure \propto]\ \ 2cv\frac{dv}{dx} = a - v^2$$

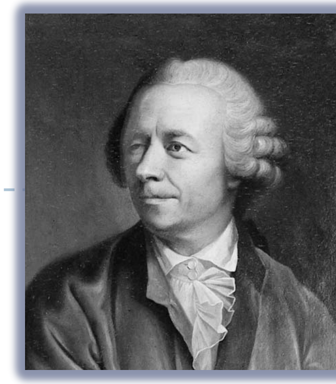
$$pressure \propto a - v^2$$

Modern form: $pressure = C - \frac{v^2}{2g} - z$



Euler's Earliest Fluid Mechanics

A letter from Euler to Johann Bernoulli, 16 May 1739:



Sit igitur [Fig. 1] vas cujuscunque figurae $ABDC$, initio, quo aqua per foramen CD effluere incipit, usque ad AB aqua repletum, cujus altitudo AC sit $= a$, et foraminis CD amplitudo $= n$. Ponamus aquam jam usque in PS subsedisse hocque tempore aquam per foramen CD effluere celeritate altitudini z debita, minimo autem tempusculo subsidere superficiem PS per spatiolum Pp; sitque $Pp = dp$ atque amplitudo vasis $PS = s$, denotabit $z + dz$ altitudinem celeritati aquae effluentis debitam, cum suprema aquae superficies usque in ps subsedit. Ut nunc mutatio motus hoc tempusculo genita innotescat, concipiatur superficies quaecunque aquae RY, ponaturque $AR = r$, et amplitudo $RY = y$; erit celeritas hujus strati RY debita altitudini $\frac{nnz}{yy}$; et quia tempusculo infinite parvo superficies RY descendit in ry, erit tum ejus celeritas debita altitudini

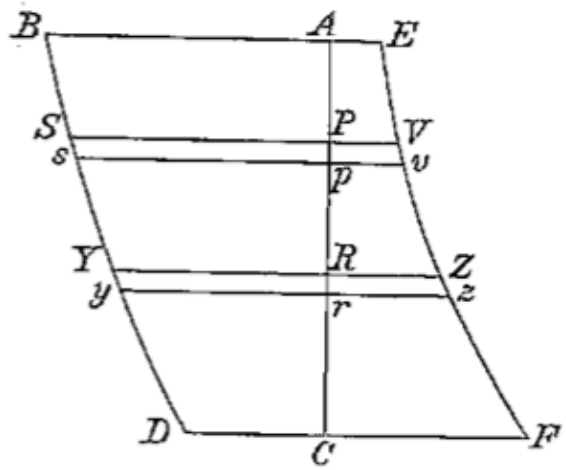
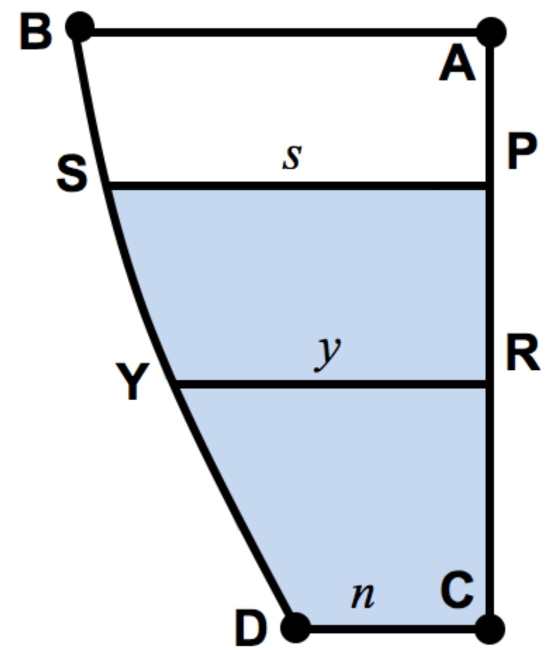


Fig. 1.

Euler's Earliest Fluid Mechanics

Initially, surface is at PS; water flows out of hole at CD.

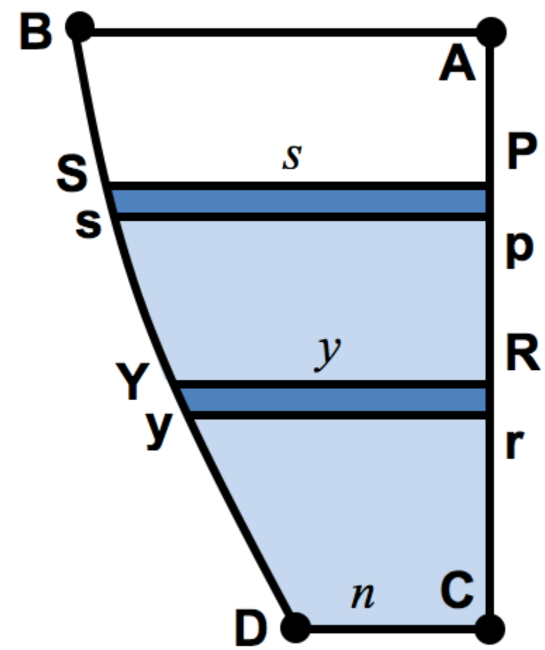
- z = velocity head at PS $[= V^2/2g]$
- $(nn/yy)z$ = velocity head at RY $[= h]$



Euler's Earliest Fluid Mechanics

Initially, surface is at PS; water flows out of hole at CD.

- z = velocity head at PS [$= V^2/2g$]
- $(nn/yy)z$ = velocity head at RY [$= h$]



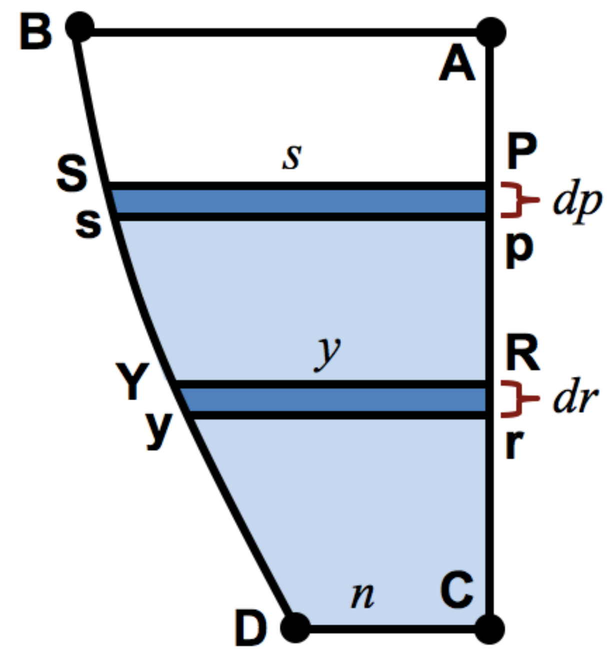
In an infinitesimal span of time:

- PS sinks to ps
- RY sinks to ry

Euler's Earliest Fluid Mechanics

Initially, surface is at PS; water flows out of hole at CD.

- z = velocity head at PS [$= V^2/2g$]
- $(nn/yy)z$ = velocity head at RY [$= h$]



In an infinitesimal span of time:

- PS sinks to ps (change = dp)
- RY sinks to ry (change = dr)
- $z + dz$ = velocity head at ps
- Velocity head at ry
  $= nn/(y+dy)^2\ (z+dz)$ [$= h + dh$]

Change in velocity head from RY to $ry = (nn/yy)\ dz - 2nnz\ dy/y^3$. [$= dh$]

Modernizing Euler's Conclusion

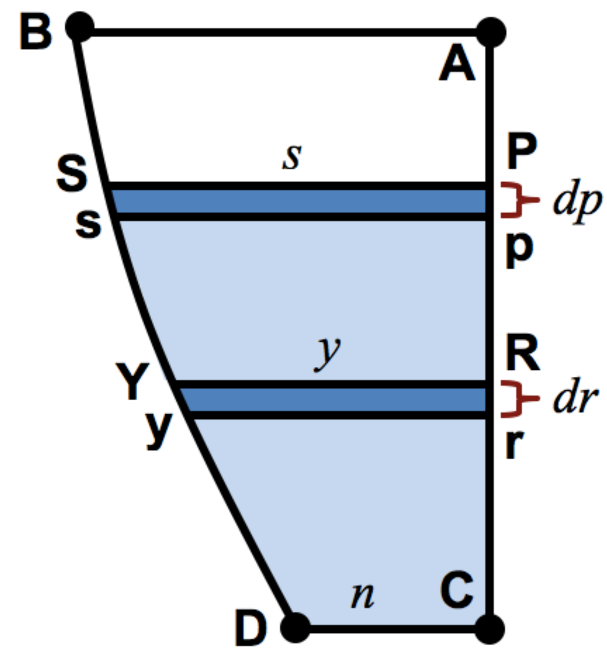
Fluid density is taken to = 1; velocity head measured as square of velocity (imagine velocity units are ***m/s***):

dh $(\boldsymbol{m^2/s^2})$

dh/dr $(\boldsymbol{m/s^2})$ "accelerating force"

$dh/dr \bullet y\, dr$ $(\boldsymbol{kg \bullet m/s^2})$ "moving force"

$dh/dr \bullet y\, dr/y = dh$ $(\boldsymbol{N/m})$ [pressure dp]



Modernizing Euler's Conclusion

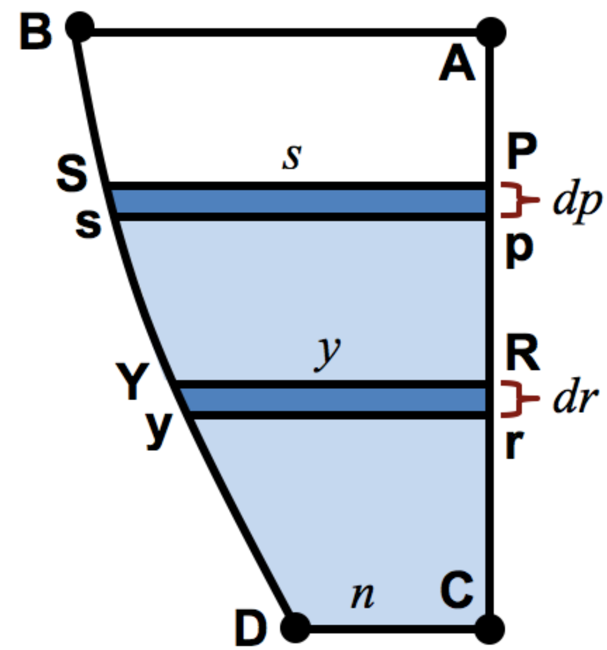
Fluid density is taken to = 1; velocity head measured as square of velocity (imagine velocity units are ***m*/*s***):

dh ($\boldsymbol{m^2/s^2}$)

dh/dr ($\boldsymbol{m/s^2}$) "accelerating force"

$dh/dr \bullet y\, dr$ ($\boldsymbol{kg \bullet m/s^2}$) "moving force"

$dh/dr \bullet y\, dr/y = dh$ ($\boldsymbol{N/m}$) [pressure dp]



Euler thus obtains pressure differential:

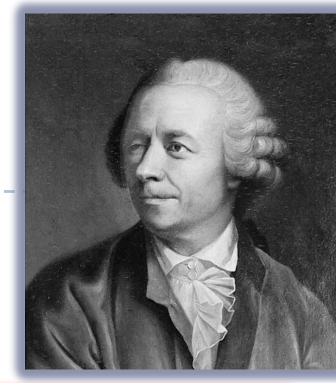
$[\ dp\ ] = (nn/yy)dz - 2nnz\ dy/y^3$.

Then he uses $1/y = dr/(s\ dp)$ to get

$[\ dp\ ] = nn[dz/(s\ dp) \bullet dr/y - 2z\ dy/y^3]$.

Modern integration yields: $p(y) - p(s) = \dfrac{n^2 dz}{s\,dp} \displaystyle\int_s^y \frac{dr}{y} + \frac{n^2}{s^2} z - \frac{n^2}{y^2} z$

Euler's Fluid Mechanics Papers



E332
Written 1751
Published 1767

RECHERCHES
SUR
LE MOUVEMENT DES RIVIERES.

E258
Written 1752
Published 1761

PRINCIPIA
MOTVS FLVIDORVM.

E226
Written 1755
Published 1757

PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.

Euler's Fluid Mechanics Papers

E332
Written 1751
Published 1767

RECHERCHES
SUR
LE MOUVEMENT DES RIVIERES.

