

Promoting Conceptual Thinking in Four Upper-Elementary Mathematics Classrooms

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Abstract

Informed by theory and research in inquiry-based mathematics, this study examined how classroom practices create a press for conceptual learning. Using videotapes of a lesson on the addition of fractions in 4 primarily low-income classrooms from 3 schools, we analyzed conversations that create a high or lower press for conceptual thinking. We use examples of interactions from these fourth- and fifth-grade lessons to propose that a high press for conceptual thinking is characterized by the following sociomathematical norms: (a) an explanation consists of a mathematical argument, not simply a procedural description; (b) mathematical thinking involves understanding relations among multiple strategies; (c) errors provide opportunities to reconceptualize a problem, explore contradictions in solutions, and pursue alternative strategies; and (d) collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

For over a decade, the mathematics education community has encouraged teachers to shift their classroom practices away from an exclusive focus on computational accuracy and toward a focus on deeper understandings of mathematical ideas, relations, and concepts (Hiebert & Carpenter, 1992; Lampert, 1991, National Council of Teachers of Mathematics, 1989, 2000). Implementing recommendations from research in the classroom is a complex task for teachers. The NCTM standards (1989, 2000) ask teachers to create conditions of learning for their students and to engage in practices that most teachers did not experience themselves as students and were not initially trained to do as teachers (Cohen & Ball, 1990; Fullan, 1991).

Studies of teacher learning have dem-

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onstrated that teachers are often able to implement some aspects of reform-minded mathematics instruction. They give multi-level problems that are connected to real-world experiences, provide manipulatives for students to use, and offer opportunities for children to work collaboratively in pairs and small groups. They ask students to present their strategies and solutions to the class, and they try to make mathematics activities interesting. To go beyond superficial implementation of the NCTM standards, however, it is important to stimulate students' conceptual understanding of mathematics (e.g., Ball, 1993; Cobb, Wood, & Yackel, 1993; Cobb, Wood, Yackel, & McNeal, 1993; Cohen, 1990; Fennema, Carpenter, Franke, & Carey, 1993; Prawat, 1992). The more superficial changes mentioned above are necessary, but they are not sufficient for helping students build sophisticated understandings of mathematics. Many teachers find it easy to pose questions and ask students to describe their strategies; it is more challenging pedagogically to engage students in genuine mathematical inquiry and push them to go beyond what might come easily for them (Ball & Bass, in press; Chazan & Ball, 1995; Fennema et al., 1996; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998; Heaton, 1993).

In this study we examined ways in which classroom practices press students for conceptual mathematical thinking (Blumenfeld, Puro, & Mergendoller, 1992). The study was guided by a sociocultural theory of learning, which asserts that one can understand individual learning by studying how the social environment is organized and how individuals participate in social practices (Forman, Minick, & Stone, 1993; Moll, 1990; Rogoff, 1997; Vygotsky, 1978). The theoretical framework contrasts with approaches that focus on specific teaching techniques. A sociocultural perspective can help researchers examine whether teaching practices engage learners in purposeful and in-depth inquiry (e.g., Rogoff, 1994). The primary goal of this study was to describe

vividly how teachers can promote student participation in a classroom community where conceptual understandings are valued and developed.

Drawing from the work of Cobb and Yackel and their colleagues, we differentiate between social norms and sociomathematical norms. Social norms refer to the general ways that students participate in classroom activities. Sociomathematical norms are specific to students' mathematical activities (Cobb, Wood, & Yackel, 1993; Cobb & Yackel, 1996; Yackel & Cobb, 1996). For example, whereas explaining one's thinking is a social norm, what counts as a mathematical explanation is a sociomathematical norm (Yackel & Cobb, 1996). Similarly, although the more superficial practice of discussing different strategies is a social norm, comparing the mathematical concepts underlying different strategies is a sociomathematical norm. Finally, working on tasks in small groups is a social norm; requiring students to achieve consensus using mathematical arguments is a sociomathematical norm.

The distinction between social and sociomathematical norms is useful for studying how classroom practices move beyond superficial features of reform. We sought to identify the sociomathematical norms in classrooms that promote students' engagement in conceptual mathematical thinking and conversation. We draw examples from four classrooms to generate hypotheses about the ways in which subtle differences in teaching practices can affect students' opportunities to engage in conceptual thinking.

In many respects, the lessons we observed were similar. The social norms of describing and sharing strategies, accepting errors as a normal part of learning, and working collaboratively with classmates existed in all four classrooms. A closer look at classroom talk, however, revealed subtle but important differences in the quality of students' engagement with mathematics. We use examples of classroom exchanges to suggest how sociomathematical norms governed classroom discussions.

Method

Data Source

This study involved four teachers in grades 4 and 5, all teaching the same lesson on the addition of fractions. The teachers were selected from a larger project of reform-minded mathematics instruction, the Integrated Mathematics Assessment (IMA) project (see Gearhart et al., 1999; Saxe, Gearhart, & Seltzer, 1999; Stipek, Givvin, Salmon, & MacGyvers, 1998; and Stipek, Salmon, et al., 1998). The project was designed to study the role of curriculum and professional support in enhancing students' opportunities to learn mathematics with understanding. The larger project included 23 upper-elementary teachers from schools in a large, ethnically diverse, urban area in California. All of the teachers taught in schools serving predominantly low-income children.

Selection of Classroom Interactions

All four teachers selected for this study had experience implementing reform-oriented curricula and a variety of practices consistent with the social norms of mathematics reform. Ms. Carter and Ms. Martin both taught in the same school. Both had master's degrees and 22 years of teaching experience. Ms. Andrew and Ms. Reed taught in two other schools. Ms. Andrew had a bachelor's degree and 2 years of teaching experience. Ms. Reed had a master's degree and 17 years of teaching experience. (All names are pseudonyms.)

Videotaped lessons were originally coded using a scheme guided by research on effective motivational strategies for engaging students in instruction, and the classroom cases for this study were selected based on scores from the motivation coding scheme (see Stipek, Salmon, et al., 1998, for a full description of coding schemes and data set). The lessons had been coded reliably on nine motivation dimensions by two raters. For each dimension, lessons were rated on a scale from 1 ("not at all like this teacher") to 5 ("very much like this teacher")

on the basis of a description of the practices associated with each dimension. From the results of a factor analysis, the dimensions were collapsed into composite variables, which yielded scores from 1 to 5.

Two composite variables were used to select classroom cases for this study. The first was labeled "press for learning." Scores on this composite variable reflected the degree to which teachers engaged students in mathematical thinking, specifically, how much they (*a*) emphasized student effort (e.g., encouraging students to work through difficult problems and find multiple solutions); (*b*) focused on learning and understanding (e.g., emphasizing the development of better understandings, asking students to explain their strategies); (*c*) supported students' autonomy (e.g., encouraging student self-evaluation, giving students choices, encouraging personal responsibility); and (*d*) deemphasized performance (e.g., getting answers right). Quantitative findings, reported elsewhere, showed a significant positive correlation between the degree of press in the observed lessons and growth in students' conceptual understanding of fractions ($r = .51, p < .05$) (see Stipek, Salmon, et al., 1998). The positive affect scale was not significantly related to change in conceptual understanding.

The press for learning scale came closest to reflecting the substantive focus on mathematical understanding that we wanted to investigate further. Thus, we selected the two lessons that scored the highest in press (Ms. Carter, 4.4; Ms. Martin, 4.63), and two lessons that were lower in press (Ms. Andrew and Ms. Reed, both equaling 3.31). We purposefully did not select cases that were coded lowest in press because we were interested in capturing subtle differences among lessons that appeared to adhere to some of the superficial features of inquiry-oriented mathematics.

