Read Ch 2 in Chopra

Single-Degree-of-Freedom System

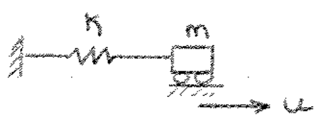
In general $\overset{m}{\square} \longrightarrow F(u, \dot{u}, \ddot{u}, t, u^2 \ldots)$

Special Cases -

Free-vibration < undamped, $F = -ku$
damped - viscous - $F = -ku - c\dot{u}$
- coulomb - $F = \pm f_0$

Forced Vibration (transient - / periodic -) $F = -ku - c\dot{u} + F(t)$

Undamped Free Vibration



EOM $m\ddot{u} = -ku$

$$\boxed{m\ddot{u} + ku = 0}$$ EOM

linear, homogenous, 2nd-order differential equation

Try solution of form: $u = e^{\lambda t}$ ← λ to be determined later

$\dot{u} = \lambda e^{\lambda t}$

$\ddot{u} = \lambda^2 e^{\lambda t}$

Substitute into EOM $\quad m\lambda^2 e^{\lambda t} + K e^{\lambda t} = 0$

$$(m\lambda^2 + K)\, e^{\lambda t} = 0$$

But $e^{\lambda t} \neq 0$, So $\quad m\lambda^2 + K = 0$

$$\lambda^2 = -\frac{K}{m}$$

$$\lambda = \pm\, i\sqrt{\frac{K}{m}}$$

Either λ will satisfy EOM, so general solution must be a linear combination of the two solutions

$$u = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$= C_1 e^{i\sqrt{\frac{K}{m}}\, t} + C_2 e^{-i\sqrt{\frac{K}{m}}\, t}$$

C_1, C_2 to be determined from boundary conditions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}_{\cos x} + \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) i}_{i \sin x}$$

So $\quad e^{ix} = \cos x + i \sin x$

$$u = C_1 \left\{ \cos\sqrt{\tfrac{K}{m}}\, t + i \sin\sqrt{\tfrac{K}{m}}\, t \right\} + C_2 \left\{ \cos -\sqrt{\tfrac{K}{m}}\, t + i \sin -\sqrt{\tfrac{K}{m}}\, t \right\}$$

$$= \underbrace{(C_1 + C_2)}_{A} \cos\sqrt{\tfrac{K}{m}}\, t + \underbrace{(iC_1 - iC_2)}_{B} \sin\sqrt{\tfrac{K}{m}}\, t$$

$$u = A \cos\sqrt{\tfrac{K}{m}}\, t + B \sin\sqrt{\tfrac{K}{m}}\, t$$

But what are A, B?

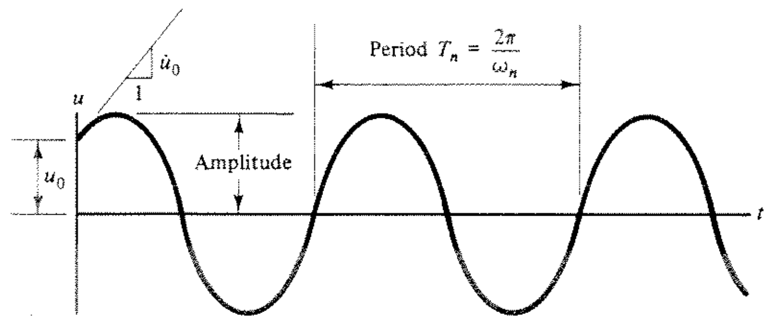
$$u(0) = A(1) + B(0) \longrightarrow A = u_o$$

$$\dot{u} = -A\sqrt{\tfrac{k}{m}}\sin\sqrt{\tfrac{k}{m}}t + B\sqrt{\tfrac{k}{m}}\cos\sqrt{\tfrac{k}{m}}t$$

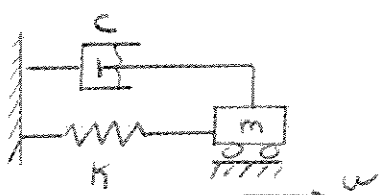
$$\dot{u}(0) = B\sqrt{\tfrac{k}{m}} \longrightarrow B = \frac{\dot{u}_o}{\sqrt{\frac{k}{m}}} = \frac{\dot{u}_o}{\omega_n} \leftarrow \text{By definition}$$

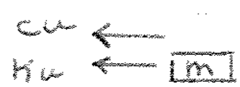
$$\boxed{u = u_o\cos(\omega_n t) + \left(\frac{\dot{u}_o}{\omega_n}\right)\sin(\omega_n t)}$$

What kind of motion does this equation describe?