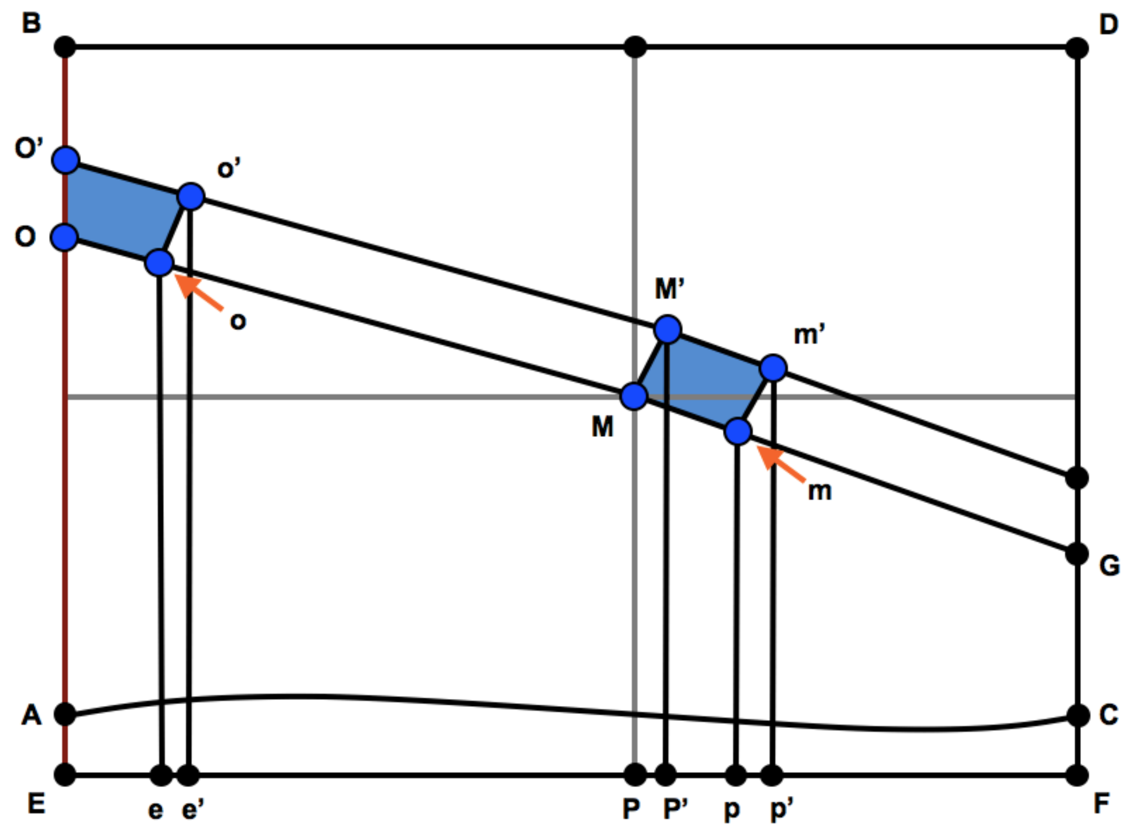
§. 5. Soit donc AC le lit d'une telle fection de riviere, ou d'une riviere infiniment étroite, qui ait partout la même largeur infiniment petite. Que ce lit AC foit une ligne courbe quelconque, qui eft fuppofée être connue; pour cet effet je conçois une ligne horizontale EF qui ferve d'axe pour y rapporter la ligne AC par des coordonnées orthogonales EP & PQ. Que ABCD foit la riviere, qui fe meut fur ce lit AC, & BD fa fuperficie fupreme; de plus je fuppofe, que la riviere fe trouve déjà dans un état permanent ou d'équilibre, de forte que fa fuperficie BD demeure continuellement la même, & qu'aux mêmes points, comme M, les particules d'eau qui y paffent ayent toujours les mêmes viteffes, & qu'elles foient affujetties aux mêmes preffions. Fig. 1.

Euler's Geometry in E332

Water in the river has constant density = 1.

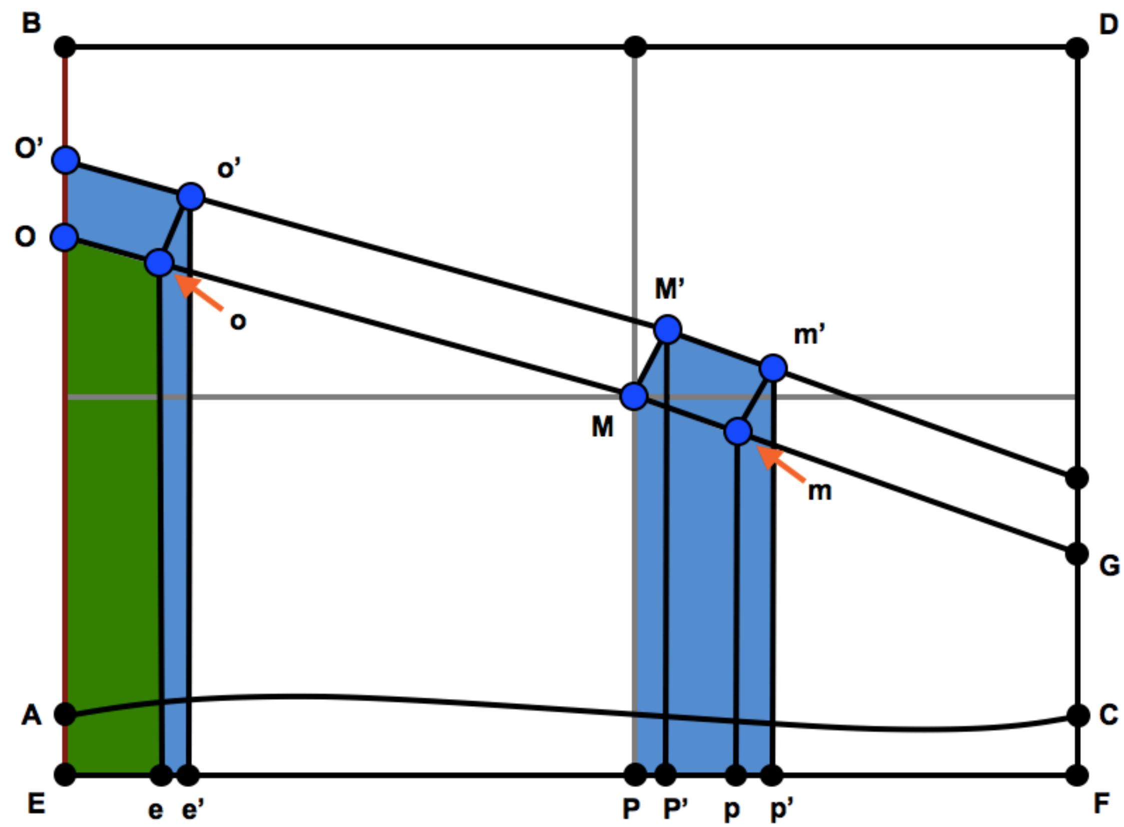
Fluid particle flows downstream from left (OO'o'o) to right (MM'm'm).

Since the fluid is *incompressible*, the area of OO'o'o must equal the area of MM'm'm.



Euler's Geometry in E332

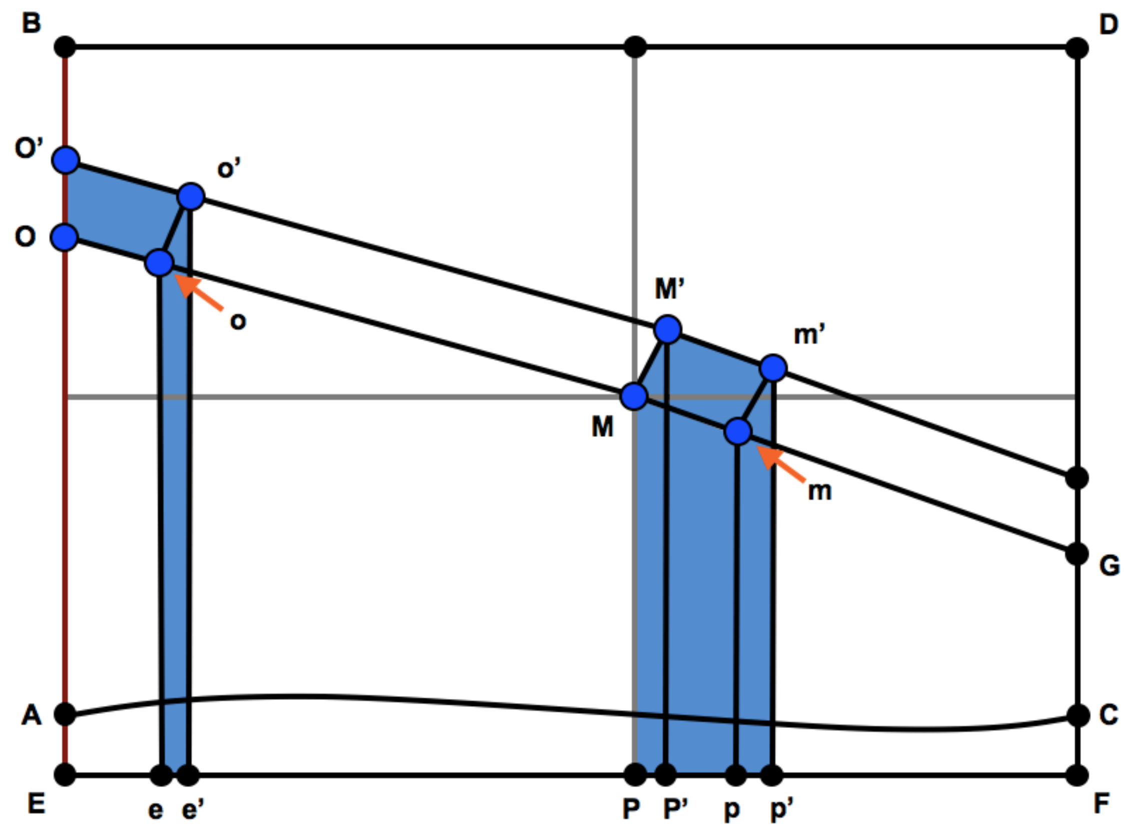
OO'o'o = EO'o'e'
– EOoe …



Euler's Geometry in E332

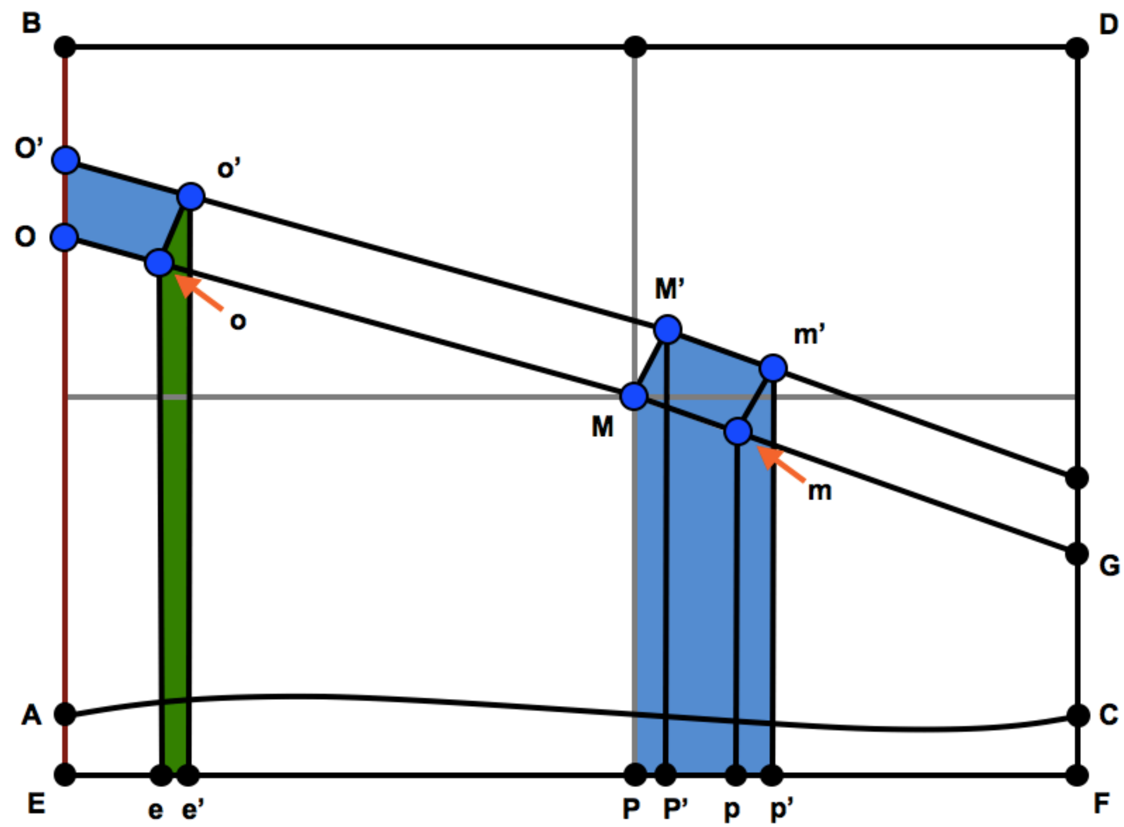
OO'o'o = EO'o'e'

– EOoe …



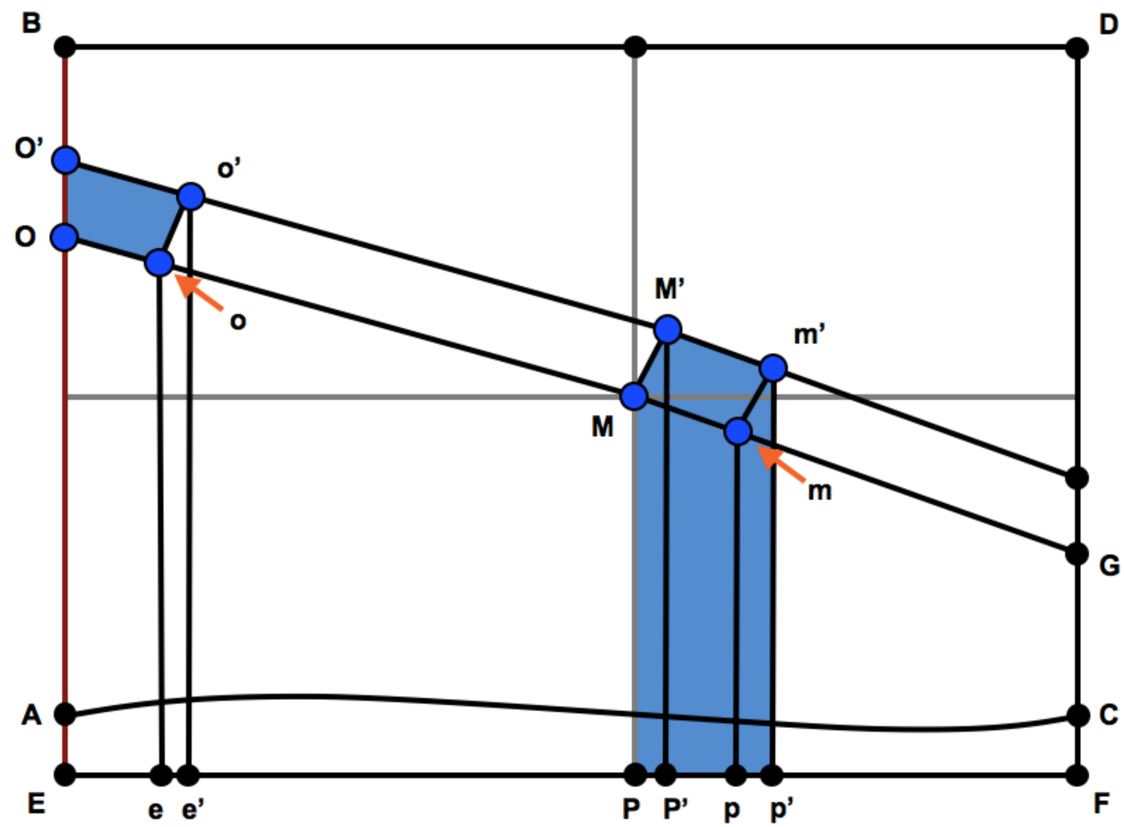
Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e' …