The second motivation composite we used to select cases was labeled "positive affect" because it conveyed the degree to which the classroom appeared as a positive

social environment. It was composed of the following dimensions, which reflected to what extent teachers (*a*) displayed a positive demeanor toward students (e.g., sensitivity, respect, interest); (*b*) displayed enthusiasm and interest in mathematics; and (*c*) fostered a nonthreatening environment (e.g., conveying that mistakes are okay, providing scaffolding for children having difficulty, not tolerating students putting one another down). The component of fostering a supportive environment overlaps with the social norm of allowing mistakes to be a normal part of learning. To ensure that the comparisons focused on differences in the mathematical norms (roughly represented by the press for learning subscale) and were not confounded with the general motivational climate, we selected cases for this study that all had high scores on the positive affect scale (Ms. Carter, 4.00; Ms. Martin, 4.33; Ms. Andrew, 4.25; Ms. Reed, 4.17). The high-press cases were both fifth-grade classrooms, and the low-press cases were fourth-grade classrooms. We were initially concerned that the differences in grade would interfere with the study's purpose. Yet, our goal was not to measure students' understandings of fractions but the nature of mathematical conversations. Although it may be reasonable to expect fifth-grade students to know more than fourth-grade students, it is also reasonable to expect that both fourth and fifth graders could engage in conceptual conversations about mathematics.

The coding scheme used in these quantitative analyses reduced complicated and subtle differences among classrooms to a simple five-point scale. The rating scales provided scores based on the kinds of interactions teachers and students had in the classroom, but they could not provide a detailed or vivid picture of classroom talk and were therefore not useful in illustrating the subtle differences in the ways in which classroom talk could move beyond superficial features of explaining and comparing solution strategies.

We recast the similarities among these

four classrooms by drawing from Cobb and his colleagues' use of the construct "social norms." We propose that the notion of norms provides the basis for a more descriptive framework that allows us to capture the differences in mathematical conversations in the classrooms. We begin by enumerating the social norms evident in all four cases that we argue are necessary but not sufficient to achieve conceptual thinking and learning. These four norms are reflected across both the positive affect and press for learning dimensions: (*a*) students describe their thinking; (*b*) students find multiple ways to solve problems, and they describe their strategies to their classmates and teacher; (*c*) students can make mistakes, which are a normal part of the learning process; and (*d*) students collaborate to find solutions to problems. Those social norms are consistent with the NCTM standards for mathematical inquiry. Together they provide the opportunity for students and teachers to create an intellectual climate that values exploration of challenging mathematical concepts.

The Lesson and Tasks

All four teachers independently shaped and adapted their lesson on the addition of fractions from *Seeing Fractions* (Corwin, Russell, & Tierney, 1990), a replacement unit designed to be consistent with California's mathematics framework (California State Department of Education, 1992). The lesson involves the partitioning of brownies. The unit plan provides one sample problem that the teacher may present to the entire class, followed with three similar problems that students solve independently or in small groups. A sample problem reads as follows:

I invited 8 people to a party (including me), and I had 12 brownies. How much did each person get if everyone got a fair share? Later my mother got home with 9 more brownies. We can always eat more brownies, so we shared these out equally too. This time how much brownie did

each person get? How much brownie did each person eat altogether? (p. 76)

By working on the problem, students grapple with equivalence. Teachers may choose to distribute sheets of paper with 16 pre-drawn squares for students to use in solving the problems. The curriculum guide encourages teachers to allow students to work in pairs or small groups.

The brownie lesson uses an area model of fractions as a way for students to develop understandings of part-whole relations, equivalence, and the addition of fractions (Gearhart et al., 1999; Saxe et al., 1999). The brownie problems ask students to partition areas into fair shares and add areas that are partitioned differently. Students often draw out their solutions and thus must link their graphical representations to appropriate numerical ones.

Data Analysis

The study involved qualitative analyses of videotaped instruction. Graduate students working on the project focused one camera on the teacher and another camera on several groups of students throughout the lesson. The authors were not present during the taping. Transcripts were created for each tape. All of the teachers taught the lesson over 2 days, except Ms. Reed, who taught it in 1 day. Each day of instruction lasted approximately 1 hour, creating 2 hours of videotape per teacher per day.

We analyzed transcripts of the videotaped lessons systematically in several phases. In the first phase, transcripts from all of the teacher-focused videotapes were read line by line. Notes were taken in the margins to keep a running record of what was occurring during the lesson. We then created formal summaries on teacher moves and teacher-student interactions during whole-class and small-group activities. The summaries served as a descriptive overview of the lesson. Whole-class discussions were summarized using the following categories: (a) setting up the lesson; (b) shar-

ing student strategies; and (c) concluding the lesson. We created summaries of small-group interactions using the following categories: (a) nature of the activity and (b) nature of teacher monitoring and feedback.

In the second phase, transcripts from all of the student-focused tapes were reviewed in detail. Summaries were created using the following categories: (a) the nature of student-student interactions during small-group activities and (b) the effects of teacher feedback to pairs or groups of students.

We analyzed summaries from the first and second phases for similarities and differences among the cases. We created a chart to compare first the teacher's role and then the students' roles in classroom conversations on the following dimensions: developing the mathematics lesson, the use of mathematical language, any modifications the teacher made to the original *Seeing Fractions* unit, questions posed, and the enactment of classroom discussions. Next, we analyzed the students' level of engagement in the mathematics lesson. We noted the depth and breadth of student participation during whole-class segments, the interactions among students during whole-class and small-group discussions, and evidence of student frustration or persistence.

The summaries and charts allowed us to become familiar, in a detailed way, with the interactions during the lessons. The content of summaries and charts was then related to the social norms that we identified. We used our summary notations as well as verbatim exchanges from the transcripts to make a final compilation of classroom interactions and events that revealed what happened when students (a) described strategies, (b) compared strategies, (c) made mistakes, and (d) worked together. We looked across cases to develop our characterization of the sociomathematical norms that derived from each of the social norms. We looked for both confirming and disconfirming evidence by examining each type of interaction captured by the videos.

Results

A casual observer of the lessons we videotaped would have seen students working together, led by positive, supportive, caring teachers. On the surface, students appeared to be focused on understanding mathematics. A deeper analysis of these lessons, however, revealed important differences in the quality of mathematical discourse. We found exchanges in which students were engaged in deeper mathematical inquiry, using mathematical arguments to debate and test strategies, in two of the four lessons (Ms. Carter's and Ms. Martin's classes). Analyses of the differences between the high- and low-press teacher-student interactions suggested that the higher standard was achieved by sociomathematical norms that corresponded to the social norms we identified: (a) an explanation consists of a mathematical argument, not simply a procedural description or summary; (b) mathematical thinking involves understanding relations among multiple strategies; (c) errors provide opportunities to reconceptualize a problem, explore contradictions in solutions, or pursue alternative strategies; and (d) collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

Our argument here is that our high-press examples demonstrated how the sociomathematical norms were enacted. We cannot demonstrate how these norms were established, but we can show what it means for them to be in place during classroom conversations. Using descriptions and transcripts of classroom activity, we describe next how each of the sociomathematical norms was evident in high- but not low-press exchanges, drawing distinctions between whole-class discussions and small-group activity settings when appropriate. It is important to note that we elaborate on the sociomathematical norms by using illustrative exchanges in each of the four cases. We use these cases to hypothesize about the kinds of conversations and interactional patterns that can promote conceptual think-

ing in a classroom. We cannot claim that the patterns of discourse were consistent in each classroom. Nor do we claim that every interaction followed a similar pattern for each teacher in the lesson we observed. There was, however, a clear pattern of differences in exchanges between the two sets of classrooms in the nature and quality of mathematical conversation, which we describe next.

