CEE502

Duhamel's Integral

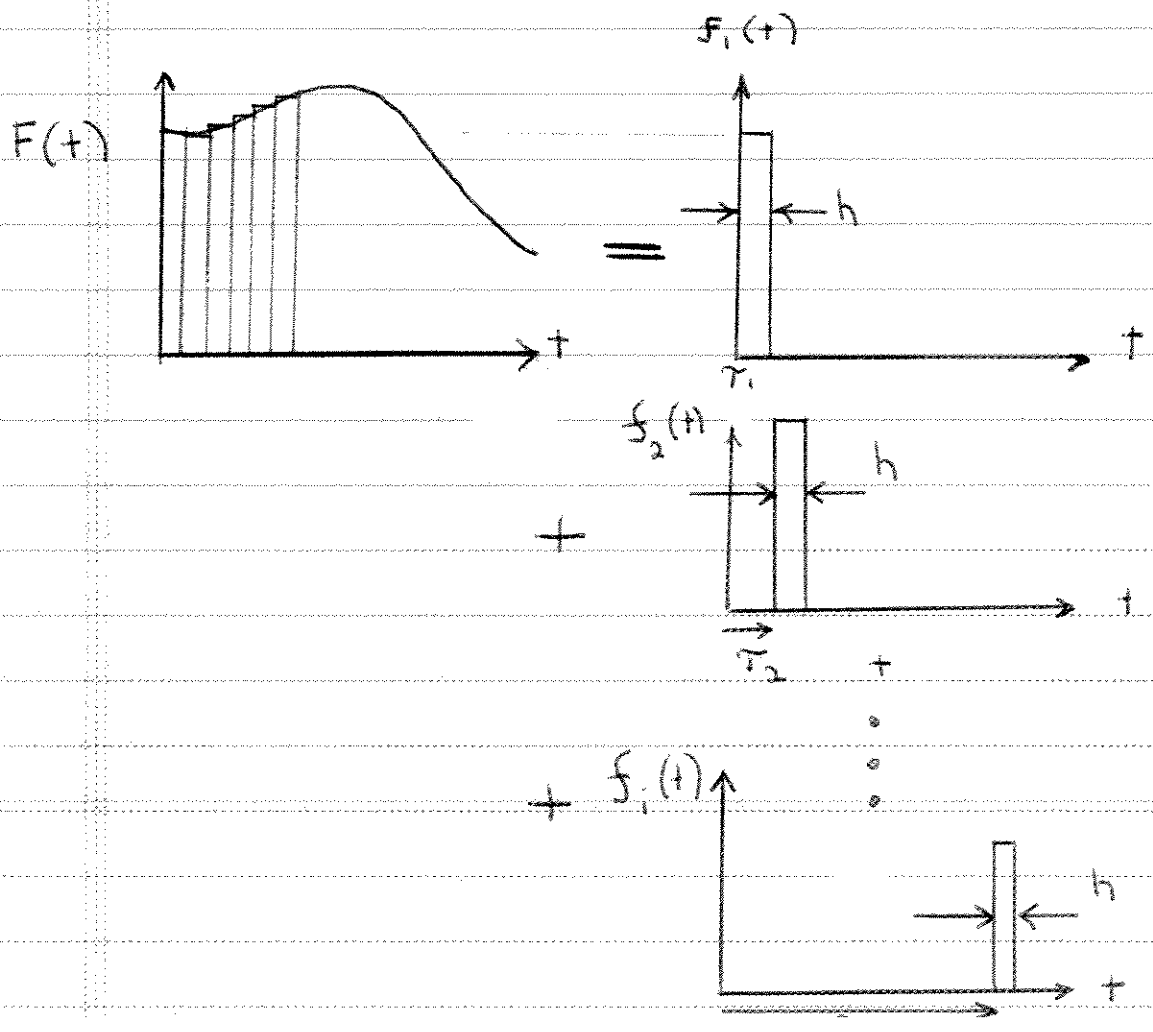
Consider the EOM: $m\ddot{u} + c\dot{u} + ku = F(t)$