Euler's Geometry in E332

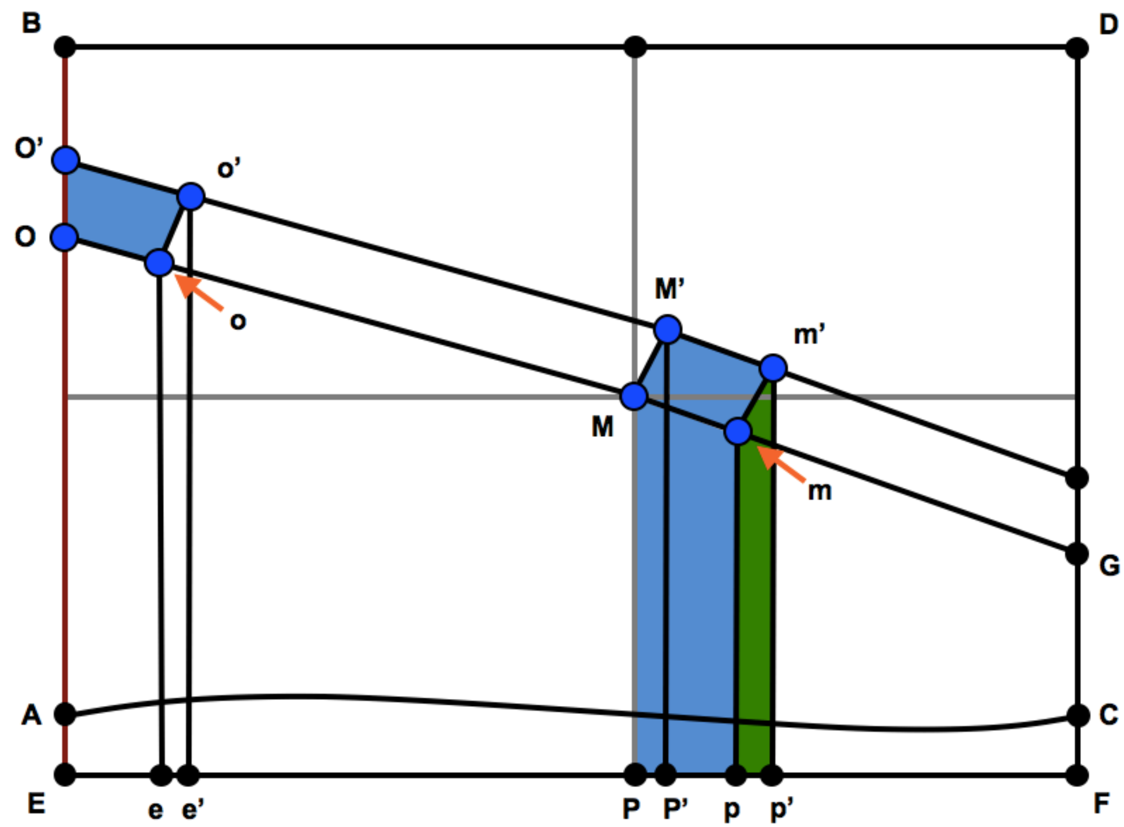
OO'o'o = EO'o'e'
– EOoe
– eoo'e'.



Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e'.

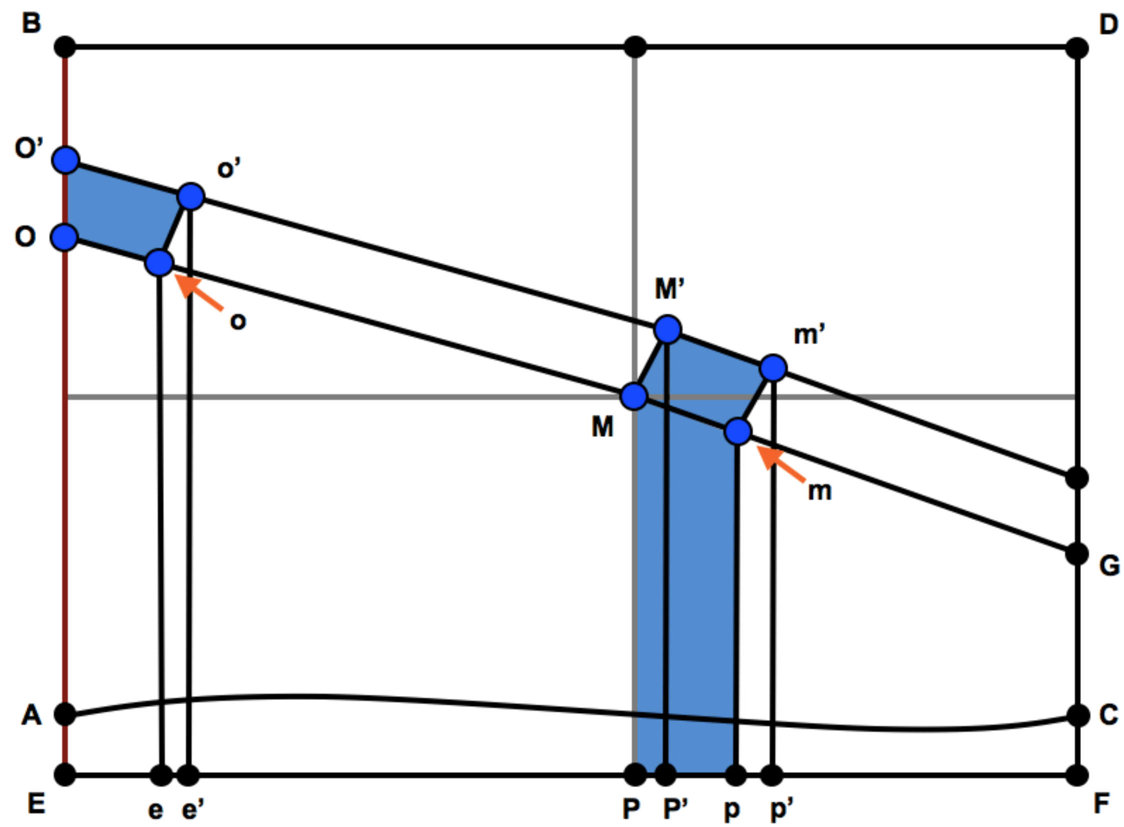
MM'm'm = PMM'P'
+ P'M'm'p'
– pmm'p' …



Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e'.

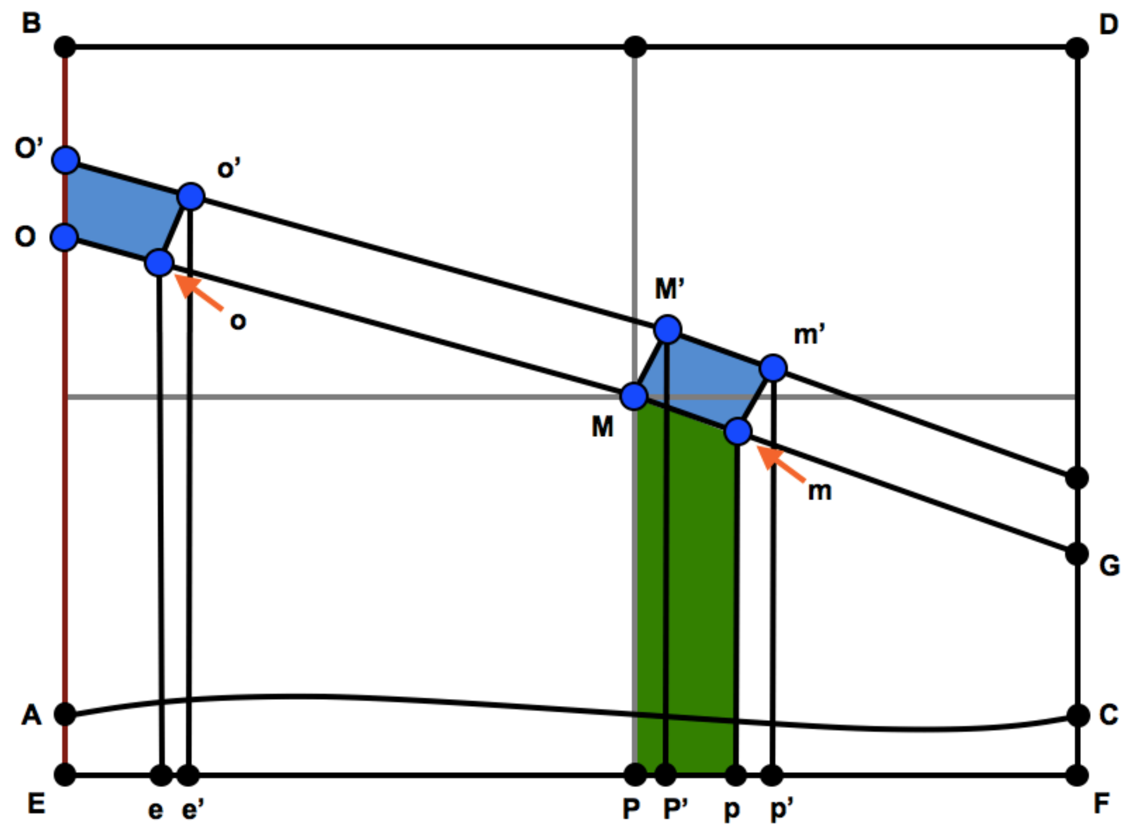
MM'm'm = PMM'P'
+ P'M'm'p'
– pmm'p' …



Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e'.

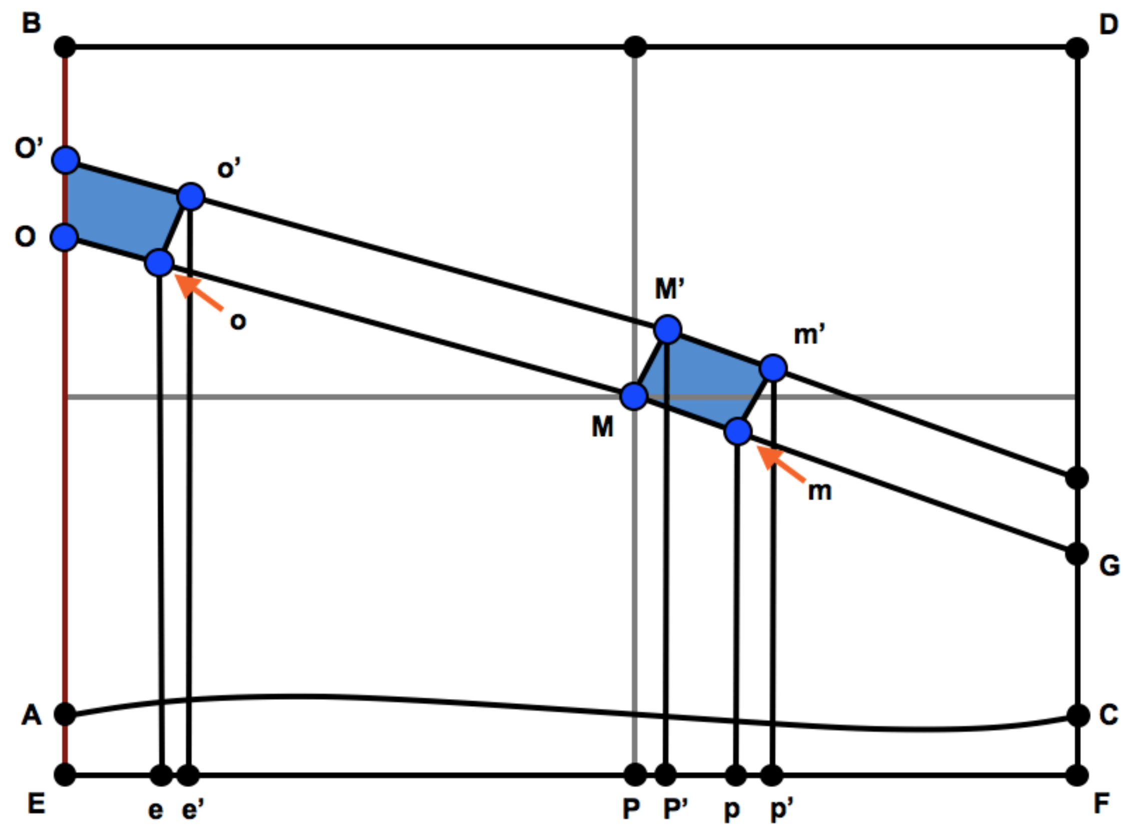
MM'm'm = PMM'P'
+ P'M'm'p'
– pmm'p'
– PMmp …



Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e'.

MM'm'm = PMM'P'
+ P'M'm'p'
– pmm'p'
– PMmp.