An Explanation Consists of a Mathematical Argument

A fundamental focus of inquiry-based mathematics is that students explain their thinking. Many classrooms are governed by that norm. But, students can describe the steps they took to solve a problem without explaining why the solution works mathematically. To push students' conceptual thinking, teachers should ask students to justify their strategies mathematically.

High press. In high-press exchanges, students went beyond descriptions or summaries of steps to solve a problem; they linked their problem-solving strategies to mathematical reasons. In whole-class settings, teachers expected presentations of strategies to include both explanation and justification. This is illustrated in the following excerpt, in which Ms. Carter asked Sarah and Jasmine to describe how they divided nine brownies equally among eight people and to explain why they chose particular partitioning strategies (see Fig. 1).

- | | |
|--------------------|---|
| <i>Sarah:</i> | The first four, we cut them in half. [Jasmine divides squares in half on an overhead transparency.] |
| <i>Ms. Carter:</i> | Now as you explain, could you explain why you did it in half? |
| <i>Sarah:</i> | Because when you put it in half, it becomes . . . eight halves. |
| <i>Ms. Carter:</i> | Eight halves. What does that mean if there are eight halves? |

$$1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8$$

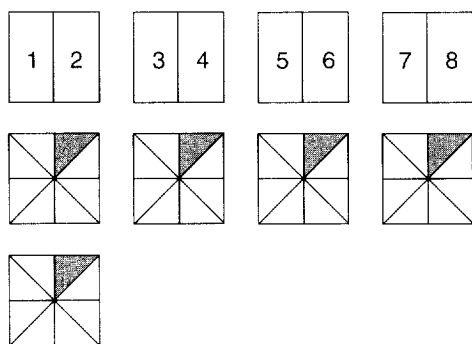


FIG. 1.—Sharing nine brownies among eight people

Sarah: Then each person gets a half.

Ms. Carter: Okay, that each person gets a half. [Jasmine labels halves 1–8 for each of the eight people.]

Sarah: Then there were five boxes [brownies] left. We put them in eighths.

Ms. Carter: Okay, so they divided them into eighths. Could you tell us why you chose eighths?

Sarah: It's easiest. Because then everyone will get ... each person will get a half and [whispers to Jasmine] How many eighths?

Jasmine: [Quietly to Sarah] $\frac{5}{8}$.

Ms. Carter: I didn't know why you did it in eighths. That's the reason. I just wanted to know why you chose eighths.

Jasmine: We did eighths because then if we did eighths, each person would get each eighth, I mean $\frac{1}{8}$ out of each brownie.

Ms. Carter: Okay, $\frac{1}{8}$ out of each brownie. Can you just, you don't have to number, but just show us what you

mean by that? I heard the words, but ... [Jasmine shades in $\frac{1}{8}$ of each of the five brownies not divided in half.]

Jasmine: Person one would get this ... [Points to one eighth.]

Ms. Carter: Oh, out of each brownie.

Sarah: Out of each brownie, one person will get $\frac{1}{8}$.

Ms. Carter: $\frac{1}{8}$. Okay. So how much then did they get if they got their fair share?

Jasmine/Sarah: They got a $\frac{1}{2}$ and $\frac{5}{8}$.

Ms. Carter: Do you want to write that down at the top, so I can see what you did? [Jasmine writes $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ at the top of the overhead projector.]

The exchange among Sarah, Jasmine, and Ms. Carter highlighted the conceptual focus of the lesson on fair share. Ms. Carter asked Sarah to explain the importance of having eight halves and the reason why the partitioning strategy using eighths made sense. After Jasmine gave a verbal justification, Ms. Carter continued to press her thinking by asking her to link her verbal response to the appropriate pictorial representation by shading the pieces, and to the symbolic representation by writing the sum of the fractions.

Examples from Ms. Martin's teaching also illustrated the norm that descriptions needed to consist of a mathematical argument. This norm was evident in whole-class discussions, seen below as a pair of boys presented their solution to a problem:

Luis: There were six crows, and we made, like, a color dot on them ... There were four brownies, and we divided three of them into halves and the last one into sixths. One of the crows got $\frac{1}{2}$ and $\frac{1}{6}$.

Chris: Each crow got $\frac{1}{2}$ and $\frac{1}{6}$. In our second step we had three brownies

and we divided them in half. So each crow got $\frac{1}{2}$. $\frac{1}{2}$ plus $\frac{1}{6}$ equals $\frac{2}{3}$. So we have $\frac{1}{2}$ and $\frac{1}{6}$ and right here is $\frac{2}{3}$. [He points to two squares, one divided into half and then $\frac{1}{6}$ and the other into sixths. $\frac{1}{6}$ had been shaded in each brownie; see Fig. 2.]

Luis: Just to prove that it's the same. Then $\frac{1}{6}$ is what they got here, plus $\frac{1}{2}$. And $\frac{1}{2}$ is equal to $\frac{3}{6}$. $\frac{3}{6}$ plus $\frac{1}{6}$ is equal to one whole and $\frac{1}{6}$.

Luis interjected into Chris's explanation a drawing of equivalent areas (replicated in Fig. 2) to show their method for proving their conclusion. The example suggests that students understood that they needed to be prepared to demonstrate their mathematical argument for equivalence graphically as well as verbally.

After the two boys presented their solution, Ms. Martin asked the class to comment on the pair's drawings. She invited everyone, not just the students at the board, to think about how the students had solved the problem. The teacher initiated a discussion that required students to focus on the mathematical concept of equivalence and its relation to the process of adding fractional parts. It was not enough for students to notice the clarity of the drawings or how they were shaded.

Ms. Martin: What can you tell me about their drawings to represent $\frac{1}{2}$ plus $\frac{1}{6}$ equals $\frac{2}{3}$?

Step 1: 4 brownies, six crows



to show that $\frac{1}{2}$ and $\frac{1}{6}$ equal $\frac{4}{6}$

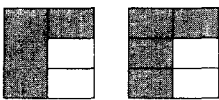


FIG. 2.—Sharing four brownies among six crows

Sam: They're very clear, and you can see exactly what they wanted you to see.

Ms. Martin: What did they want you to see?

Sam: How they added $\frac{1}{2}$ and $\frac{1}{6}$.

Ms. Martin: And what do you see about the adding of the $\frac{1}{2}$ and $\frac{1}{6}$?

Carrie: [They] shaded it in.

Ms. Martin: Okay, they shaded it in. What do you notice about the parts that are shaded in? The $\frac{1}{2}$ plus the $\frac{1}{6}$ and the $\frac{2}{3}$? Do you see anything? No. [Waits.] Jamie, you had your hand up, what do you see?

Jamie: The first section is the same as the first step, it shows $\frac{1}{2}$ and $\frac{1}{6}$. They divided the $\frac{1}{2}$ into sixths and they got $\frac{3}{6}$. They added $\frac{1}{6}$ plus $\frac{1}{2}$. They did an equivalent fraction for it. The first section is the same as the first step, and the second section is the same as the second step [showing] how much each crow got.

Sam: All of the parts they made are equal and [they showed] both of the steps that they took to add the portions.

Ms. Martin: So their diagrams and illustrations are accurately representing the fractional parts. And what did they actually do with the $\frac{1}{2}$ and the $\frac{1}{6}$? What was the process they were doing?

Sam: They were adding.

Similar patterns of interaction were evident in small-group discussions in other high-press exchanges. While students were divided into small groups, Ms. Carter asked them their reasons for agreeing or disagreeing with a solution, not just whether they agreed or disagreed. Verification was an integral part of group activities during the lesson, as is illustrated in the exchange below. Ms. Carter began by asking the group to repeat how much each person received in each step. They eventually agreed that each person received two whole brownies, $\frac{1}{2}$ and $\frac{1}{6}$ of a brownie.