We have shown that the particular solution, $u_p(t)$ for a driving force $F(t)$ can be obtained by superimposing solutions such that:

$$u_p(t) = u_1(t) + u_2(t) + \cdots u_i(t)$$
$$\text{and} \quad F(t) = F_1(t) + F_2(t) + \cdots F_i(t)$$

Where $u_i(t)$ is the particular solution for $F_i(t)$

Consider the decomposition of $F(t)$ into the sum of short impulses of length h



Obtain the solution $u_i(t)$ for the driving force $f_i(t)$, where $f_i(t)$ is a block pulse of length h which begins at $t = \tau_i$

① $t < \tau_i$ $\qquad u_i(t) = 0$

② $\tau_i \le t \le \tau_i + h$

$$u_i(t) = \frac{f_i}{k}\left\{1 - e^{-\zeta\omega_n(t-\tau_i)}\left[\cos\omega_d(t-\tau_i) + \frac{\zeta}{(1-\zeta^2)^{1/2}}\sin\omega_d(t-\tau_i)\right]\right\}$$

③ $t \ge \tau_i + h$

$$u_i(t) = e^{-\zeta\omega_n(t-\tau_i-h)}\left[u_1\cos\omega_d(t-\tau_i-h) + \frac{\dot{u}_1 + \zeta\omega_n u_1}{\omega_d}\sin\omega_d(t-\tau_i-h)\right]$$

where u_1, $\dot{u}_1$ are obtained from Eq ① evaluated a $t = \tau_i + h$

Obtain u_1

Evaluating Eq ① @ $t = \tau_i + h$:

$$u_i(\tau_i + h) = \frac{f_i}{k}\left\{1 - e^{-\zeta\omega_n h}\left[\cos\omega_d h + \frac{\zeta}{(1-\zeta^2)^{1/2}}\sin\omega_d h\right]\right\}$$

for $\omega_n h \ll 1$, $\omega_d h \ll 1$

$$\begin{pmatrix} e^{-\zeta\omega_n h} = 1 - \zeta\omega_n h + \dots \\ \cos\omega_d h \to 1 - \frac{\omega_d^2 h^2}{2} + \dots \\ \sin\omega_d h \to \omega_d h + \dots \end{pmatrix}$$

$$u_1 = u_i(\tau_i + h) = \frac{f_i}{k}\left\{1 - (1 - \zeta\omega_n h + \dots)\left[\left(1 - \frac{\omega_d^2 h^2}{2} \dots\right) + \frac{\zeta}{(1-\zeta^2)^{1/2}}(\omega_d h + \dots)\right]\right\}$$

3/4

$$u_1 = \frac{f_i}{k}\left\{ 1 - 1 + \xi\omega_n h - \frac{\xi}{(1-\xi^2)^{1/2}}\omega_d h + \mathcal{O}\, h^2 \right\}$$

"of order of"

But

$$\omega_d = (1-\xi^2)^{1/2}\omega_n \text{ , so} \qquad \boxed{u_1 = \frac{f_i}{k}\,\mathcal{O}\, h^2}$$

Obtain $\dot{u}_1$

Take derivative of Eq ①:

$$\dot{u}_i(t) = \frac{f_i}{k}\left\{ -\xi\omega_n e^{-\xi\omega_n(t-\tau_i)}\left[\cos\omega_d(t-\tau_i) + \frac{\xi}{(1-\xi^2)^{1/2}}\sin\omega_d(t-\tau_i)\right]\right.$$
$$\left. - \omega_d e^{-\xi\omega_n(t-\tau_i)}\left[-\sin\omega_d(t-\tau_i) + \frac{\xi}{(1-\xi^2)^{1/2}}\cos\omega_d(t-\tau_i)\right]\right.$$

Evaluate @ $t = \tau_i + h$:

$$\dot{u}_1 = u_i(\tau_i + h) = \frac{f_i}{k}\left\{ \xi\omega_n e^{-\xi\omega_n h}\left[\cos\omega_d h + \frac{\xi}{(1-\xi^2)^{1/2}}\sin\omega_d h\right]\right.$$
$$\left. - \omega_d e^{-\xi\omega_n h}\left[-\sin\omega_d h + \frac{\xi}{(1-\xi^2)^{1/2}}\cos\omega_d h\right]\right\}$$