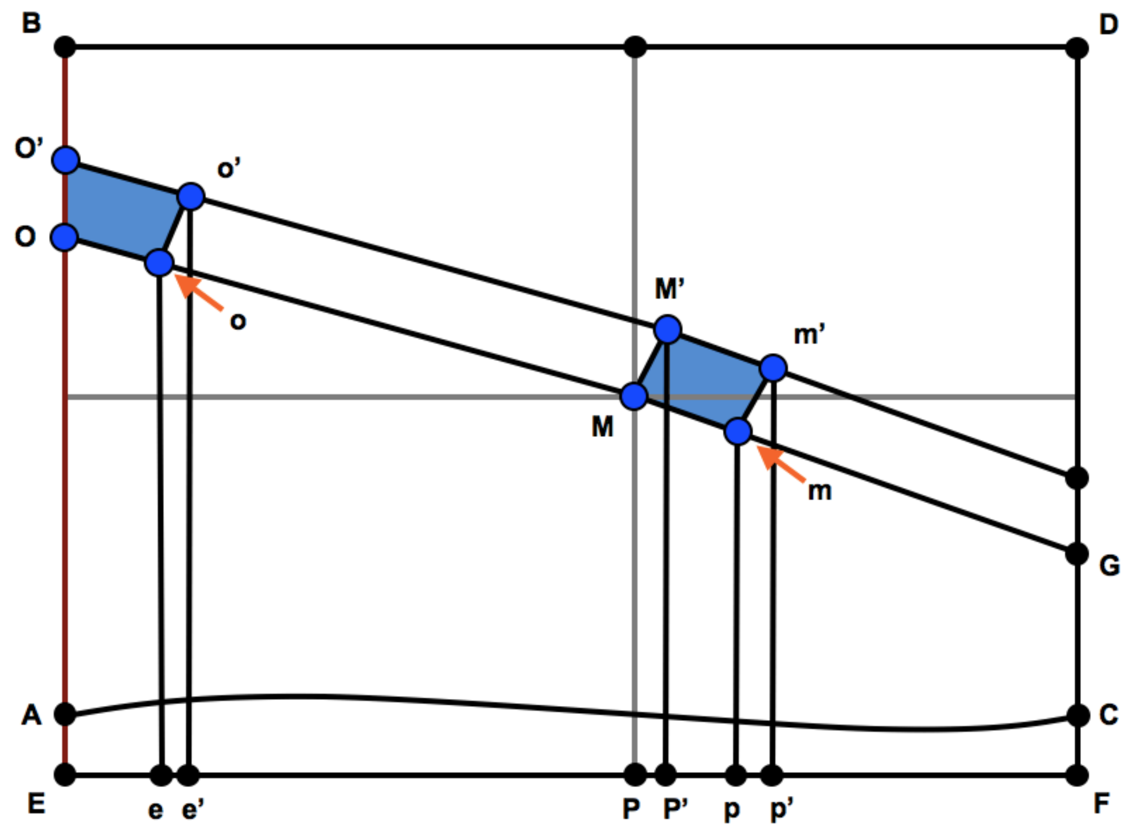


Euler's Geometry in E332

OO'o'o = EO'o'e'
– EOoe
– eoo'e'.

MM'm'm = PMM'P'
+ P'M'm'p'
– pmm'p'
– PMmp.

Area of trapezoid =
Base x *Avg of heights*

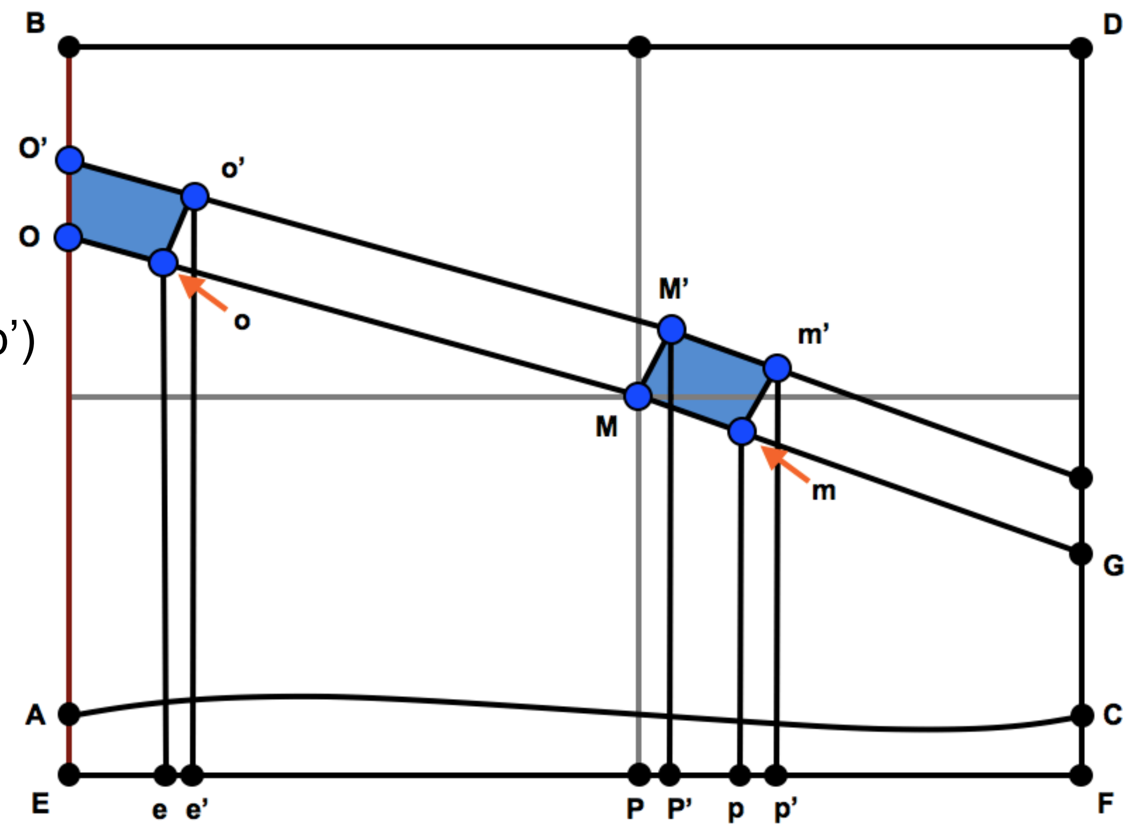


Euler's Geometry in E332

OO'o'o = ½ Ee'(EO'+o'e')
– ½ Ee(EO+oe)
– ½ ee'(eo+o'e').

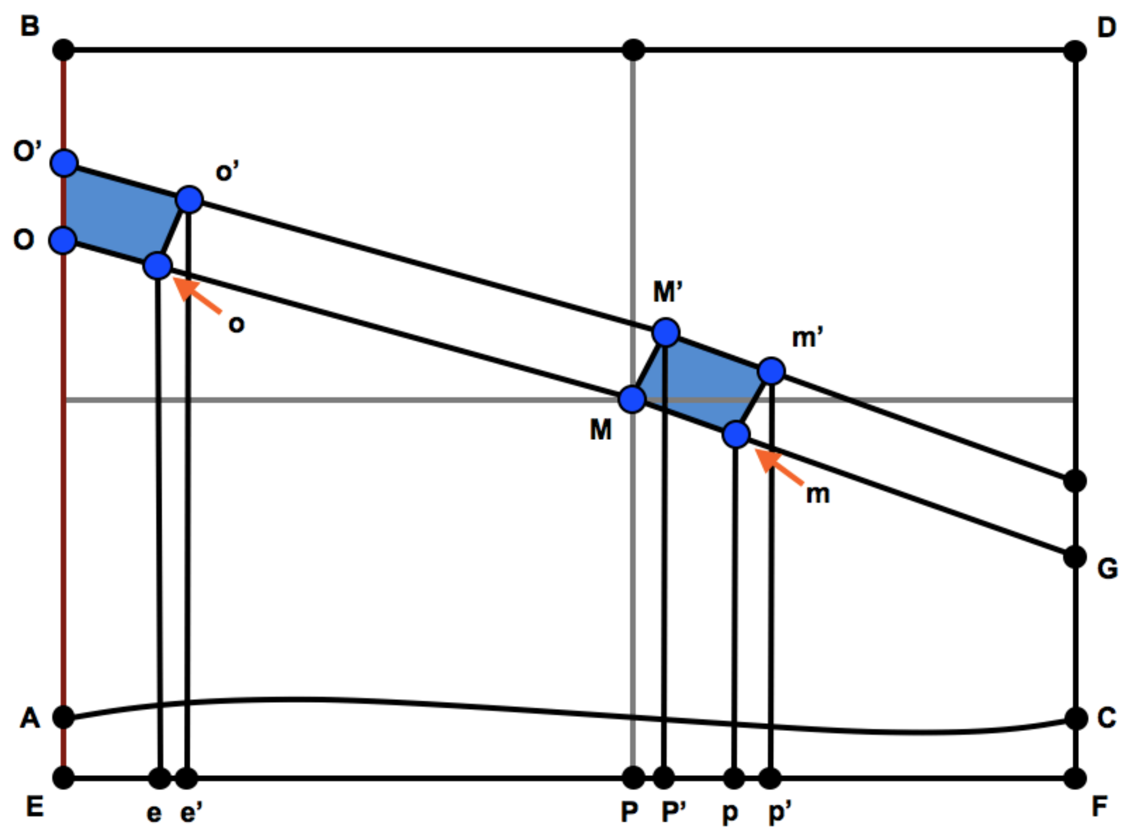
MM'm'm = ½ PP'(PM+M'P')
+ ½ P'p'(P'M'+m'p')
– ½ pp'(pm+m'p')
– ½ Pp(PM+mp).

Area of trapezoid =
Base × Avg of heights



Euler's Geometry in E332

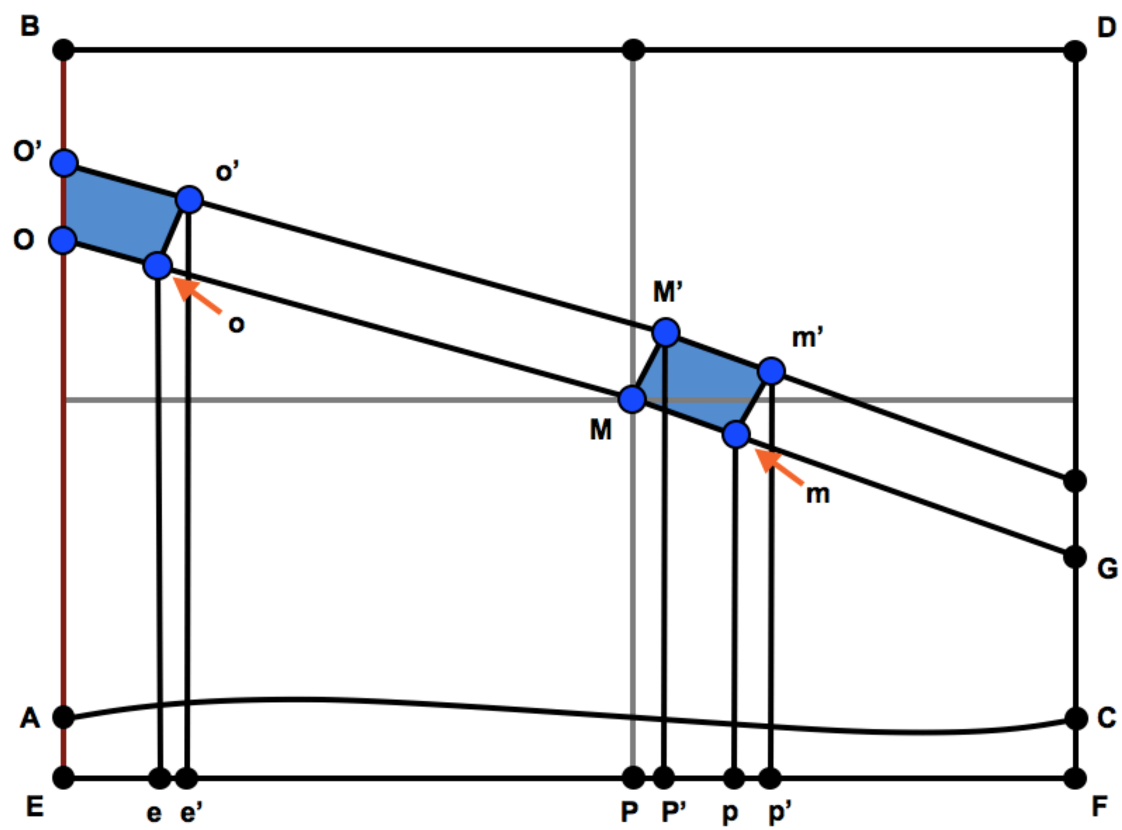
OO'o'o = ½ Ee'(EO'+o'e') – ½ Ee(EO+oe) – ½ ee'(eo+o'e')



Euler's Geometry in E332

OO'o'o = ½ Ee'(EO'+o'e') – ½ Ee(EO+oe) – ½ ee'(eo+o'e')

= ½ (Ee + ee')(EO+OO'+o'e') – ½ Ee(EO+oe) – ½ ee'(eo+o'e')