Ms. Carter: So they got two and they got a $\frac{1}{2}$ and $\frac{1}{8}$. Oh, you did a good job drawing that. But then you say they got two and $\frac{3}{8}$. . . I see the two.

Carmen: Uh huh. That's what he said [referring to Edgar], and I told him, "Why did we get $2\frac{3}{8}$?"

Ms. Carter: Why don't you agree with that?

Carmen: I don't know. Because this is $\frac{1}{2}$ [points to half shaded on square], and you put a little piece [draws next to half what looks like $\frac{1}{8}$], and that's $\frac{1}{8}$.

Ms. Carter: I see. So that's $\frac{1}{2}$ and $\frac{1}{8}$, and you're saying that's not a picture of $2\frac{3}{8}$. Or what are you saying? I'm not quite sure.

Carmen: I don't know. I told him, "How [did you get] the answer $2\frac{3}{8}$?"

Edgar: I thought that . . .

Carmen: And I told him, and he said, "Because I know."

Ms. Carter: Remember the one thing I always need is that I need you to be able to explain it.

Edgar: I thought that at first we would add these two [referring to numerators] and then take the biggest number from the bottom [denominator].

Carmen was already thinking that the answer $2\frac{3}{8}$ did not match the area of $\frac{1}{2}$ and $\frac{1}{8}$. She was not satisfied with Edgar's justification of "I know." She knew the classroom norm that required Edgar to justify his actions. Ms. Carter went on to ask the group how they could prove the correct solution by referring to their graphical representations. She brought the group back to the conceptual focus of the brownie problems to link the adding of fractional pieces to the area they represent rather than relying on the arbitrary rule that Edgar had created.

In high-press interactions, students learned that they could justify their actions by triangulating verbal, graphical, and numerical strategies, further illustrated by the following exchange. Ms. Martin approached a group of three girls who had divided seven

brownies equally among six crows and determined that each crow received $\frac{7}{6}$ of a brownie. Their strategy (shown in Fig. 3) involved dividing each brownie into sixths and distributing one-sixth of each brownie to each of the six crows. In the first part of the problem the crows found four brownies, and in the second part of the problem they found three more brownies. The girls had combined $\frac{3}{6}$ and $\frac{4}{6}$ to arrive at their solution of $\frac{7}{6}$.

Ms. Martin: You need to prove this [pointing to $\frac{7}{6}$, their answer].

Katey: How?

Ms. Martin: I don't know.

Katey: 'Cause we counted 1, 2, 3, 4, 5, 6, 7 [pointing to one-sixth from each of the seven brownies].

Ms. Martin: This is a brownie here? [She outlines one brownie. The group had drawn two brownies stuck together, shading $\frac{1}{6}$ in one and $\frac{3}{6}$ in the other; see Fig. 3.]

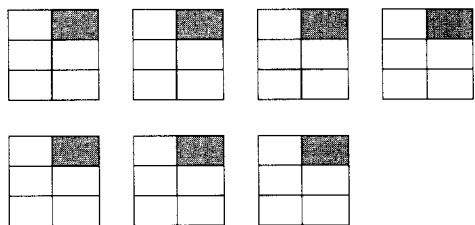
Katey: Yeah.

Ms. Martin: You need to show it in one brownie. This just shows $\frac{4}{6}$ plus $\frac{3}{6}$.

Katey: We have to make another brownie?

Ms. Martin: There are two brownies.

Each crow gets $\frac{1}{6}$ from each brownie



Solution: $\frac{4}{6}$ and $\frac{3}{6}$ equal $\frac{7}{6}$



FIG. 3.—Sharing seven brownies among six crows

- Katey:* Should they be, like, separate?
- Ms. Martin:* You need to show in one ...
- Ashley:* Oh, like in one brownie that [pointing to $\frac{1}{6}$], in one brownie that [pointing to $\frac{3}{6}$]?
- Ms. Martin:* No.
- Stephanie:* No, you need to split the brownies in half. They can't be together.
- Ms. Martin:* You sure you have to split them in half?
- Stephanie:* No.
- Ashley:* Okay, so this is one brownie and that's one brownie.
- Ms. Martin:* Well, how could you show me this in a different way, using one brownie?
- Stephanie:* That's hard.
- Katey:* Well, it would be a weird form of brownie, but you could make like $\frac{1}{6}$. Cut off this [pointing to the sixths not shaded in] and make these into sixths. And then it'd be a brownie with something hanging off the edge, actually.
- Ashley:* Like make a tiny brownie ...
- Katey:* It'd be like a brownie, it'd be like one whole brownie and ... ohhh. [Smiles.]
- Ms. Martin:* One whole brownie and ...
- Katey:* $\frac{1}{6}$.
- Stephanie:* $\frac{1}{8}$.
- Katey:* $\frac{1}{6}$, $\frac{1}{6}$ [affirming the right answer].
- Ms. Martin:* And do you think that's the same as $\frac{2}{6}$?
- Katey/Stephanie:* Yeah.
- Katey:* Yeah, 'cause if these were sixths, you could go 1, 2, 3, 4, 5, 6 [counting the sixths], plus one is 7.
- Ms. Martin:* Can you show me that?

Ms. Martin sustained this exchange for 3 minutes. Sustained exchanges allow for but

do not ensure conceptual thinking. What is important about the above exchange is that Ms. Martin was not satisfied with the group's correct solution. They had arrived at their answer, had explained that they counted the number of sixths that each row received, and they had drawn their solution. Yet the teacher pressed them to think how else they could conceptualize $\frac{2}{6}$, and without providing them with the answer, she asked a pivotal question, "How can you show me this in a different way, using one brownie?" The question helped the group conceptualize the $\frac{2}{6}$ visually, label it with the appropriate fractional name, and justify mathematically why $1\frac{1}{6}$ and $\frac{7}{6}$ were equivalent.

In summary, in high-press examples teachers pressed students to give reasons for their mathematical actions, focusing their attention on concepts rather than procedures. Teachers asked questions in sustained mathematical exchanges with students. Teachers also engaged the whole class in a conversation about a particular student's problem, thus increasing how much each student was involved in mathematical thinking.

Low press. In low-press exchanges, teachers engaged in the same social practice of having students describe their thinking. But the mathematical content was very different from the high-press exchanges. Students described solutions primarily by summarizing the steps they took to solve a problem, as demonstrated in the following exchange in which Raymond described his solution for dividing 12 brownies among eight people. Ms. Andrew had drawn 12 squares on the chalkboard.

[Raymond divides four of the brownies in half.]

Ms. Andrew: Okay, now would you like to explain to us what ... loud ...

Raymond: Each one gets one, and I give them a half.

Ms. Andrew: So each person got how much?

Raymond: One and $\frac{1}{2}$.

Ms. Andrew: $\frac{1}{2}$?

Raymond: No, one and $\frac{1}{2}$.

Ms. Andrew: So you're saying that each one gets one and $\frac{1}{2}$. Does that make sense? [Chorus of "yeahs" from students; the teacher moves on to another problem.]

Ms. Andrew did not ask students to justify why they chose a particular partitioning strategy. Instead, in the lessons we videotaped, both Ms. Andrew and Ms. Reed asked questions that required a show of hands or yes/no responses, such as: "How many people agree?" "Does this make sense?" or "Do you think that was a good answer?" There was no evidence that the teachers were looking for detailed responses; rather, they accepted global, superficial nods of agreement or disagreement. Ms. Andrew wanted to engage her students in the activity and to see if they understood, but the questions she asked yielded general responses without revealing specific information about the students' thinking or understanding of the mathematical concepts involved. When students described their solution strategies, Ms. Andrew did not probe for their reasons for choosing a particular partitioning strategy. The absence of this sociomathematical norm was evident in other contexts as well, such as when Ms. Andrew read to the class an example of a group's "good" written explanation: "First there were eight people and three brownies. We divided two brownies into fourths, and each person got $\frac{1}{4}$. And we divided the last one into eighths, and we gave each person a fourth and an eighth." The description simply summarized the steps the students took. Moreover, although Ms. Andrew had stated that the focus of the lesson was to combine fractional pieces, she accepted the answer of $\frac{1}{4}$ and $\frac{1}{8}$ as adequate.