Free Vibration with Viscous Damping





$$m\ddot{u} = -c\dot{u} - ku$$

$$\boxed{m\ddot{u} + c\dot{u} + ku = 0}$$

Try solution of form $u = e^{\lambda t}$

$$m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + ke^{\lambda t} = 0$$

$$(m\lambda^2 + c\lambda + k) = 0$$

From quadratic formula

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

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$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \underbrace{\frac{k}{m}}_{\omega_n^2}}$$

$$u = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$= e^{-\frac{c}{2m}t}\left\{ C_1 e^{\sqrt{(c/2m)^2 - k/m}\, t} + C_2 e^{-\sqrt{(c/2m)^2 - k/m}\, t} \right\}$$

Character of response will vary with the sign of the equation inside the square root.

$$\underline{(c/2m)^2 - \frac{k}{m} = 0}$$ Critical Damping

The damping that results in the equation going to zero is called critical damping, denoted by C_c

$$\left(\frac{C_c}{2m}\right)^2 - \frac{k}{m} = 0 \longrightarrow C_c^2 = 4m^2\frac{k}{m}$$

$$\boxed{C_c = 2\sqrt{mk}}$$

$$\boxed{u = e^{-\frac{c}{2m}t}(A + Bt)}$$

$$\underline{(c/2m)^2 - \frac{k}{m} > 0}$$ Over damped

Introduce definition, let $\xi = \frac{c}{C_c} \longrightarrow c = \xi C_c = 2\xi\sqrt{mk}$

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \left(\frac{2\xi\sqrt{mk}}{2m}\right)^2 - \frac{k}{m} = \underbrace{\frac{k}{m}}_{\omega_n^2}(\xi^2 - 1) > 0$$ or equivalently

$\xi > 1$

For $\zeta > 1$, solution is real,

$$u = e^{\overbrace{-2\zeta\sqrt{mk}/2m\;t}^{-\zeta\omega_n t}} \left\{ C_1 e^{\omega_n\sqrt{\zeta^2-1}\;t} + C_2 e^{-\omega_n\sqrt{\zeta^2-1}\;t} \right\}$$

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \qquad \text{Underdamped } (\zeta < 1)$$

$$u = e^{-\omega_n\zeta t}\left[C_1 e^{\sqrt{\omega_n^2(\zeta^2-1)}\;t} + C_2 e^{-\sqrt{\omega_n^2(\zeta^2-1)}\;t} \right]$$

$$= e^{-\omega_n\zeta t}\left[C_1 e^{i\omega_n(1-\zeta^2)^{1/2}t} + C_2 e^{-i\omega_n(1-\zeta)^{1/2}t} \right]$$

But this identical to what we had before for free vibration except that

$$\sqrt{\frac{k}{m}} = \omega_n \longrightarrow \omega_n\sqrt{1-\zeta^2} = \omega_d$$

and we have a factor $e^{-\omega_n\zeta t}$ in front

$$u = e^{-\omega_n\zeta t}\left\{ A\cos(\omega_d t) + B\sin(\omega_d t) \right\}$$

Now let's determine A & B in terms of initial conditions.

$$u(0) = (1)\ \{A\ (1) + B(0)\} = A \rightarrow \underline{\underline{A = u_o}}$$

$$\dot{u} = -\omega_n \xi e^{-\omega_n \xi t}\{A\cos(\omega_d t) + B\sin(\omega_d t)\}$$
$$+ e^{-\omega_n \xi t}\{-A\sin(\omega_d t)\,\omega_d + \omega_d B\cos(\omega_d t)\}$$

$$\dot{u}(0) = -\omega_n \xi \underset{\substack{\downarrow \\ u_o}}{A} + \omega_d B \quad \rightarrow \quad B = \frac{\dot{u}_o + \xi \omega_n u_o}{\omega_d}$$

$$\boxed{u(t) = e^{-\omega_n \xi t}\left\{u_o \cos\omega_d t + \left(\frac{\dot{u}_o + \xi\omega_n u_o}{\omega_d}\right)\sin\omega_d t\right\}}$$

Sec. 2.2 Damped Free Vibration

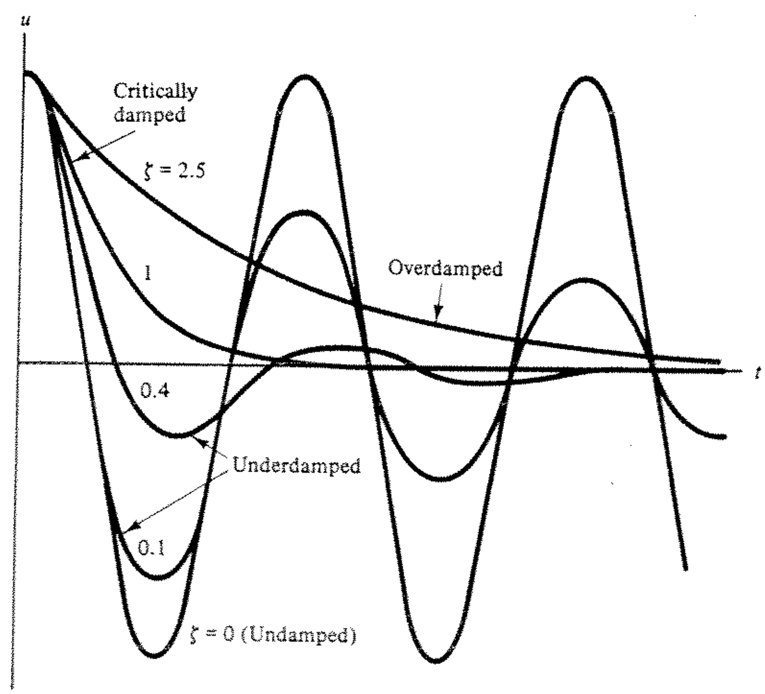

Figure 2.4 The effects of damping on free vibration.