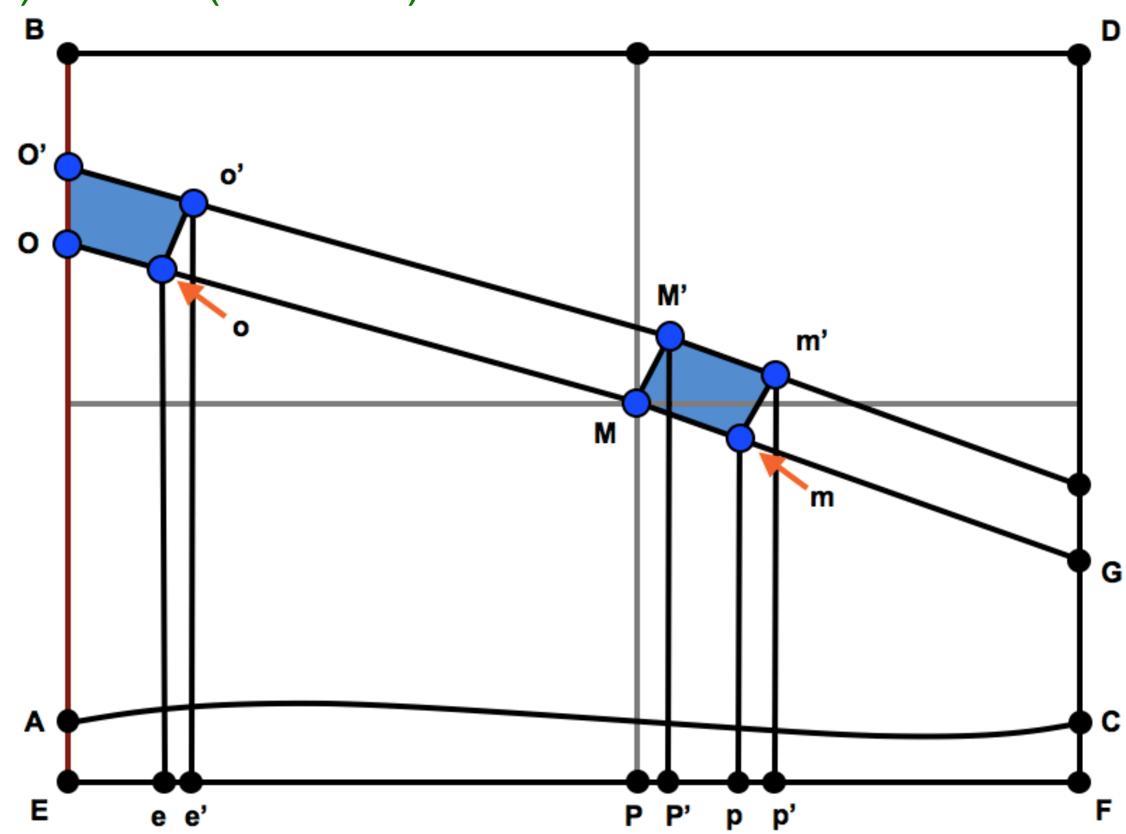


Euler's Geometry in E332

$$OO'o'o = \tfrac{1}{2}\, Ee'(EO'+o'e') - \tfrac{1}{2}\, Ee(EO+oe) - \tfrac{1}{2}\, ee'(eo+o'e')$$

$$= \tfrac{1}{2}\,(Ee + ee')(EO+OO'+o'e') - \tfrac{1}{2}\, Ee(EO+oe) - \tfrac{1}{2}\, ee'(eo+o'e')$$

$$= \tfrac{1}{2}\, Ee(OO' + o'e' - eo) - \tfrac{1}{2}\, ee'(EO' - oe)$$



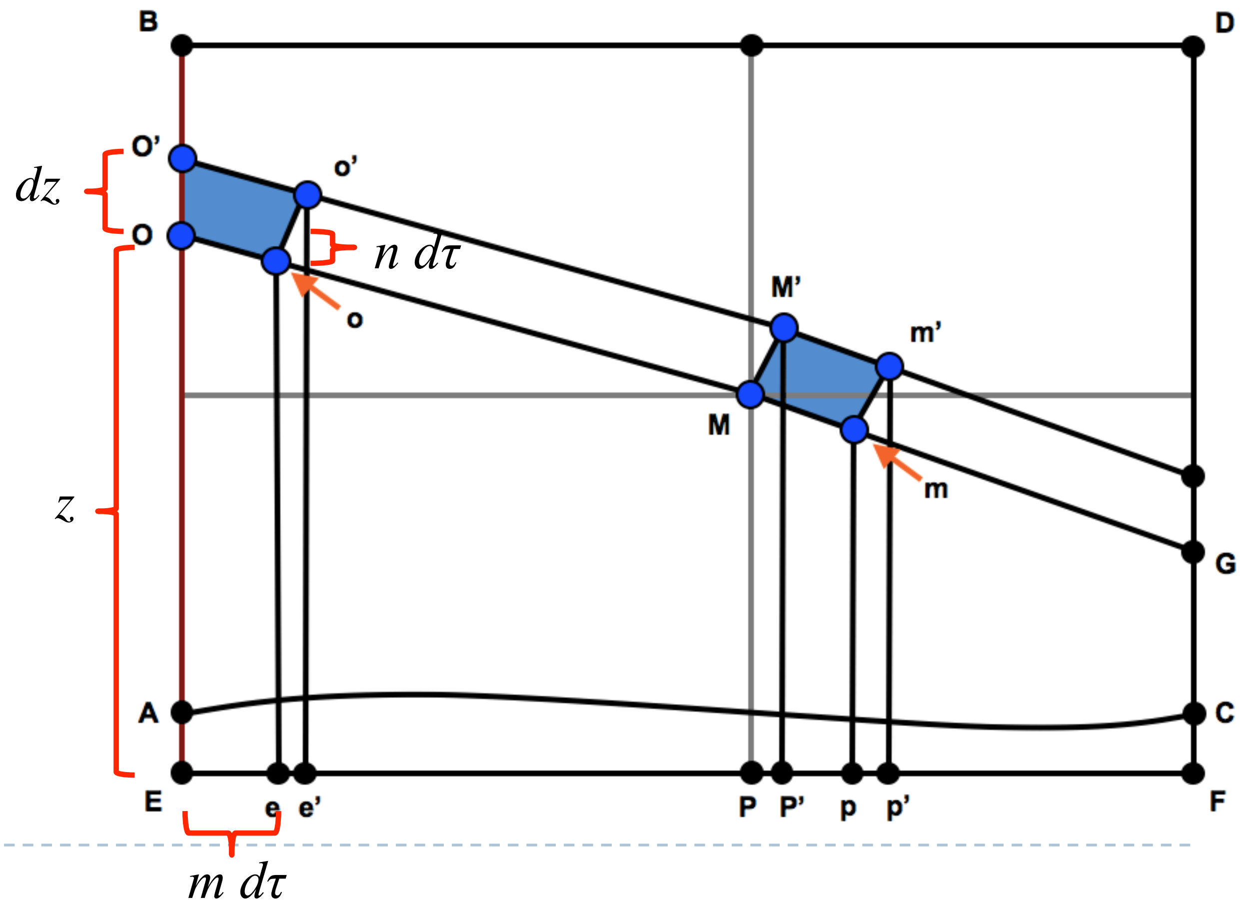
From Geometry to Calculus

OO’o’o = ½ Ee(OO’ + o’e’ – eo) – ½ ee’(EO’ – oe)

m, n = *forward & downward velocities at O*

dτ = time for O to flow to o.

Distance = Rate × Time



From Geometry to Calculus

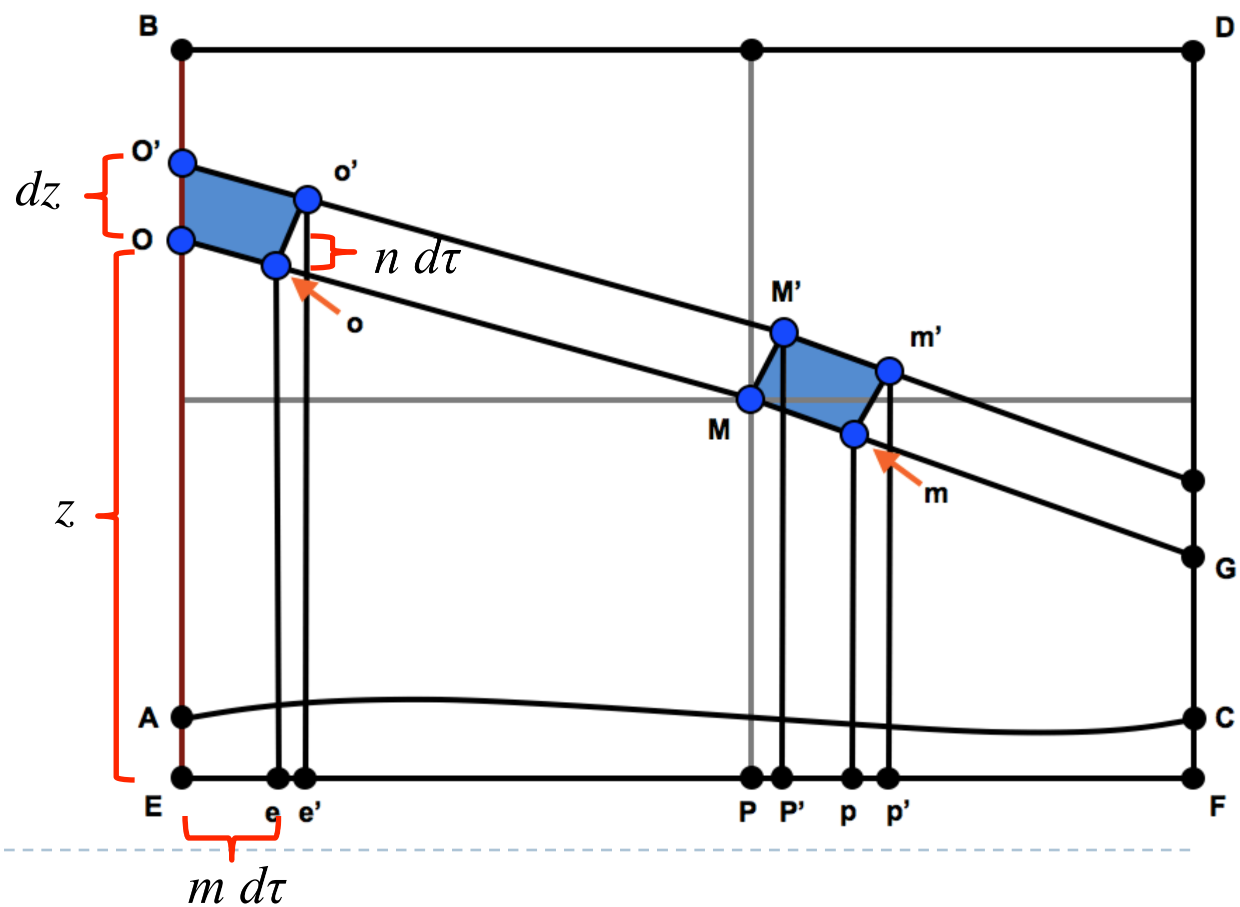
OO'o'o = ½ Ee(OO' + o'e' – eo) – ½ ee'(EO' – oe)

= ½ $m\ d\tau$ (dz + [z + dz – $n\ d\tau$] – [z – $n\ d\tau$]) – ½ ee'(EO' – oe)

m, n = forward & downward velocities at O

$d\tau$ = time for O to flow to o.

Distance = *Rate* × *Time*



From Geometry to Calculus

$$OO'o'o = \tfrac{1}{2}\,Ee(OO' + o'e' - eo) - \tfrac{1}{2}\,ee'(EO' - oe)$$

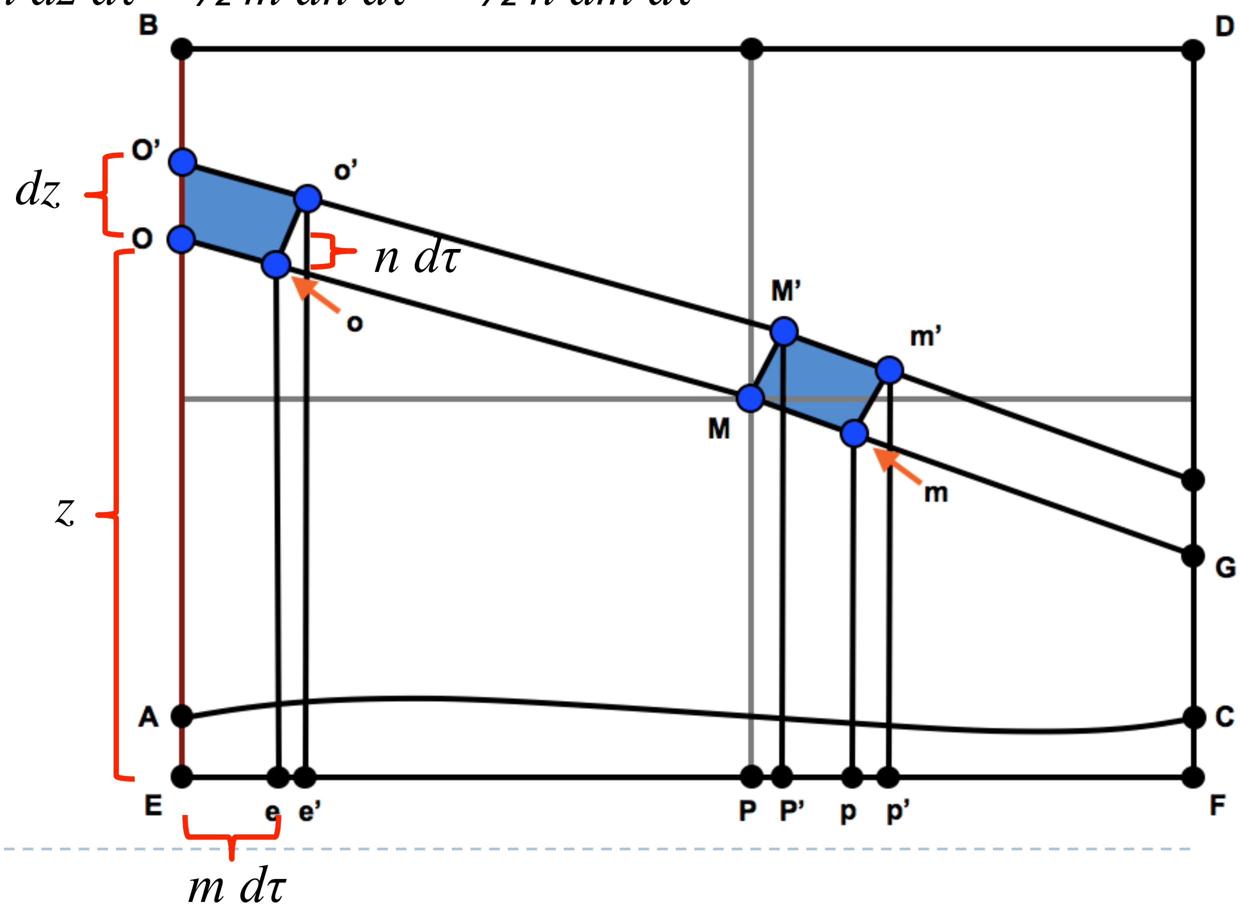
$$= \tfrac{1}{2}\, m\, d\tau\, (dz + [z + dz - n\, d\tau] - [z - n\, d\tau]) - \tfrac{1}{2}\, ee'(EO' - oe)$$

$$= m\, d\tau\, dz + \tfrac{1}{2}\, dm\, dz\, d\tau - \tfrac{1}{2}\, m\, dn\, d\tau^2 + \tfrac{1}{2}\, n\, dm\, d\tau^2$$

m, n = forward & downward velocities at O

$d\tau$ = time for O to flow to o.