Ms. Reed also did not push her students to verify their solutions in the lessons we observed. The comments below suggested that she expected such analysis to be too difficult for students. She responded to one student who had stated that $\frac{1}{2}$ and $\frac{1}{8}$ combine to make $\frac{5}{8}$ by asking, "How do you

know? Can you prove that to me? Am I asking too much?" With another group who had offered $\frac{1}{4}$ and $\frac{3}{8}$ as a solution to a problem, Ms. Reed asked, "Could you have found a way to combine those to make one number? Or is that part going to make you too stuck?" Ms. Reed seemed reluctant to press students to think conceptually about the central ideas of the fraction lesson.

During small-group activity, both Ms. Reed and Ms. Andrew were primarily concerned with managerial and procedural instructions, making sure that each group had the materials they needed and had started working. As they stopped to talk to each group, they listened to a group's solution, praised them for their good efforts, and moved on to a new group. Below, Ms. Andrew approached a pair of girls and stayed long enough to hear a brief summary of what they did, yet she did not press them to move beyond their current thinking and to combine fractional parts.

Ms. Andrew: Explain to me what you did.

Susana: We cut them in half, and then we gave them two little pieces.

Ms. Andrew: Okay, now what are those two little pieces?

Susana/Martha: Sixths.

Ms. Andrew: Okay, can you draw what you did on this one? [Susana divides an extra square into sixths.]
Ms. Andrew: Oh, I see. You cut them into sixths like that? Now, how much did each person get?

Martha: $\frac{1}{2}$ and $\frac{3}{8}$.

Ms. Andrew: Very good, and I see that you wrote that here . . . go on to the next part.

In this exchange, the teacher listened to a procedural summary of students' work and directed them to move on to the next part of the problem. Although similar interactions were evident in the lessons from which we drew the high-press examples, they served as information about concep-

tual issues that needed further attention. For example, twice during the lessons we observed, Ms. Carter chose not to press students in individual interactions. Instead, she waited for whole-class discussions to engage the entire class in thinking about a group's solution.

The sociomathematical norm that explanation consists of mathematical argument was also absent in student-student interactions in low-press examples. For example, a group of three girls divided three brownies among eight people in the following way:

- Lupe:* I know how to do that problem. I know how to do that problem. [Claudia gets up.]
- Lupe:* [As Claudia is walking away.] Put it in fourths. [But Claudia does not acknowledge.] [Claudia walks back with Maria.]
- Lupe:* I know, Claudia, look [cut] this one in . . .
- Maria:* [Interrupting.] Do one problem on one paper, and if you want to do extra, she [the teacher] says we could get more paper.
- Lupe:* Okay, just put this one in fourths or in halves. Let me see.
- Maria:* Let's see how many people there are [looks up at board and counts]. There's eight people.
- Lupe:* Put this one in fourths.
- Maria:* In fourths and then this one.
- Lupe/Maria:* In eighths.
- Maria:* No, this one in fourths, and then this one in eighths. [She's suggesting to divide the first two in fourths and the last one in eighths. Claudia has been drawing stick figures.]
- Lupe:* But we have four and four. We have three [brownies].
- Claudia:* That's too easy.
- Lupe:* Too easy?
- Claudia:* Yeah.
- Maria:* Put that one in fourths [pointing to one of three squares].
- Lupe:* In eighths.

Maria: Yeah, in eighths, and this one.

Maria/Lupe: In fourths and in fourths [referring to the other two]. [Claudia is still drawing stick figures.]

This exchange is the most extensive mathematical conversation this group had on video. Yet their talk revolved around *how* to partition the brownies, not *why*.

In summary, one dimension of classroom practice that defines a high press for conceptual learning is that explanations consist of mathematical arguments. In high-press examples, teachers shaped classroom discourse around conceptual mathematical issues by posing questions that asked for mathematical justification or verification. Linking numerical and graphical representations provided a context for extended conversations about ideas of equivalence, part-whole relations, and combining fractional parts. In low-press examples, conversations did not move beyond summaries of the steps taken to solve a problem. Conceptual discourse was limited.

Understanding Mathematical Relations among Strategies

When one or two problems become the focal point of a lesson, students commonly share their strategies for solving the same problem with the whole class. Sharing strategies, however, does not ensure conceptual, mathematical discourse. To engage students in conversations about mathematical concepts, in high-press interactions, the teachers asked students to examine the mathematical similarities and differences among multiple strategies. In low-press exchanges, strategies were offered one after the other, with discussion limited to nonmathematical aspects of student work.

High press. Ms. Martin involved all students in a group or the class in discussions of a student's presentation. Although she encouraged students to praise each other, she also focused their attention more on the mathematics of the presentation than on

product appearance, clarity, or correctness. This press for mathematical inquiry in comparing strategies is illustrated in the following dialogue. After Michelle and Sally had described their strategy and solution for a fair-sharing problem (see Fig. 4), the teacher turned to the class:

Ms. Martin: Does anyone have any questions about how they proceeded through the problem? . . . What did they use or do that was different than what you might have done?

Jeff: They used steps.

Ms. Martin: Right, they divided it into steps. But there were some steps that I haven't seen anyone else use in the classroom yet.

Carl: They added how many brownies there were altogether.

Ms. Martin: Okay, so they used . . .

Jamie: They divided into six, and there was one left over, and then they figured how they were going to divide that equally so that every crow gets a fair share.

Ms. Martin: Exactly, and that was very observant of you to see that. As I walked around yesterday, this is the only pair that

used a division algorithm to determine that there was a whole brownie and a piece left over. So they did it in two different ways.

In this exchange, the teacher asked the class to reflect on what was unique about a particular group's strategy. Students made observations about both organizational ("they divided it into steps") and mathematical ("they divided into sixths") aspects of the group's work. By also asking about how different solutions differed mathematically, students were invited to compare the strategies that had been presented thus far.

Low press. In low-press exchanges, the class applauded correct solutions without analysis, and the teacher glossed over inadequate solutions. Moreover, when students attempted to make connections among solutions, the teacher did not value them, as illustrated by the next example.

At the beginning of the lesson, Ms. Reed asked students to think of ways of combining $1\frac{1}{2}$ and $1\frac{1}{8}$. One student, Ron, combined the two wholes and provided the class with the solution of $2\frac{1}{2}$ and $\frac{1}{8}$. Ms. Reed called on one student after another until she called on a student who provided the correct solution of $2\frac{5}{8}$. Ron noticed that the correct solution was equal to his and tried to make some connection between his answer and the correct one. His claim could have been a good opportunity for Ms. Reed to ask the class to verify whether the two solutions were equivalent, an idea central to the conceptual focus of the fair share lesson. Ms. Reed also had the opportunity to bring in Zoey's estimation that the sum of $\frac{1}{2}$ and $\frac{1}{8}$ was close to $\frac{2}{3}$. The teacher's response to Zoey's thinking was, "Is this a real kind of answer? Sorta $\frac{2}{3}$?" And her response to Ron was the following:

Ms. Reed: [Addressing Ron.] Is [$2\frac{5}{8}$] the same as your answer? Is this the same?

Students: Yes, no.

