Juggling With Numbers

Erik R. Tou

School of Interdisciplinary Arts & Sciences
University of Washington, Tacoma
Juggling Is Old!

Oldest known depictions appear in an Egyptian temple at Beni Hasan (c. 1994-1781 BCE).
Everybody Juggles!

Mughal Emperor Babur wrote of jugglers and acrobats in India (c. 1528):

... the Hindustani bazigers [jugglers] were brought in and performed their tricks, and the lulis [tumblers] and rope-dancers exhibited their feats... One of these is the following—they take seven rings, one of which they suspend over their forehead, and two on their thighs; the other four they place, two on two of their fingers, and the other two on two of their toes, and then whirl them all around with a quick uninterrupted motion.

Georg Forster wrote of jugglers in Tonga (1773):

A girl of 10-12 years... had five ball shaped fruits, which she threw permanently high and caught by an admirable skill and quickness.
A Numerical Description for Juggling Patterns

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- Only in the 1980s did jugglers develop a way to keep track of different juggling patterns mathematically.
- Idea: use a numerical code to describe the throws.
- Measure height of throw according to number of “beats” until it comes back down (usually, “beats” = “thuds”)

![Diagram of juggling pattern]
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[Diagram showing juggling pattern representation]
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\[
\text{\ldots} \quad 3333 \quad \text{\ldots}
\]
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- This is the *siteswap* for the juggling pattern.
Introduction to Siteswap Notation

 Siteswap Notation 

$$\text{(441)}$$

$$\text{(531)}$$
Properly Defining A Siteswap

Some remarks:

- The beats always alternate between left and right hands.
- The *length* (or, *period*) of a siteswap is the number of beats that occur before it repeats.
- We are only interested in *monoplex* juggling: at most one ball caught/thrown at once.
- Balls which land simultaneously are *collisions*, and are not allowed.
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*More Precisely*: If two balls are thrown at times $i$ and $j$, and remain in the air for $t_i$ beats and $t_j$ beats, respectively, it cannot be the case that $t_i + i = t_j + j$ (since this would create a collision).
Definition

A siteswap is a finite sequence of nonnegative integers. A valid siteswap is one with no collisions, i.e., the quantities $t_i + i \pmod{n}$ are distinct for $1 \leq i \leq n$. 

Question: Given a valid siteswap, how do you know the number of balls required to juggle it?

Lemma: Let $s$ be a valid siteswap with successive throws $a$ and $b$, so that $s = (\ldots, a, b, \ldots)$. Then the siteswap $s' = (\ldots, b+1, a-1, \ldots)$ is also valid and requires the same number of balls as $s$. 
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The “Height Swap”

Example: \((\ldots, 6, 2, \ldots) \longrightarrow (\ldots, 3, 5, \ldots)\) switches the landing times of successive throws.

So, no new collisions can be introduced from a height swap (and no old collisions can be removed).
A Game: Number Morphs

Rules for number morphs:

1. Choose five numbers, each between 1 and 6. Be sure to include at least one 1 and at least one 6.
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2. Do either of the following “height swaps,” treating the last digit as being adjacent to the first digit.
   - Find the largest number, and if the number to its right is at least two less, do a height swap.
   - Find the smallest number, and if the number to its left is at least two more, do a height swap.

3. Continue until no further moves are possible. Record the difference between the largest and smallest numbers—this is your score.

Try it!

Play number morphs for 5, 1, 6, 3, 5.
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My version of the game:

- Starting siteswap: (51635)
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Final score: 4

\[ 4 - 4 = 0 \] — win!

