Sinusoidal Driving Force (Damped Response)

EOM – $m\ddot{u} + c\dot{u} + ku = P\sin\omega t$

Divide by m

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{P}{m}\sin\omega t$$

$\frac{k}{m} \to \omega_n^2$

From before $\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n}$

$$\boxed{\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{P}{m}\sin\omega t}$$

Try solution of form $u = \underbrace{C_1 \sin\omega t}$ $- C_2\cos\omega t$

This term is not enough since you have a first derivative. This means that you will have a phase lag between the driving frequency and the response when $\zeta \neq 0$

$$\dot{u} = \omega C_1\cos\omega t + \omega C_2\sin\omega t$$
$$\ddot{u} = -\omega^2 C_1\sin\omega t + \omega^2 C_2\cos\omega t$$

Substitute into EOM –

$$\frac{P}{m}\sin\omega t = \left(-\omega^2 C_1 + 2\zeta\omega_n\omega C_2 + \omega_n^2 C_1\right)\sin\omega t + \left(+\omega^2 C_2 + 2\zeta\omega_n\omega C_1 - \omega_n^2 C_2\right)\cos\omega t$$

If this true for all times then it must be true when either $\sin\omega t = 0$ or $\cos\omega t = 0$. Get 2 eq, 2 unknowns.

$$\left(-\omega^2 + \omega_n^2\right)C_1 + 2\zeta\omega_n\omega\, C_2 = P/m$$
$$2\zeta\omega_n\omega\, C_1 + \left(\omega^2 - \omega_n^2\right)C_2 = 0$$

Solve simultaneous equations

$$C_1 = \frac{P}{K}\;\frac{1 - (\omega/\omega_n)^2}{\{1 - (\omega/\omega_n)^2\}^2 + \{2\zeta\omega/\omega_n\}^2}$$

$$C_2 = \frac{P}{K}\;\frac{2\zeta\,\omega/\omega_n}{\{1 - (\omega/\omega_n)^2\}^2 + \{2\zeta\omega/\omega_n\}^2}$$

$$u = C_1 \sin \omega t - C_2 \cos \omega t$$

Compare

But this can be rewritten as:

$$u = X \sin(\omega t - \phi)$$

$$= X \sin \omega t \cos \phi - X \cos \omega t \sin \phi$$

Then

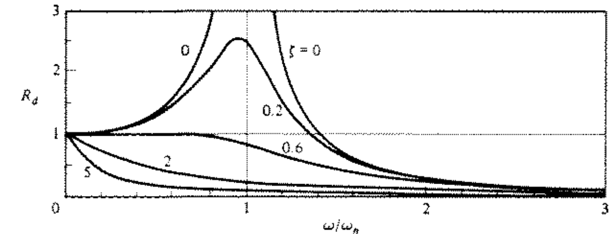
$$C_1 = X \cos \phi \qquad \longrightarrow \qquad (C_1^2 + C_2^2)^{1/2} = X$$
$$C_2 = X \sin \phi \qquad \qquad \tan \phi = C_2 / C_1$$

Therefore:

$$\boxed{u_p = \overbrace{\frac{P}{k} R_d}^{X} \sin(\omega t - \phi)}$$

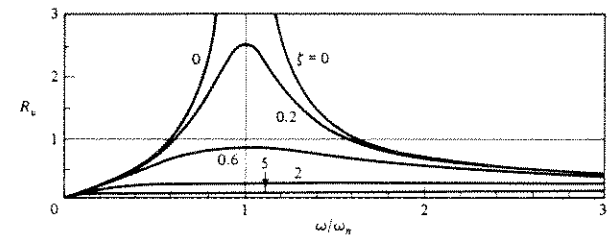
where, $$R_d = \frac{1}{\left\{ \left[1 - (\omega/\omega_n)^2\right]^2 + \left[2 \xi \omega/\omega_n\right]^2 \right\}^{1/2}}$$

$$\tan \phi = \frac{2 \xi \omega / \omega_n}{1 - (\omega/\omega_n)^2}$$



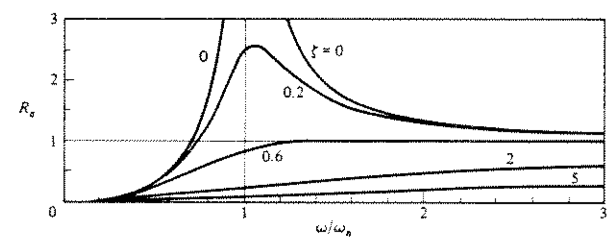
R_d
$\zeta = 0$
0
0.2
0.6
2
5
ω/ω_n
0
1
2
3

(a)



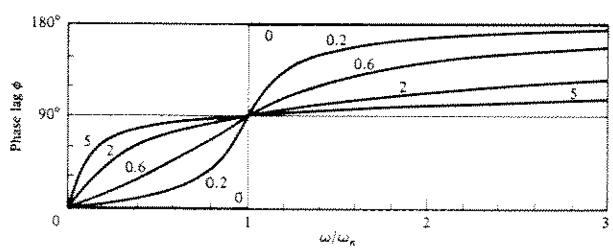
R_v
$\zeta = 0$
0
0.2
0.6
2
5
ω/ω_n
0
1
2
3

(b)



R_a
$\zeta = 0$
0
0.2
0.6
2
5
ω/ω_n
0
1
2
3

(c)



Phase lag ϕ
180°
90°
0
0.2
0.6
2
5
ω/ω_n
1
2
3

(d)