Distance = Rate × Time



From Geometry to Calculus

$OO'o'o = \tfrac{1}{2}\, Ee(OO' + o'e' - eo) - \tfrac{1}{2}\, ee'(EO' - oe)$

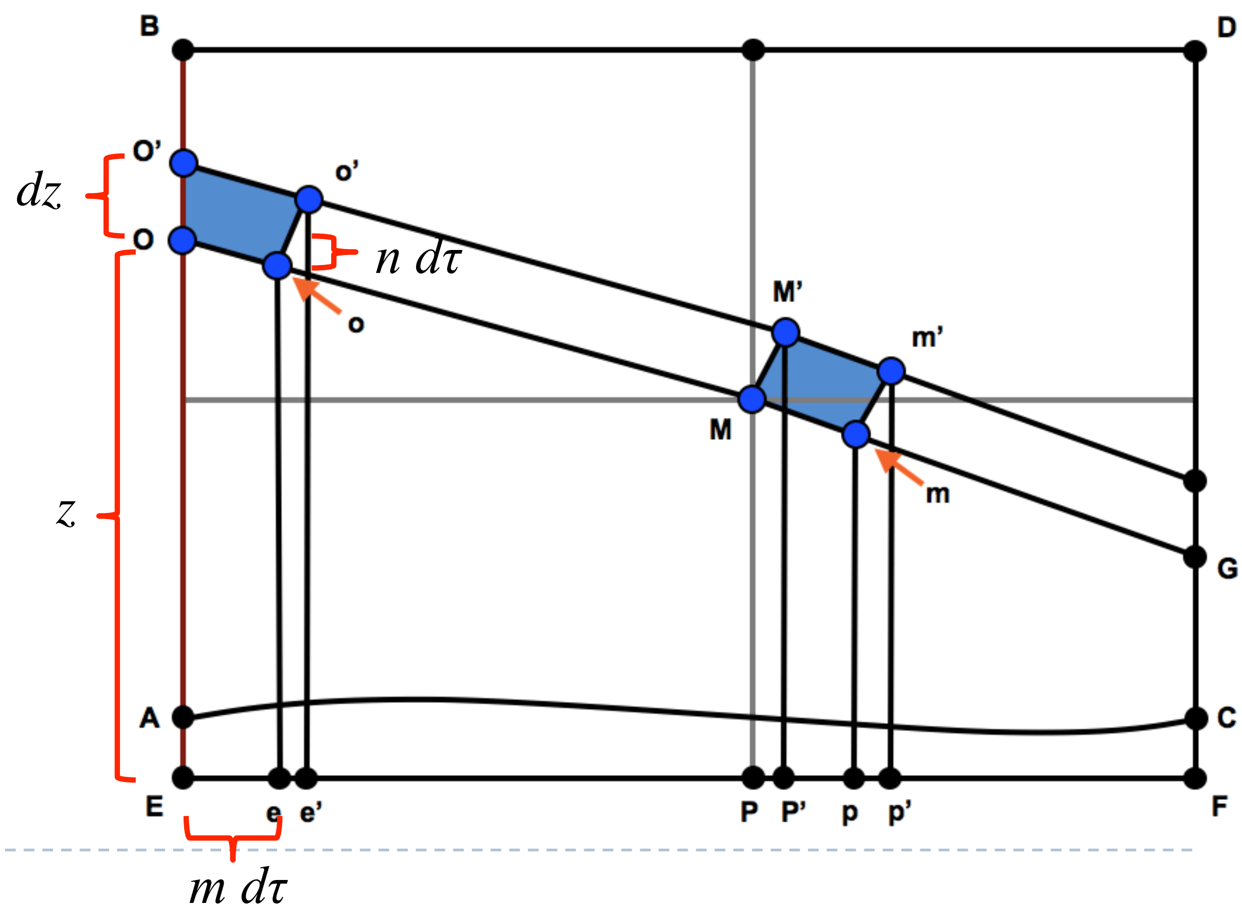
$\qquad = \tfrac{1}{2}\, m\, d\tau\, (dz + [z + dz - n\, d\tau] - [z - n\, d\tau]) - \tfrac{1}{2}\, ee'(EO' - oe)$

$\qquad \approx m\, dz\, d\tau$

m, n = forward & down-ward velocities at O

$d\tau$ = time for O to flow to o.

Distance = Rate × Time



From Geometry to Calculus

OO'o'o $\approx$ $m\, dz\, d\tau$

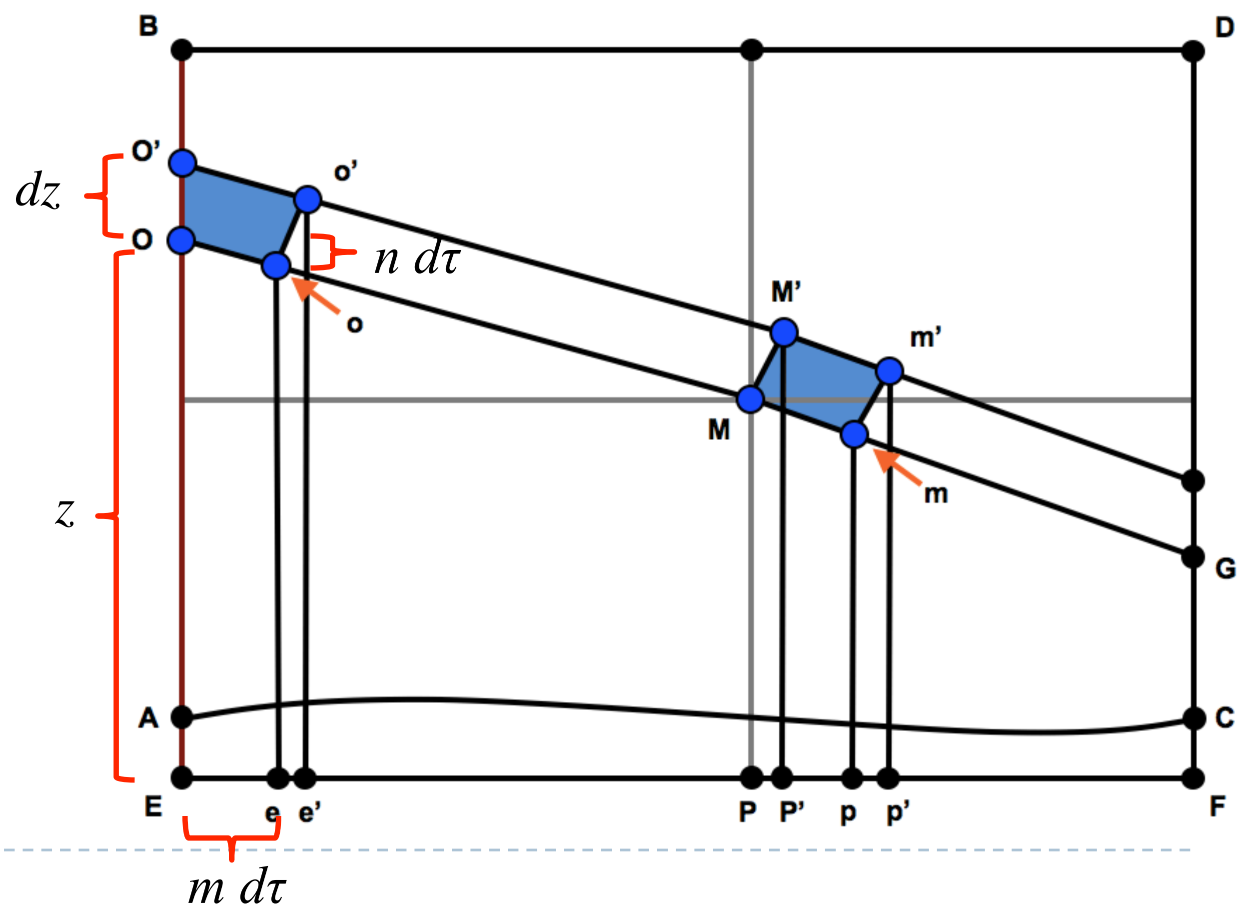
MM'm'm $\approx PS\, dz\, d\tau - QR\, dz\, d\tau = (PS - QR)\, dz\, d\tau$

m, n = *forward & downward velocities at O*

$d\tau$ = *time for O to flow to o.*

$Distance = Rate \times Time$

P, Q, R, S = partial derivatives of x & y with respect to t & z



From Geometry to Calculus

OO'o'o $\approx$ $m\,dz\,d\tau$

MM'm'm $\approx PS\,dz\,d\tau - QR\,dz\,d\tau = (PS - QR)\,dz\,d\tau$

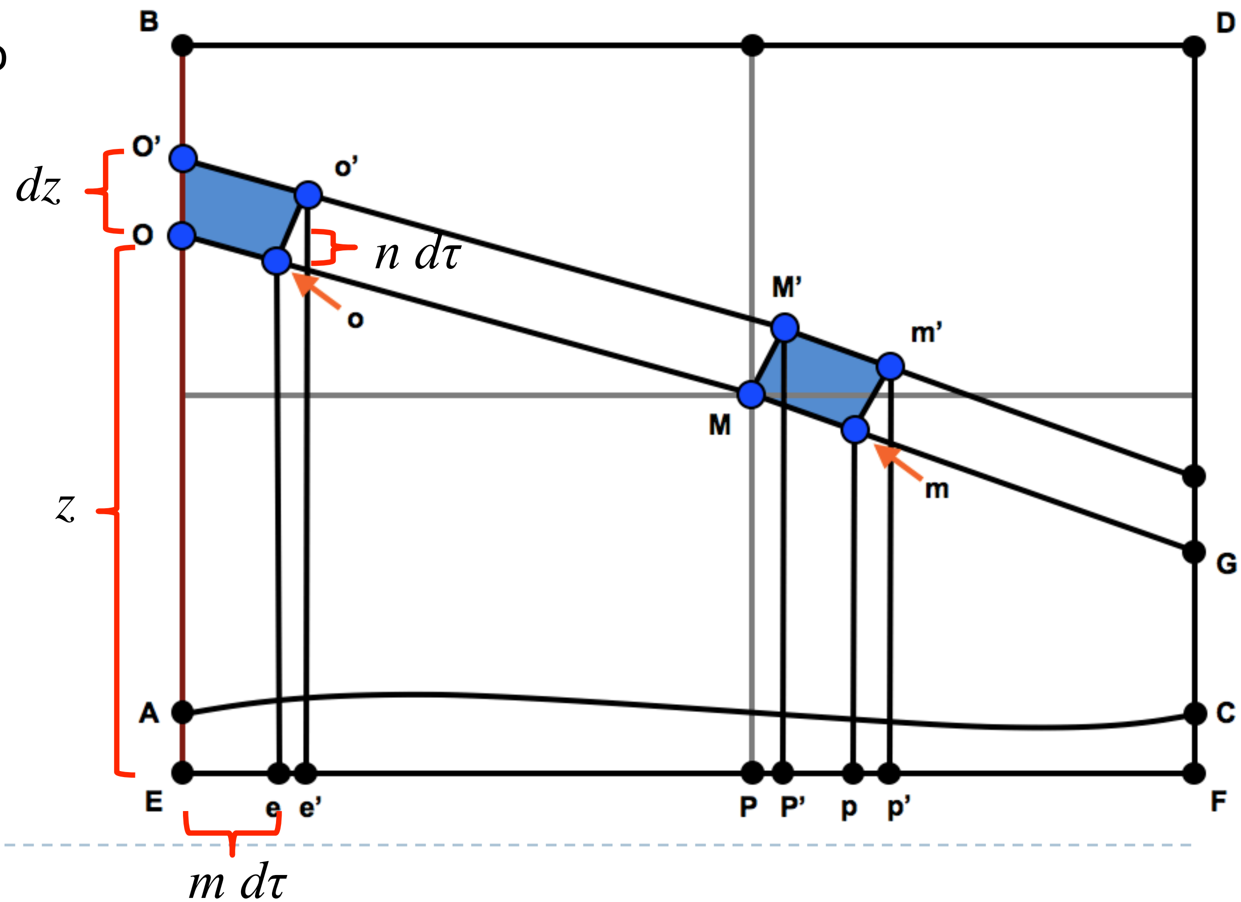
Since the area of OO'o'o equals the area of MM'm'm,

$$m = PS - QR$$

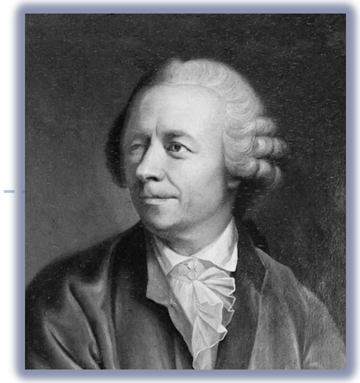
Or, in modern notation:

$$\left.\frac{\partial x}{\partial t}\right|_{t=0} = \frac{\partial(x,y)}{\partial(t,z)}$$

(Depends only on z.)