For $\omega_n h \ll 1$
$\omega_d h \ll 1$

$$\dot{u}_1 = \frac{f_i}{k}\left\{ \xi\omega_n\,(1-\xi\omega_n h \cdots)\left[\left(1 - \frac{\omega_d^2 h^2}{2} + \cdots\right) + \frac{\xi}{(1-\xi^2)^{1/2}}(\omega_d h + \cdots)\right]\right.$$
$$\left. - \omega_d\,(1-\xi\omega_n h \cdots)\left[(-\omega_d h + \cdots) + \frac{\xi}{(1-\xi^2)^{1/2}}\left(1 - \frac{\omega_d^2 h^2}{2} + \cdots\right)\right]\right.$$

$$\dot{u}_1 = \frac{f_i}{k}\left\{ \xi\omega_n - \omega_d\frac{\xi}{(1-\xi^2)^{1/2}} - \xi^2\omega_n^2 h + \xi\omega_n\frac{\xi}{(1-\xi^2)^{1/2}}\omega_d h \right.$$
$$+\omega_d^2 h + \xi\omega_n\omega_d h \ldots$$

$$\dot{u}_1 = \frac{f_i}{k}\left\{ +\omega_d^2 h + \xi^2 \frac{\omega_d \omega_n h}{(1-\xi^2)^{1/2}} \right\} + \mathcal{O} h^2$$

$$= \frac{f_i}{k} \underset{\uparrow\, k/m}{\omega_n^2} h = \boxed{\frac{f_i h}{m} = \dot{u}_1}$$

(Neglecting higher-order terms)

So, for $t \geq \tau_i + h$, Eq (3) becomes
(Neglecting higher-order terms)

$$u_i(t) = e^{-\xi\omega_n(t-\tau_i-h)}\left[\frac{f_i h}{\omega_d m} \sin \omega_d (t-\tau_i-h)\right]$$

But we have many pulses spaced at $d\tau_i$
with a length of pulse $h = d\tau_i$

$$u_p(t) = \sum_{i=1}^{\tau_i = t} e^{-\xi\omega_n(t-\tau_i-d\tau_i)}\left[\frac{f_i}{m\omega_d} \sin \omega_d (t-\tau_i-d\tau_i)\right] d\tau_i$$

Let $d\tau_i \to 0$, then

$$\boxed{u_p(t) = \int_0^t \frac{f(\tau)}{m\omega_d} e^{-\xi\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau}$$

Note that $u(t) = \underbrace{u_h(t)}_{\text{Required to fit nonzero initial conditions.}} + u_p(t)$

Example -

Consider an undamped single-degree-of-freedom system subjected to a base motion of the form

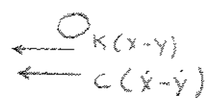
$$\dot{y}(t) = v_o e^{-t/t_o} \, q(t)$$

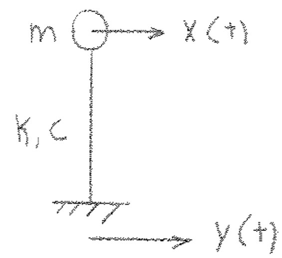
where v_o, t_o are constants
$\dot{y}(t)$ is the velocity of the base
$q(t)$ is a unit step function

Determine the motion of the system as a function of time if the system is initially at rest

Determine EOM -

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$$



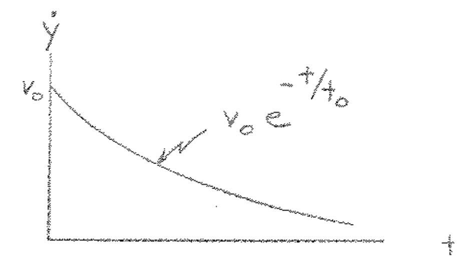


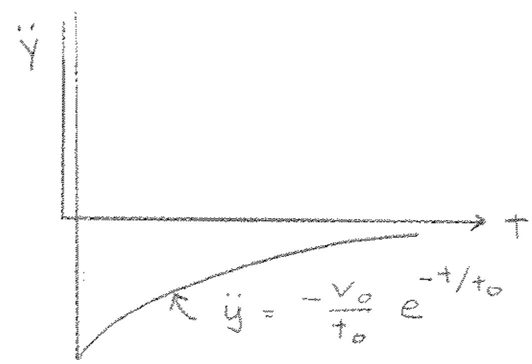
let $z = x - y$ then
$\dot{z} = \dot{x} - \dot{y}$
$\ddot{z} = \ddot{x} - \ddot{y}$

$$\boxed{m\ddot{z} + kz + c\dot{z} = -m\ddot{y}}$$

(in our case, $c = 0$)