Theorem

A sequence \( s \) of nonnegative integers is a valid siteswap if, and only if, the number morphs game results in a constant sequence \((b, b, \ldots, b)\).
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- Every height swap reduces the spread by either 1 or 2.
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- No height swap can introduce or remove a collision.
- The sequence $s'$ obtained from any height swap will be a valid siteswap if, and only if, $s$ is a valid siteswap.
- Every height swap reduces the spread by either 1 or 2.
- Unless you get stuck, the spread will eventually equal 0.
An Average Theorem

- If you get stuck with spread $\geq 1$, then there exist adjacent throws $t + 1$ and $t$; these throws will land at the same time (a collision).
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Theorem

*The number of balls required to juggle a valid siteswap $s$ is equal to the average of the nonnegative integers appearing in $s$.***
The Reverse Question

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**Better Question:** Given $b$ balls and some $n \geq 1$, how many valid siteswaps are there of length $n$?
The Reverse Question

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Answer: Infinitely many — the length \( n \) can be as large as you want!

Better Question: Given \( b \) balls and some \( n \geq 1 \), how many valid siteswaps are there of length \( n \)?

Examples:

- For \( b = 2 \) and \( n = 2 \), there are five: \((22), (40), (04), (31), (13)\).
- For \( b = 3 \) and \( n = 3 \), there are 37:

\[(900), (090), (009), (630), (603), (063), (360), (036), (306), (333), (711), (171), (117), (441), (414), (144), (522), (252), (225), (720), (180), (126), (450), (423), (153), (027), (018), (612), (045), (342), (351), (702), (801), (261), (504), (234), (135)\]
Juggling Cards

Take $b = 4$, and consider this set of five “juggling cards.”

You can build any 4-ball juggling diagram from these cards.
Example: What siteswap corresponds to this sequence of cards?
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\[ C_3 \quad C_2 \quad C_4 \quad C_1 \quad C_2 \]

\textit{Answer:} (53192).
Playing Around With Juggling Cards

\[ C_1, C_1, C_1 \]
\[ C_3, C_3, C_3, C_3 \]
\[ C_4, C_0, C_4, C_0 \]

Answers: (1), (3), (80)
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Counting Siteswaps

Theorem

Given an integer \( n \geq 1 \), there exist \((b + 1)^n\) valid siteswaps with \( \leq b \) balls and length \( n \), counting repetitions and cyclic permutations separately.

Idea of Proof: For any \( b \), there are \( b + 1 \) juggling cards. Each siteswap can be represented by setting \( n \) cards in a row (with repetitions possible). The total number of siteswaps will then be \((b + 1)^n\).

Corollary

Given an integer \( n \geq 1 \), there exist \((b + 1)^n - b^n\) valid siteswaps with \( b \) balls and length \( n \), counting repetitions and cyclic permutations separately.

Examples: For \( b = 2 \) and \( n = 2 \), there are \(3^2 - 2^2 = 5\) valid siteswaps. For \( b = 3 \) and \( n = 3 \), there are \(4^3 - 3^3 = 37\) valid siteswaps.
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How to Multiply Juggling Patterns?

*Juggling cards:* it’s easy to “concatenate” old patterns to get new ones.

However, this isn’t compatible with siteswaps: $(531) \otimes (51) = (46131)$
How to Multiply Juggling Patterns?

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\[ (531) \otimes (51) = (46131) \]

However, this isn’t compatible with siteswaps: \((531)(51) = (53151)\), but \((53151)\) is not a valid siteswap.
Solution: Restrict to sets of “compatible” patterns.

Definition
A juggling pattern is a *ground state* pattern if there is a moment when the juggler can stop juggling, after which $b$ “thuds” are heard as the balls hit the ground on each of the next $b$ beats.
Ground State Siteswaps

Facts about ground state siteswaps:

- They are all compatible with the “standard” siteswap \((b)\).
- Ground state patterns for \(b = 3\): \((3), (42), (423), (441), (531), (522), (6231), \text{etc.}\)
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- Ground state patterns for \(b = 3\): \((3), (42), (423), (441), (531), (522), (6231)\), etc.
- Any two ground state siteswaps (with same \(b\)) can be “multiplied” via concatenation: \((441)(6231) = (4416231)\).
- Multiplication isn’t always commutative: \((3)(42) = (42)(3)\), but \((3)(42)(522) \neq (42)(3)(522)\).
- Most ground state siteswaps can be “factored” into shorter ones: \((53403426231) = (5340)(3)(42)(6231)\).
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- Most ground state siteswaps can be “factored” into shorter ones: \((53403426231) = (5340)(3)(42)(6231)\).
- If a siteswap can’t be factored, it is “primitive.”
- The “identity” siteswap is \((\)\).
Counting Ground State Siteswaps