Euler's Fluid Mechanics Papers



E332
Written 1751
Published 1767

RECHERCHES
SUR
LE MOUVEMENT DES RIVIERES.

E258
Written 1752
Published 1761

PRINCIPIA
MOTVS FLVIDORVM.

E226
Written 1755
Published 1757

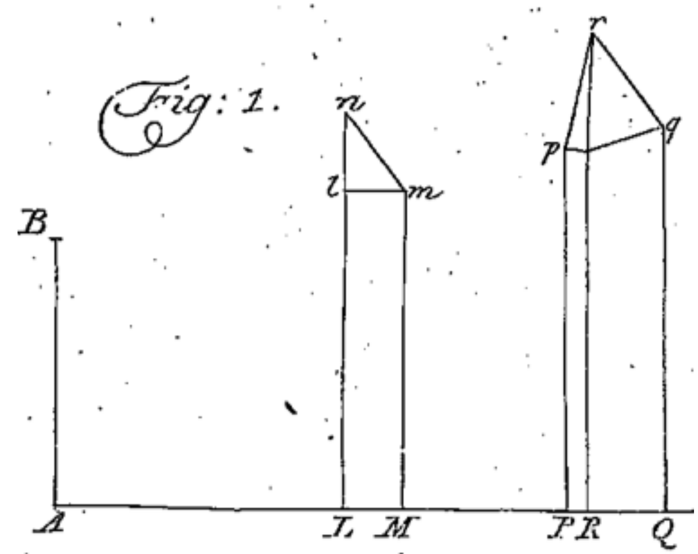
PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.

Euler's Geometry in E258

E258
Written 1752
Published 1761

PRINCIPIA
MOTVS FLVIDORVM.

13. Cum igitur ſit $du = L\,dx + l\,dy$ et $dv = M\,dx + m\,dy$, hinc geminas celeritates cuiusque alius puncti, quod quidem infinite parum a puncto l diſtat, aſſignare licebit; ſi enim talis puncti a puncto l diſtantia ſecundum axem AL ſit $= dx$, et ſecundum axem $AB = dy$, tum huius puncti celeritas ſecundum axem AL erit $= u + L\,dx + l\,dy$; celeritas autem ſecundum alterum axem $AB = v + M\,dx + m\,dy$. Tempuſculo ergo infinite paruo dt hoc punctum proferetur ſecundum directionem axis AL per ſpatiolum $= dt(u + L\,dx + l\,dy)$ et ſecundum directionem alterius axis AB per ſpatiolum $= dt(v + M\,dx + m\,dy)$.



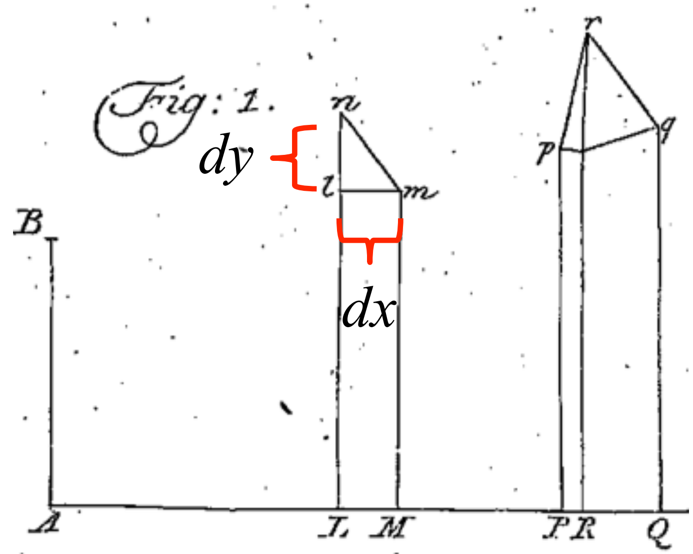
Euler's Geometry in E258

Fluid triangle ℓmn flows to fluid triangle pqr in infinitesimal time dt.

$$u = dx = \ell m$$

$$v = dy = \ell n$$

ſit $L + m = 0$. Seu cum ſit $L = \frac{du}{dx}$ et $m = \frac{dv}{dy}$, celeritates u et v, quae puncto l ſecundum directiones axium AL et AB ineſſe concipiuntur, eiusmodi functiones coordinatarum x et y eſſe debent, vt ſit $\frac{du}{dx} + \frac{dv}{dy} = 0$, ſicque motuum poſſibilium criterium in hoc conſiſtit, vt ſit $\frac{du}{dx} + \frac{dv}{dy} = 0$; niſi enim haec condi-



Euler's Geometry in E258

Fluid triangle ℓmn flows to fluid triangle *pqr* in infinitesimal time dt.

$$u = dx = \ell m$$

$$v = dy = \ell n$$

$$L = du/dx, \quad \ell = du/dy$$

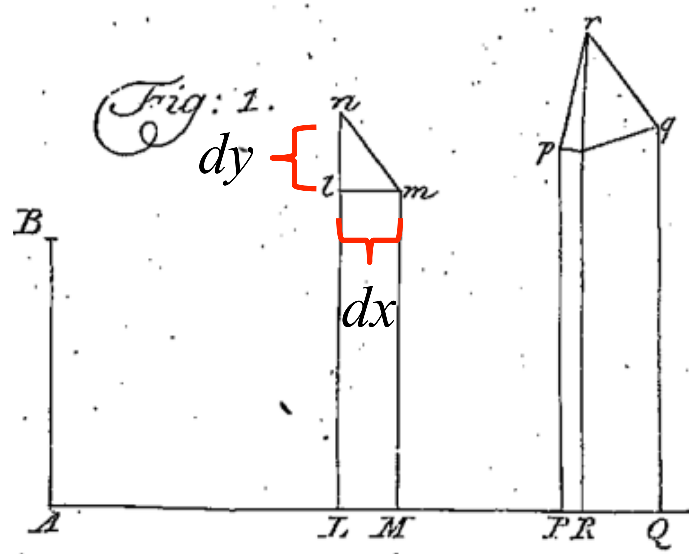
$$M = dv/dx, \quad m = dv/dy$$

$$du = L\,dx + \ell\,dy, \quad dv = M\,dx + m\,dy$$

Argument similar to E332 shows that

$$L + m = 0.$$

ſit $L + m = 0$. Seu cum ſit $L = \frac{du}{dx}$ et $m = \frac{dv}{dy}$, celeritates u et v, quae puncto l ſecundum directiones axium AL et AB ineſſe concipiuntur, eiusmodi functiones coordinatarum x et y eſſe debent, vt ſit $\frac{du}{dx} + \frac{dv}{dy} = 0$, ſicque motuum poſſibilium criterium in hoc conſiſtit, vt ſit $\boxed{\frac{du}{dx} + \frac{dv}{dy} = 0}$, niſi enim haec condi-



Euler's Geometry in E226

E226
Written 1755
Published 1757

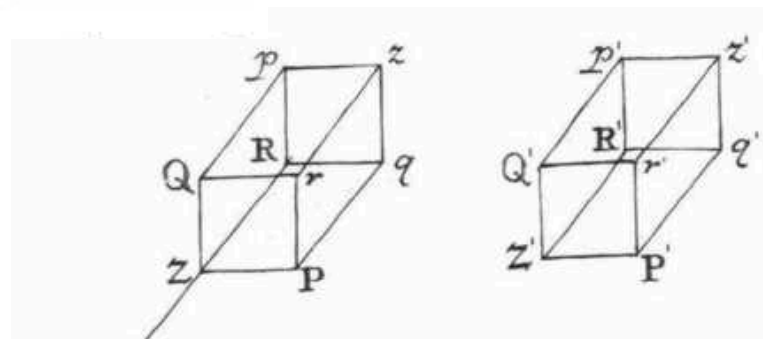
PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.

niment peu incliné à l'axe OC. Donc, ſi nous conſidérons un parallelepipede rectangle $ZPQRzpqr$ formé des trois côtés $ZP = dx$, $ZQ = dy$, & $ZR = dz$, le fluide qui occupoit cet eſpace ſera tranſporté pendant le tems dt à remplir l'eſpace $Z'P'Q'R'z'p'q'r'$, infiniment peu différent d'un parallelepipede rectangle, dont les trois côtés ſeront :

$$Z'P' = dx\left(1 + dt\left(\frac{du}{dx}\right)\right);$$

$$Z'Q' = dy\left(1 + dt\left(\frac{dv}{dy}\right)\right);$$

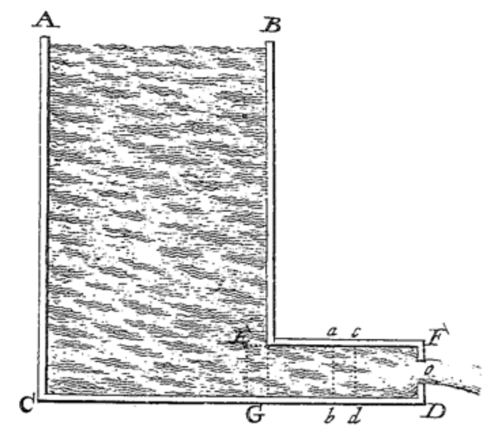
$$Z'R' = dz\left(1 + dt\left(\frac{dw}{dz}\right)\right)$$



Cubical fluid particles, 3 dimensions; techniques are more algebraic.

Development of Euler's Fluid Mechanics

1738-39 – D. Bernoulli & Euler use method of parallel sections. Bernoulli argues using actual descent & potential ascent; Euler uses "moving forces".



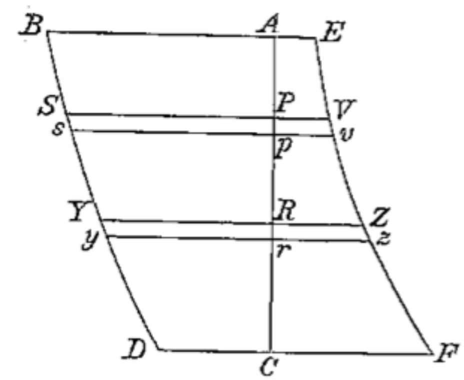
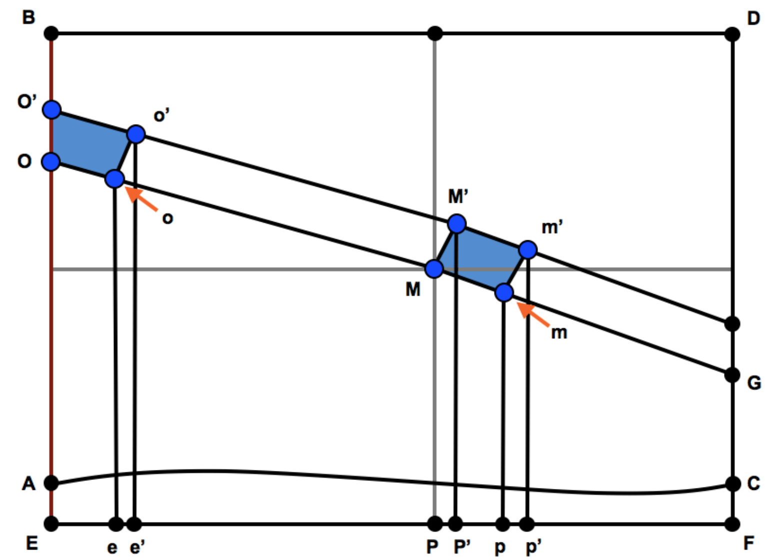


Fig. 1.

Development of Euler's Fluid Mechanics

1738-39 – D. Bernoulli & Euler use method of parallel sections. Bernoulli argues using actual descent & potential ascent; Euler uses "moving forces".

1751 – Euler uses quadrilateral fluid particle method, derives formulas with Euclidean geometry in 2D. Awkward notation involving initial conditions.

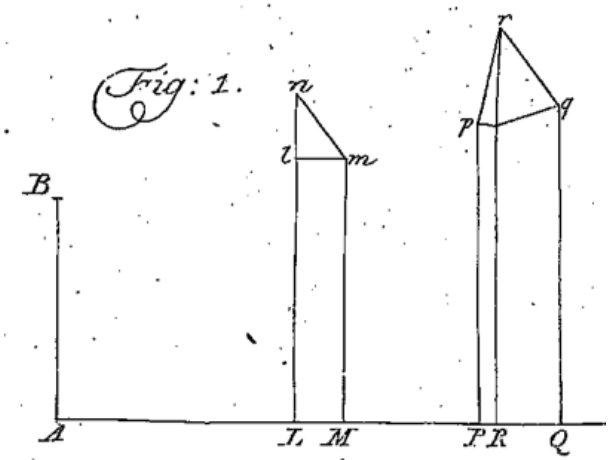


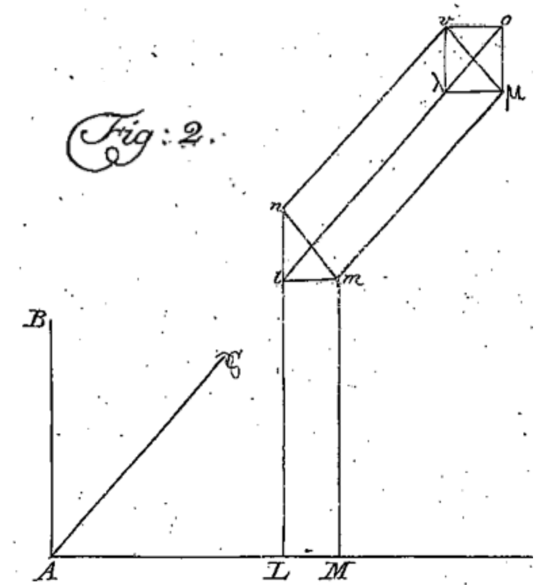
Development of Euler's Fluid Mechanics

1738-39 – D. Bernoulli & Euler use method of parallel sections. Bernoulli argues using actual descent & potential ascent; Euler uses "moving forces".

1751 – Euler uses quadrilateral fluid particle method, derives formulas with Euclidean geometry in 2D. Awkward notation involving initial conditions.