Ms. Reed: How many think that this is the same as this [Ron's vs. correct

Step 1: 4 brownies, 6 crows



Step 2: 3 more brownies



solution: 7 brownies divided by 6 crows.
1 whole and 1/6



FIG. 4.—Sharing four brownies among six crows and then three more brownies among six crows.

solution]—the same amount?
[Hands raised.]

Ms. Reed: Yes, Ron, it is the same amount. But this [$2\frac{2}{8}$] isn't really written as mathematically as this [$2\frac{1}{2}$ and $\frac{1}{8}$]. $2\frac{1}{2}$ and $\frac{1}{8}$ does equal $2\frac{2}{8}$ —you're right. But this time I wanted you to try and make it into this [pointing to $2\frac{2}{8}$].

Thus, the teacher told Ron that he was right, yet not "mathematical" enough, just as she had told Zoey that her answer was not "real." Although the topic of the lesson was combining fractional parts, she ignored Zoey's estimate, and she did not press the students to verify Ron's observation. Instead, she provided the answer for them, losing an opportunity to engage students in conceptual thinking.

In the lessons we observed, sharing strategies during whole-class discussions looked like a string of presentations, each one followed by applause and praise. We observed an instance in which students made links between strategies, but their links consisted of nonmathematical aspects of students' strategies. In Ms. Andrew's class, a pair of students said they cut the brownies and distributed the pieces to each individual instead of drawing lines from the fractional parts of the brownies to the individuals who received them. Although the students' partitioning strategy was the same as the one a group had just presented, they viewed their strategy as mathematically different based on the way they handed out the pieces.

In the lessons we observed, students in all four classrooms described their work and were praised for their efforts. In high-press exchanges, teachers focused students' attention on mathematical differences among shared strategies, thus directing conversations to encompass mathematical connections among various solution paths. In low-press exchanges, connections were limited to nonmathematical aspects of students' strategies.

Using Errors to Reconceptualize Problems and Pursue Alternatives

In traditional mathematics instruction, students use a standard algorithmic procedure to get the right answer. When students get the wrong answer, they may look back to see which step they missed. Because the goal in inquiry-based mathematics classrooms is to build conceptual understandings, errors can help teachers and students identify misunderstandings.

The social norm that mistakes are acceptable and even useful is commonly established in inquiry-based classrooms. Teachers can press for conceptual thinking by promoting the sociomathematical norm that mistakes are opportunities to reconceptualize a problem, explore contradictions to a solution approach, and try out alternative strategies. Thus, inadequate solutions serve as entry points for further mathematical discussion involving justification and verification.

High press. By requiring students to give mathematical reasons for their problem-solving strategies, Ms. Carter created opportunities for her students to verify whether solutions were correct. In the first section of the results, we described a class presentation in which Sarah and Jasmine offered $\frac{1}{2}$ and $\frac{5}{8}$ as the solution to a problem (see Fig. 1). The interaction between Ms. Carter and the girls continued as follows:

Ms. Carter: Do you want to write that down at the top [of the overhead projector], so I can see what you did? [Jasmine writes $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$.]

Ms. Carter: Okay, so that's what you did. So how much was that in all?

Jasmine: It equals $1\frac{1}{8}$ or $\frac{9}{8}$.

Ms. Carter: So she says it can equal 6 and $\frac{2}{8}$?

Jasmine: No, it can equal $\frac{6}{8}$ or it can equal $1\frac{1}{8}$.

Ms. Carter: Okay, so you have two different answers. Could you write them down so people can see it? And boys and

girls, I'd like you to respond to what they've written up here. She says it either could equal $\frac{5}{8}$ or $1\frac{1}{8}$.

Ms. Carter: [Notes who has a hand raised and who is thinking.] She's given us two answers here: $\frac{5}{8}$ or it can equal $1\frac{1}{8}$. Okay, could those four people right now . . . do you agree with both answers?

Students: No.

Ms. Carter: Do you have a reason why you don't agree? Don't explain it to me, but do you have a reason? Raise your hand if you have a reason why you don't agree. [Hands.] 1, 2, 3, 4, 5, 6 . . . okay. Would those six people please stand? . . . You're going to be in charge of explaining why you don't agree to your team. [She assigns those six students to teams. Camera focuses on group seated in front of class.]

Jessica: I don't believe that it is $\frac{5}{8}$ because one more eighth would equal, like . . . There are $\frac{5}{8}$ in a half. So it wouldn't work. [She goes on to explain.] It can't be $\frac{5}{8}$ because . . . they have $\frac{5}{8}$ up there, see, and there's one here, and only one more . . .

Carlos: One more half will make a whole.

Jessica: One more half will make a whole. It wouldn't be that way. I agree with the second answer.

James: $\frac{5}{8}$ is right.

Jessica: $\frac{5}{8}$ isn't right. 'Cause look, there's $\frac{5}{8}$. So the answer to the first question is $\frac{5}{8}$. So the answer to the first question is $\frac{5}{8}$ instead of $\frac{5}{8}$. [Pause.] Wait, let me see . . . five . . . yeah. It should only be $\frac{5}{8}$. [Ms. Carter walks over to group.]

Ms. Carter: What did your team come up with?

Jessica: We figured out that $\frac{5}{8}$ wasn't the right answer because only $\frac{5}{8}$ could get the answer of $\frac{5}{8}$ instead of a $\frac{1}{2}$. A $\frac{1}{2}$ and a $\frac{1}{2}$ is a whole, so we saw in the picture that $\frac{5}{8}$ equals a

half. So $\frac{5}{8}$ and $\frac{5}{8}$ is $\frac{5}{8}$. [Ms. Carter leaves.]

The teacher could have stepped in and pointed out why $\frac{5}{8}$ and $1\frac{1}{8}$ are not equal. Instead, her response to this mistake was to encourage students to explore the error by explaining why $\frac{5}{8}$ and $1\frac{1}{8}$ are not equal. By engaging the entire class in thinking about which solution was correct, she created an opportunity for all students to engage in mathematical analysis.

There was also evidence that Ms. Carter used her observations of inadequate solutions during group work to plan whole-class instruction. As mentioned earlier, twice we observed her ask the entire class to address a question that she had observed a small group previously struggling to answer. Ms. Martin, in contrast, spent most of her time with small groups, encouraging group members to debate strategies within their group. Although the organization of instruction differed, both Ms. Martin and Ms. Carter created opportunities for students to clarify and extend their knowledge of fractions by exploring in depth why solutions were inadequate as well as adequate.

Ms. Martin listened patiently to how each group solved the problem. She questioned them about how they arrived at their answers and asked them to verify their mathematical actions. Mistakes were used as opportunities for students to think further. For example, when she approached a group of boys who had two answers for partitioning seven brownies among six crows, she did not validate either solution. Instead, she asked the student with the incorrect solution to explain his thinking. It was difficult for her to understand exactly what the student had done, but another group member who was listening to his explanation understood his strategy; he reexplained it, pointing out where the mistake was. To verify that the student understood his own mistake, Ms. Martin asked him to explain why his solution was inadequate and why the other one was correct. She en-

gaged in an extended mathematical conversation with the group, encouraging the boys to listen to each other and to use their mistakes as a way to explore contradictions in the two solutions and provide a mathematical argument for the adequate solution.

Persistence appeared to be an important norm in Ms. Martin's class. We observed exchanges when students stuck with difficult problems until they thought they understood them. For example, two girls struggled for 10 minutes with a complex fair-share problem that they had written themselves. Several times, Ms. Martin approached the pair to check on their progress and ask questions. They debated about whether to make the problem easier, and Ms. Martin voiced confidence in their ability and encouraged them to pursue the problem even though it might be frustrating. Later, when one of the students said to the other, "Can't we just change the problem?" her partner reiterated the norm for high expectations and persistence that the teacher had just reinforced, "No, we can't change it. We can do it. If we change it, that's the same thing as we can't do it." By the end of the lesson, they had decided to work on the problem independently at home (one of them suggested: "Just lock yourself in the bathroom") and discuss it again in class the next day. They seemed to have internalized the sociomathematical norm that when solutions are inadequate, students need to continue to explore alternative ways of thinking about the problem.

