# FOURIER SERIES REPRESENTATION

A periodic functions, F(t), can be represented as a Fourier series, with the following form.

$$F(t) = a_o + \sum_{j=1}^{\infty} \left( a_j \cos \frac{j2\pi t}{T_0} + b_j \sin \frac{j2\pi t}{T_0} \right)$$

where $T_0$ is the period of the function. This series solution can also be expressed as:

$$F(t) = a_o + \sum_{j=1}^{\infty} \left( a_j \cos j\omega_0 t + b_j \sin j\omega_0 t \right)$$

where $\omega_0 = \frac{2\pi}{T_0}$

By pre-multiplying both sides of this equation by $\cos(j\omega_0 t)$ and by $\sin(j\omega_0 t)$ and integrating over one full cycle, one can derive expressions for the constants that multiply the cosine and sine terms of the series.

$$a_o = \frac{1}{T_0} \int_{t=0}^{T_0} F(t)\, dt$$

$$a_j = \frac{2}{T_0} \int_{t=0}^{T_0} F(t) \cos(j\omega_0 t)\, dt$$

$$b_j = \frac{2}{T_0} \int_{t=0}^{T_0} F(t) \sin(j\omega_0 t)\, dt$$

The response of a single-degree-of-freedom system to such a periodic force can be obtained by determining the displacement response to each term of the series individually, $u_j(t)$, and then summing the individual displacement response.

The particular (steady-state) solution for the first force term is $u_0(t)=a_0/k$. The particular solution for each sine force term was determined earlier.

$$u_{p,\sin,j}(t) = \frac{b_j}{k} \left\{ \frac{\left(1-\beta_j^2\right)\sin(j\omega_0 t) + 2\zeta\beta_j \cos(j\omega_0 t)}{\left[1-\beta_j^2\right]^2 + \left[2\zeta\beta_j\right]^2} \right\}$$

where $\beta j = j\omega_0/\omega_n$ .

This solution can also be written as:

$$u_{p,\sin,j}(t) = \frac{b_j / k}{\sqrt{\left[1-\beta_j^2\right]^2 + \left[2\zeta\beta_j\right]^2}} \sin(j\omega_0 t - \phi_j)$$

where

$$\tan\phi_j = \frac{2\zeta\beta_j}{1-\beta_j^2}$$

The particular solution for the cosine force terms is similar. Combining the constant, sine and cosine solutions, one obtains:

$$u_p(t) = \frac{a_o}{k} + \sum_{j=1}^{\infty} \frac{1}{k} \left( \frac{\sqrt{a_j^2 + b_j^2}}{\sqrt{\left[1-\beta_j^2\right]^2 + \left[2\zeta\beta_j\right]^2}} \sin(j\omega_0 t - \phi_j) \right)$$

where $\tan\phi_j = \dfrac{-a_j\left[1-\beta_j^2\right] + b_j\left[2\zeta\beta_j\right]}{b_j\left[1-\beta_j^2\right] + a_j\left[2\zeta\beta_j\right]}$

This expression can be rewritten as follows

$$u_p(t) = \frac{a_o}{k} + \sum_{j=1}^{\infty} \frac{R_{d,j}}{k} \sqrt{a_j^2 + b_j^2} \sin(j\omega_0 t - \phi_j)$$

where:

$$R_{d,j} = \frac{1}{\sqrt{\left[1-\beta_j^2\right]^2 + \left[2\zeta\beta_j\right]^2}}$$

The relative contributions of each response term depends on the amplitude of the components of the forcing function ($a_j$, $b_j$), and on the magnitude of the amplification term, $R_{d,j}$, which in turn depends on the frequency ratio, $\beta_j$.