**Question:** Given $b$, how many ground state siteswaps are there with length $n \geq 0$?

**Theorem (Chung & Graham, 2008)**

Given $b, n \geq 0$, the number of ground state juggling patterns with $b$ balls and length $n$ is given by

$$J_b(n) = \begin{cases} 
  n! & \text{if } n \leq b \\
  b! \cdot (b + 1)^{n-b} & \text{if } n > b.
\end{cases}$$

Examples:

- $b = 3, n = 0$: 0! = 1 — ()
- $b = 3, n = 3$: 3! = 6 — (333), (342), (423), (441), (531), (522).
- $b = 4, n = 7$: $4! \cdot 5 = 120 \cdot 5 = 600$ siteswaps!
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**Examples**:

- $b = 3, n = 0$: $0! = 1$ — ()
- $b = 3, n = 3$: $3! = 6$ — (333), (342), (423), (441), (531), (522).
- $b = 4, n = 7$: $4! \cdot 5^3 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5 \cdot 5 = 3000$ siteswaps!
Counting Primitive Ground State Siteswaps

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This is a much harder question to answer!
Counting Primitive Ground State Siteswaps

**Question:** Given $b$, how many *primitive* ground state siteswaps are there with length $n \geq 0$?

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**Theorem (τ, 2019)**

Given $b \geq 4$, the number of primitive, ground state juggling patterns with $b$ balls and length $n$ is approximated by

$$P_b(n) \sim \frac{b + 1 - \rho}{|s'_b(1/\rho)|} \cdot \rho^n,$$

where $s_b(z)$ is a $b$-degree polynomial and $\rho$ is a constant satisfying

$$0.73 \cdot \frac{1}{e^b \sqrt{b}} < 1 - \frac{\rho}{b+1} < 6.04 \cdot \frac{\sqrt{b}}{e^b}.$$
An Analogy Instead of a Proof

The Classic Question: Given a positive integer $n$, what proportion of the numbers from 1 to $n$ are prime?

The Answer (1896): The proportion is approximately $1\log n$, i.e., the primes are "sparse" in the integers since $\lim_{n \to \infty} 1/\log n = 0$.

Our Question: Given $b$, what proportion of ground state siteswaps of length $n$ are primitive?

The Answer (2019): The proportion is approximately $C b \cdot (\rho^b + 1)n$, i.e., the primitive siteswaps are sparse since $\rho^b + 1 < 1 - 0.73 e^b \sqrt{b} < 0.994$.

$$\lim_{n \to \infty} C b \cdot (\rho^b + 1)n < \lim_{n \to \infty} C b \cdot (0.994)n = 0.$$
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$$e^{b \sqrt{b}} < 0.994.$$ 

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The Answer (2019): The proportion is approximately \( C_b \cdot \left( \frac{\rho}{b+1} \right)^n \), i.e., the primitive siteswaps are sparse since \( \frac{\rho}{b+1} < 1 - \frac{0.73}{e^b \sqrt{b}} < 0.994 \):

\[
\lim_{n \to \infty} C_b \cdot \left( \frac{\rho}{b+1} \right)^n < \lim_{n \to \infty} C_b \cdot (0.994)^n = 0.
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Some Open Questions

- What other arithmetic properties does the set of ground state juggling sequences have?
- What happens when you allow for a ball to be added or dropped (i.e., what if $b$ can change)?
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- What happens when you allow for a ball to be added or dropped (i.e., what if $b$ can change)?
- What irrational numbers are the most “jugglable”?

Slides online: https://tinyurl.com/JugglingWithNumbers
Also see: *The Mathematics of Juggling* by Burkard Polster (Springer, 2003).