Low press. In contrast, although inadequate solutions were accepted as a normal part of learning in low-press teacher-student interactions, they were either passed over until an adequate solution was offered, or teachers provided the reasons why strategies were mathematically incorrect. In the following exchange, Ms. Reed took little advantage of an error two students made and provided the answers for them. The problem involved dividing five brownies among eight people. Initially, the group had offered a solution of $\frac{1}{4}$ and $\frac{3}{8}$, but the teacher had

asked them to try to combine the fractional parts. The group offered the following answer.

Laura: We think it's $\frac{3}{16}$.

Ms. Reed: Okay, how did you get that?

Laura: These are the eighths, so then each person gets $\frac{1}{8}$. They get $\frac{1}{8}$ from here and $\frac{1}{8}$ from here and $\frac{1}{8}$ from here.

Ms. Reed: Shhhh! Excuse me just a minute. We need to all pay attention!

Laura: So, so far each person gets $\frac{3}{8}$, plus then they found two more [brownies], so each person gets $\frac{1}{8}$ from them again. So there's $\frac{1}{8}$. . .

Ms. Reed: And another $\frac{1}{8}$, don't they? Or no?

Laura: Three plus one equals four and then the sixteenths. [The blackboard around their solution is covered with messy numerics, some crossed out and written over.]

Ms. Reed: Oh, okay. Don't do the sixteenths. Sixteenths means something different. Just leave it as eighths. We'll talk about that in a few minutes. But then there's actually $\frac{3}{8}$. There's $\frac{1}{8}$ here and then there's $\frac{1}{8}$ there [referring to the two additional brownies], so that's $\frac{3}{8}$. Okay. [Class applauds.]

It is unclear whether Ms. Reed understood how the group got $\frac{3}{16}$ as the answer; she did not ask Laura to clarify what she meant by "three plus one equals four and then the sixteenths." The teacher's statement, "Sixteenths means something different," invalidates the students' mathematical activity rather than giving them an opportunity to critically analyze their mathematical thinking. The class applause (which, ironically, came right after the teacher solved the problem) seemed to serve only as a signal to end the discussion of this group's strategy.

In addition to not using student errors as the bases for further mathematical inquiry, in the low-press examples, teachers praised adequate solutions and ignored

wrong ones. Below, Ms. Reed approached a group of girls who explained that they combined $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{4}$ by adding the numerators and denominators and arriving at $\frac{3}{16}$. Ms. Reed responded:

Ms. Reed: Okay, that's a good guess. But you don't get to add this $\frac{1}{4}$ and this $\frac{1}{8}$ like that. $\frac{1}{4}$ is one size and $\frac{1}{8}$ is another. It's like adding an apple and an orange together. You just don't say this four and eight is 12. It's not the same thing.

Antonia: Maybe we could add . . .

Rachel: Ahh, so we could split the fourths into eighths.

Ms. Reed: Ahh, listen, listen. She's [referring to Rachel] got a good idea. [Moves on to another group.]

Ms. Reed did not use the group's inadequate solution as an opportunity to encourage them to think about and compare the area of $\frac{3}{16}$ to $\frac{1}{4}$ and $\frac{1}{8}$. Instead, she pushed them directly to the right strategy and then moved on to another group of students.

In the next exchange, Ms. Andrew did not push students to rethink their answers. A group of three boys divided five brownies among six people and arrived at an answer of $\frac{1}{2}$, $\frac{1}{6}$, and $\frac{1}{6}$. Note how the teacher provided the mathematical reasoning for the boys.

Ms. Andrew: They got $\frac{1}{2}$. You already said that. And then $\frac{1}{6}$ and then another sixth. So, how many sixths did they get?

Anthony: One, two.

Ryan: One, two.

Joe: $\frac{1}{2}$.

Ms. Andrew: What did you say? [to Joel] They got two.

Ryan: Sixths.

Anthony: $\frac{2}{12}$.

Joe: $\frac{2}{6}$.

Ms. Andrew: $\frac{2}{6}$ [confirming the right answer].

Ms. Andrew: Why did you say $\frac{2}{12}$? Because there are 12 parts altogether?

Anthony: Yeah.

Ms. Andrew: Okay, be sure not to get confused. Because there are two brownies, not one. Perfect. Good good job.

At first, the boys appeared to be guessing. Ms. Andrew focused on Joe once he stated the right answer. Although she predicted accurately why Anthony said $\frac{2}{12}$, she did not ask him to think about why his answer may not work. Instead, she asked and answered the question herself. Her statement, "Because there are two brownies, not one," was left unexplained. She did not encourage the members of the group to think about what $\frac{1}{6}$ and $\frac{1}{12}$ signify or to use graphical representations to clear up their confusion.

The sociomathematical norm of using mistakes as a basis for mathematical inquiry was absent in student-student as well as teacher-student interactions in the lessons we observed. Earlier, we presented the example in which Ms. Reed told a student (Antonia) that her strategy of combining $\frac{2}{4}$ and $\frac{1}{8}$ by adding the numerators and denominators was wrong. Another student in her group (Rachel) suggested that they split the fourths into eighths. After Ms. Reed left the group, the two girls had the following conversation:

Antonia: Split the fourths into eighths, how?

Rachel: Like, um, here's a fourth, right?

Antonia: Let's use a clean sheet of paper.

Rachel: There are four brownies or just a fourth?

Antonia: Fourths into eighths.

Rachel: You mean . . . [Begins writing.]

Antonia: Oh yeah, those two fourths into eighths.

Rachel: Yeah, split them into eighths.

Antonia: Oh, those two into eighths. I get it.

Rachel: I told you.

Antonia: Thanks. So that'll be . . . Could you carry on from now because you got the idea?

Although at first Antonia was not sure how fourths and eighths were related, she

claimed to understand as Rachel began writing. Yet, Antonia faltered again when she tried to combine the eighths. Although mistakes were generally acceptable in her class, it appeared that she had not internalized a norm that they offer opportunities for conceptual thinking. Antonia disengaged from the project of adding the fractions and left the task to her partner, who she believed would find the right answer. The collaboration disintegrated.

In summary, although in both high- and low-press examples mistakes were viewed as a normal part of learning, only in the high-press exchanges did teachers press students to critically analyze their strategies and solutions, conveying clearly that the goal was to understand mathematical concepts. In the low-press cases, in contrast, teachers precluded further mathematical inquiry by giving the answer themselves.

Collaboration Includes Accountability and Consensus through Argumentation

Collaborative work is common in classrooms in which teachers are attempting to implement reform-minded mathematics instruction. The purpose of peer collaboration is for students to construct mathematical understandings in a social context and to become skilled in communicating in mathematical language by describing and defending their differing mathematical interpretations and solutions (Cobb et al., 1993). Holding each student accountable for thinking through the mathematics involved in a problem and promoting the idea that consensus should be reached through mathematical argumentation can help establish sociomathematical norms that promote students' full participation in mathematical discussions.

In all four lessons we observed, students worked in groups of two to four for most of the instructional time, as recommended by the *Seeing Fractions* unit. The sociomathematical norms, described above, were evident in the high-press examples. Students

in the low-press examples were seen working together and agreeing on a solution without debating the mathematics involved, with members of the group often deferring to a student perceived to be the most skilled.