Determine $\ddot{y}(t)$





To instantaneously go from an initial velocity $= 0$ to $\dot{y} = v_o$ requires $\ddot{y}(0) = \infty$. But since the velocity is not infinite, the integral of the acceleration must be finite.

Introduce the Dirac Delta Function, $\delta(x)$. $\delta(x)$ is defined to have the following properties.

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{t=-\infty}^{t=\infty} \delta(t)\, dt = 1$$

(Can think of $\delta(t)$ as a unit impulse and the derivative of the unit step function.)

The acceleration is then

$$\boxed{\ddot{y}(t) = v_o e^{-t/t_o} \delta(t) - \frac{v_o}{t_o} e^{-t/t_o}}$$

The displacement history can then be calculated using Duhamel's integral, sometimes referred to as the convolution integral or the superposition integral.

In general $\quad u(t) = \dfrac{1}{m\omega_d}\displaystyle\int_0^t f(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$

In this problem, $\zeta = 0$, $\omega_d = \omega_n$, $u \rightarrow z$, $f(\tau) = -m\ddot{y}(\tau)$

$$z(t) = \frac{1}{m\omega_n}\int_0^t \left(-m v_o e^{-\tau/t_o}\delta(\tau) + \frac{m v_o}{t_o} e^{-\tau/t_o}\right) \sin\omega_n(t-\tau)\, d\tau$$

$$= -\frac{v_o}{\omega_n}\int_{\tau=0}^t e^{\tau/t_o} \sin\omega_n(t-\tau)\delta(\tau)\, d\tau + \frac{v_o}{\omega_n t_o}\int_0^t e^{-\tau/t_o}\sin\omega_n(t-\tau)\, d\tau$$

$$= -\frac{v_o}{\omega_n}\sin\omega_n t + \frac{v_o}{\omega_n t_o}\int_0^t e^{-\tau/t_o}\sin\omega_n(t-\tau)\, d\tau$$

From integral tables $\quad \displaystyle\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

Let $\quad x = t - \tau$

$\quad dx = -d\tau$ (t is constant within the integral)

$$z(t) = -\frac{v_o}{\omega_n}\sin\omega_n t + \frac{v_o}{\omega_n t_o}\int_t^0 e^{\frac{x}{t_o}} e^{-t/t_o} \sin\omega_n x\, (-dx)$$

$$= \frac{-v_o}{\omega_n}\sin\omega_n t - \frac{v_o}{\omega_n t_o} e^{-t/t_o}\int_t^0 e^{x/t_o}\sin\omega_n x\, dx$$

Comparing result with integral table $\begin{cases} a = 1/t_o \\ b = \omega_n \end{cases}$

Then $\quad z(t) = -\dfrac{v_o}{\omega_n}\sin\omega_n t - \dfrac{v_o}{\omega_n t_o} e^{-t/t_o}\left[\left(\dfrac{1}{1/t_o^2 + \omega_n^2}\right) e^{x/t_o}\left(\dfrac{1}{t_o}\sin\omega_n x - \omega_n\cos\omega_n t\right)\right]_{x=t}^{x=0}$

Evaluating the bracketed term at $x = t$ & $x = 0$

$$z(t) = -\frac{v_o}{\omega_n}\sin\omega_n t - \frac{v_o}{\omega_n t_o} e^{-t/t_o}\left[\frac{t_o^2}{1+(\omega_n t_o)^2}\right]\left\{-\omega_n - e^{t/t_o}\left(\frac{1}{t_o}\sin\omega_n t - \omega_n\cos\omega_n t\right)\right\}$$

Simplifying

$$\boxed{z(t) = \frac{v_o t_o}{1+(\omega_n t_o)^2}\left[e^{-t/t_o} - \omega_n t_o \sin\omega_n t - \cos\omega_n t\right]}$$