1752 – Euler uses triangular & tetrahedral fluid particles, and same Euclidean techniques. Begins to use consistent notation for partial derivatives.





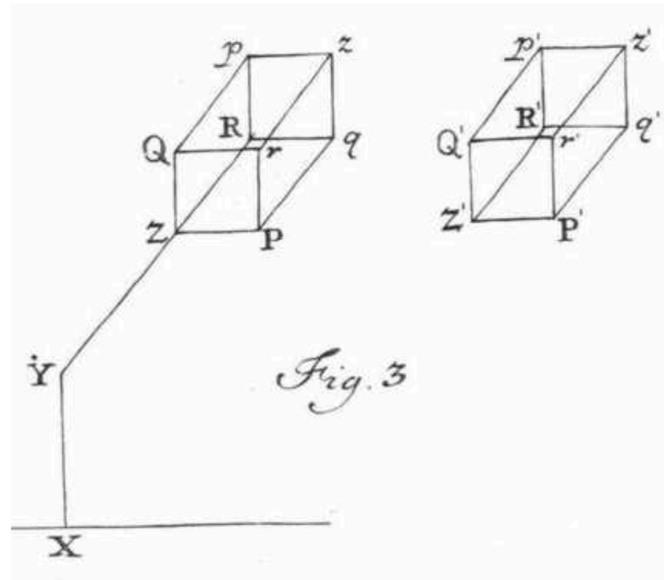
Development of Euler's Fluid Mechanics

1738-39 – D. Bernoulli & Euler use method of parallel sections. Bernoulli argues using actual descent & potential ascent; Euler uses "moving forces".

1751 – Euler uses quadrilateral fluid particle method, derives formulas with Euclidean geometry in 2D. Awkward notation involving initial conditions.

1752 – Euler uses triangular & tetrahedral fluid particles, and same Euclidean techniques. Begins to use consistent notation for partial derivatives.

1755 – Euler uses cubical fluid particles. Less geometric than previous two works.



Epilogue – Lagrange's *Mécanique Analytique* (1788)

Some Conclusions:

- Serious notational deficiencies; key concepts still in development
- Potential ascent & actual descent eventually replaced by modern notion of force.
- **Earlier:** geometry drives the algebraic argument
- **Later:** algebra and analysis dominate, geometry reduced to more of an illustration
- **Euler's work** in 1739-1755 followed this pattern, becoming ***more flexible*** and ***less dependent on geometry***.

Nous ſuppoſerons que le fluide ſoit homogène & peſant, & qu'il parte du repos, ou qu'il ſoit mis en mouvement par l'impulſion d'un piſton appliqué à ſa ſurface; ainſi les vîteſſes p, q, r, de chaque particule, devront être telles que la quantité $p\,dx + q\,dy + r\,dz$ ſoit intégrable (art. 18); par conſéquent on pourra employer les formules de l'article 20.

Soit donc φ une fonction de x, y, z & t, déterminée par l'équation

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0,$$

on aura d'abord pour les vîteſſes de chaque particule, ſuivant les directions des coordonnées x, y, z, ces expreſſions,

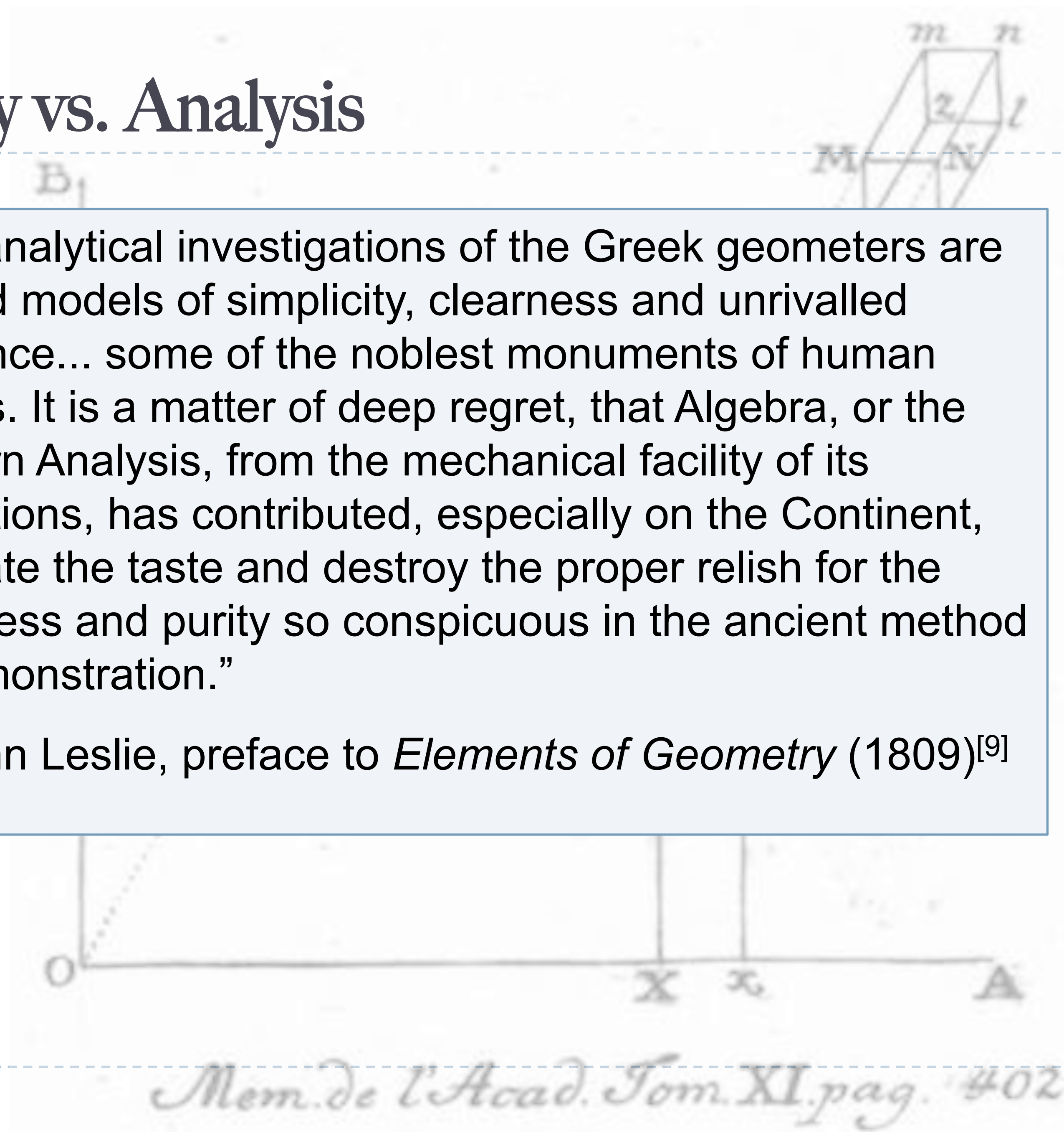
$$p = \frac{d\varphi}{dx},\ q = \frac{d\varphi}{dy},\ r = \frac{d\varphi}{dz}.$$

Enſuite on aura

$$\lambda = V + \frac{d\varphi}{dt} + \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dy}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dz}\right)^2,$$

quantité qui devra être nulle à la ſurface extérieure libre du fluide (art. 2).

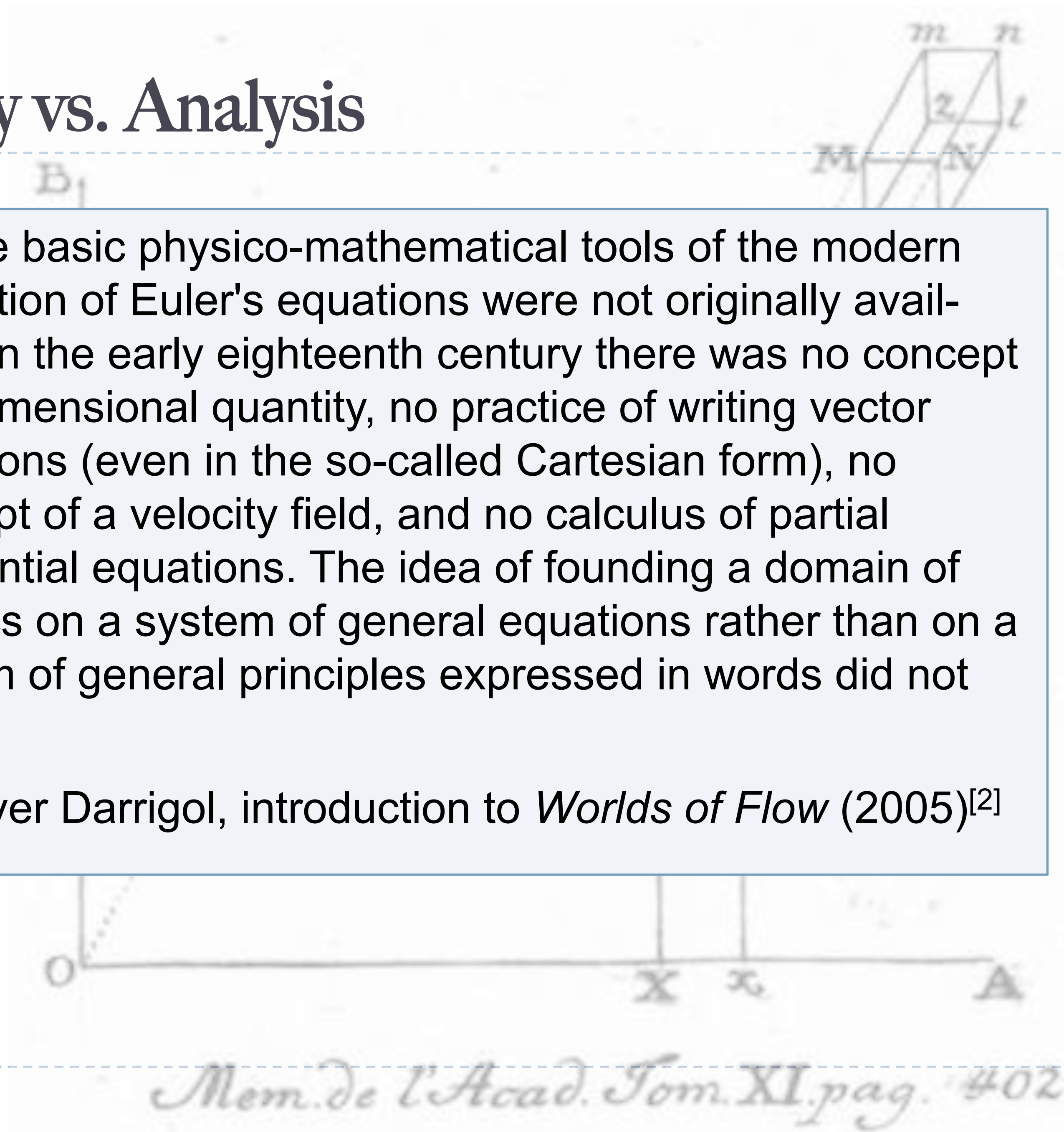
Geometry vs. Analysis



> "The analytical investigations of the Greek geometers are indeed models of simplicity, clearness and unrivalled elegance... some of the noblest monuments of human genius. It is a matter of deep regret, that Algebra, or the Modern Analysis, from the mechanical facility of its operations, has contributed, especially on the Continent, to vitiate the taste and destroy the proper relish for the strictness and purity so conspicuous in the ancient method of demonstration."
>
> — John Leslie, preface to *Elements of Geometry* (1809)[9]

Geometry vs. Analysis

> "… the basic physico-mathematical tools of the modern derivation of Euler's equations were not originally available. In the early eighteenth century there was no concept of a dimensional quantity, no practice of writing vector equations (even in the so-called Cartesian form), no concept of a velocity field, and no calculus of partial differential equations. The idea of founding a domain of physics on a system of general equations rather than on a system of general principles expressed in words did not exist."
>
> — Oliver Darrigol, introduction to *Worlds of Flow* (2005)[2]



References

[1] Bernoulli, Daniel. *Hydrodynamica*. Basel: Dulsecker, 1738. (English Translation by T. Carmody and H. Kobus, Dover, 1968.)

[2] Darrigol, Oliver. *Worlds of Flow: A History of Hydrodynamics From the Bernoullis to Prandtl*. New York: Oxford University Press, 2005.

[3] Euler, Leonhard. "Recherches sur le mouvement des rivières". *Mémoires de l'académie des sciences de Berlin* **16** (1767), pp. 101-118. Numbered E332 in Eneström's index. Available at eulerarchive.maa.org.

[4] ——. "Principia motus fluidorum". *Novi commentarii academiae scientiarum Petropolitanae*, **6** (1756/7) 1761, pp. 24-26, 271-311. Numbered E258 in Eneström's index. Available at http://eulerarchive.maa.org.

[5] ——. "Principes generaux du mouvement des fluides". *Mémoires de l'académie des sciences de Berlin* **11** (1755), 1757, pp. 274-315. Numbered E226 in Eneström's index. Available at http://eulerarchive.maa.org.

[6] ——. Letter to Johann Bernoulli, 16 May 1739. Published in *Bibl. Math* **63** (1905), pp. 24-33. Numbered E863 in Eneström's index. Available at http://eulerarchive.maa.org.

[7] Lagrange, Joseph Louis. *Traité de méchanique analytique*. Paris: Desaint, 1788.

[8] Mikhailov, Gleb. *Introduction to Die Werke von Daniel Bernoulli*, Vol. 5. Basel: Birkhäuser, 2002.

[9] Timmons, Todd. *Mathematics in Nineteenth Century America: The Bowditch Generation*. Chestnut Hill, MA: Docent Press, 2013.

[10] Truesdell, Clifford. "Rational Fluid Mechanics, 1657–1765." In *Euleri Opera Omnia*, Ser. 2, Vol. 12. Lausanne: Orell Füssli Turici, 1964. pp. IX–CXXV.