High press. Ms. Carter's and Ms. Martin's approaches to collaborative work in the lessons we observed were consistent with those strategies documented by Cobb and his colleagues. Both teachers instructed students on how to talk about mathematics with classmates (Cobb et al., 1993; Williams & Baxter, 1996). Ms. Martin began her lessons by going over the guidelines, written on the board, that she expected students to follow during small-group activity. One guideline read, "Did you come to a consensus with your partner or partners about that solution?" The following conversation with students demonstrates how she promoted the norm of using mathematical thinking to reach a consensus.

Karen: What does consen . . . consensus [sic] . . . mean or whatever?

Ms. Martin: I was waiting for someone to ask me what consensus means. Does anybody know or think they know what the word consensus means?

Julie: Agreement?

Ms. Martin: Agreement, so if Janet and Louisa are working together, and they cannot agree on a solution, then what do they do?

Karen: Have to use a consensus or whatever.

Ms. Martin: Well consensus is the same as an agreement. They either come to a consensus or they come to an agreement. But what happens if they can't seem to come to a consensus or agreement as to the solution? What do you think they do?

Ricky: Try again?

Ms. Martin: They need to try again. And how could they try again and try to prove to each other a solution? Julie?

- Julie:* Draw illustrations of what their answer was.
- Ms. Martin:* Okay, you could use illustrations to show what you think the solution to the problem is.
- Mark:* You could both, like, write down the answer you think it is and compare, and if you have the same answer, then you figure that that's the answer.

Similarly, when Ms. Martin began the second day of class work on the fair-share problems, she made the following statement regarding individual accountability:

Everyone in your group, whether it's just the two of you or the three of you, everyone in your group needs to understand the process that you all were supposed to go through together. Because when you make a presentation, you don't know whether or not you are going to be asked a question. So you don't know if you're going to be asked by me or by your classmates. So you need to make sure that each person understands each part of the process you went through.

Transcripts of whole-class discussions showed that both Ms. Martin and Ms. Carter invited all members of the group to contribute to the explanation of their group's work. Furthermore, students were not sure when they might have to answer a question and therefore had to be prepared. Ms. Carter also used a cooperative learning strategy referred to as "numbered heads together" (Stone, 1989), which helped promote full participation of all group members. Each member of a group was assigned a number between one and four. During whole-class discussion, Ms. Carter called out a number randomly to determine who was responsible for responding to the question. Thus, each student was invested in all members of the group fully understanding the strategy they used.

The norms of individual accountability and consensus applied within the context of small-group activity. Ms. Martin reminded

student groups that each member must understand the group's strategies and be prepared to discuss them. As she asked questions about students' solutions, she invited and expected every student to be a part of the conversation.

Students appeared to have internalized this expectation. When students worked together in pairs or small groups, the distribution of labor was fairly equal, with all students engaged in solving the problem. Below, a pair of students in Ms. Martin's classroom discussed their written explanation of dividing seven brownies for three crows:

Keisha: See, this is how I explained it. [Reads.] "What we did is we took three brownies and cut them into half because three plus three equals six. And there are six crows."

Mark: This is what I put so far. [Reads.] "We knew that there were three more brownies, and we divided each one in half. One brownie had two halves, and another brownie had two halves, and another one had two halves."

Keisha: Just write, "And all three had two halves." And all . . .

Mark: [Starts writing.] "And all three . . .," I don't have to write.

Keisha: Okay, and each crow got $\frac{1}{2}$ and . . . just write, " $3 + 3$ is 6 " so each crow got $\frac{1}{2}$ of the brownie.

Mark: Yeah, but these are halves. [Counts halves in each brownie.] 2, 2.

Keisha: Yeah, I know. 1, 2, 3. [Counts brownies.] Cut them in half, 1, 2, 3, 4, 5, 6. [Counts halves.] There were six crows. Each crow got $\frac{1}{2}$.

Mark: [Writes "Each crow got one-half."]

Having written about one of the steps in the problem, the pair evaluated and expanded their written explanations. They were attentive to each other's comments. Mark responded to each of Keisha's suggestions, deciding whether they were needed in the written explanation. When Keisha directed

Mark to write " $3 + 3 = 6$," Mark's response indicated that he was not merely taking orders and finishing the activity but was thinking about what she said and how it applied to what they had done. Thinking about their partitioning strategy of halves, he did not understand how " $3 + 3 = 6$ " applied until Keisha explained that she was referring not to the halves themselves, but that the number of halves corresponded to the number of crows. They arrived at mutual understanding through a mathematical argument.

Low press. In the low-press examples, we observed teachers give only general directives such as "work with a partner" or "remember to work together." The distribution of work across a group was not equal. In most instances of group work, we observed one or two students completing the work of the group while others were more peripheral, as in the following example from Ms. Reed's class.

Lisa: We need five brownies. So see, 1, 2, 3, 4. So we cut these into half. So 1 . . . 8. They get a half each. And then there's one more cookie and eight people, so we just cut into eighths, and it'll be even for everybody.

Ellen: Wow, you did that fast. I didn't even do anything.

Lisa: I knew there's 5, and I knew 4; 2 times 4 is 8.

Ellen: Oh, I get it.

Ellen did not have an opportunity to think about the problem and instead listened to and agreed with Lisa's explanation. Individual accountability was not a normative aspect of working together. In low-press examples, students did not question each other or make sure that each person understood the mathematical relations involved in the problem. Instead, students who became unclear about what to do often withdrew and allowed another student to take over. In the high-press exchanges, every student in a group or pair was expected to participate in mathematical problem solv-

ing and to use mathematical arguments to achieve consensus.

Discussion

The aims of this study were to analyze and provide vivid images of classroom practices that create a press for conceptual learning. The fair-share, multistep problems provide an appropriate context for students to grapple with the mathematical concepts of equivalence and addition as applied to fractions.

On the surface, all four teachers we observed implemented qualities of inquiry-oriented mathematics instruction. Students solved open-ended problems in groups, documented their work graphically and numerically, and shared their different strategies enthusiastically. The teachers and classmates accepted and supported students who made mistakes. All four teachers encouraged their students to describe how they solved the problems and circulated in the room during small-group activity to talk to students about their work. Thus, a number of social norms that characterize inquiry-based instruction were in place in all the lessons we observed.

The differences among the high- and low-press exchanges provide evidence for the need to go beyond these superficial teaching practices to examine the nature and degree of conceptual thinking. The social norms shared by all four cases were not sufficient for engaging students in mathematical inquiry. We propose that the concept of sociomathematical norms provides a useful framework for thinking about what teachers need to do to promote the development of students' mathematical ideas. Differences among our high- and low-press examples suggest the importance of the following sociomathematical norms: (a) an explanation consists of a mathematical argument, not simply a procedural description; (b) mathematical thinking involves understanding relations among multiple strategies; (c) errors provide opportunities to reconceptualize a problem, explore contra-

dictions, and pursue alternative strategies; and (d) collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

The notion of sociomathematical norms is very different from specific prescriptions for educational practice. Sociomathematical norms concern a set of expectations about what constitutes mathematical thinking. Supporting teachers to create sociomathematical norms in their classrooms requires more than describing discrete behaviors. Rather, teachers need to understand what a sociomathematical norm is and construct pedagogical strategies that can be applied in a variety of contexts. Although we propose that at least four sociomathematical norms worked together to create a press for conceptual learning, continued research may reveal other norms that contribute to a high press. It is also important to investigate, with longitudinal data, how sociomathematical norms are created and sustained, and how they influence students' mathematical understanding.

The notion of engaging children in what Wood, Cobb, and Yackel (1991) call "genuine conversations" about mathematics means that teachers take students' ideas seriously in their attempts to support student understanding. In the exchanges showing a high press for conceptual learning, all students were accountable for participating in an intellectual climate characterized by argument and justification. Classroom practices that are characterized by a high press for conceptual thinking allow the mathematics to drive students' engagement in activities.

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