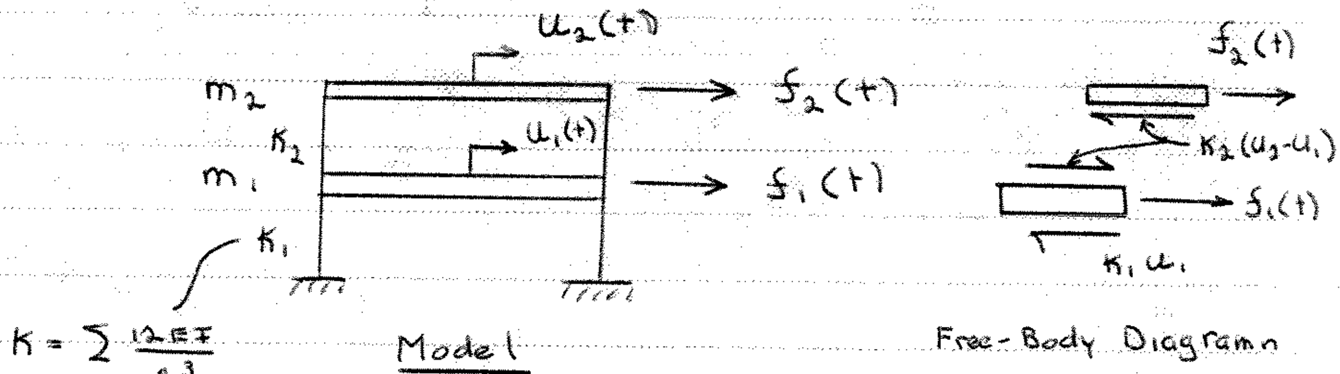
2 DOF System

Let's begin our discussion of multi-degree-of-freedom systems by considering an undamped system with 2 DOF. Consider a simple model of a two-story building.



What assumptions have we made in this model?

1) rigid beams
2) weightless columns (lumped mass approximation)
3) no column axial deformation
4) linear material law (small strain)
5) small deflection (No P-Δ effect)
6) fixed supports
7) no contribution to stiffness by nonstructural elements
8) no damping

EOM

DOF #1: $m_1 \ddot{u}_1 = -K_1 u_1 + K_2 (u_2 - u_1) + f_1(t)$

DOF #2: $m_1 \ddot{u}_2 = -K_2 (u_2 - u_1) + f_2(t)$

In matrix notation:

Eq (1) -
$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{[M]} \underbrace{\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}}_{\{\ddot{u}\}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{\{u\}} = \underbrace{\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}}_{\{f\}}$$

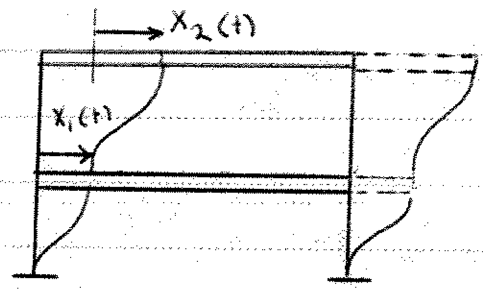
$[M]$ inertia matrix — Diagonal Matrix

$[K]$ Stiffness Matrix — Not a diagonal matrix (Static coupling)

$\{u\}$ displacement vector

$\{f\}$ driving force vector

Let's consider an alternate coordinate system.



X_1, X_2 are related to u_1, u_2 by:
$$u_1 = X_1$$
$$u_2 = X_1 + X_2$$

In matrix notation:
$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad \text{Eq 2a}$$

Similarly
$$\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} \qquad \text{Eq 2b}$$

Substitute Eqn's 2a and 2b into Eqn 1.
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Performing the matrix multiplication:
$$\begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_2 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

We have seen that the form of the mass and stiffness matrices change with the choice of coordinate system. This 2 DOF system would be much easier to analyze if both the mass and stiffness matrices were uncoupled, i.e.

$$\begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1^* \\ f_2^* \end{Bmatrix}$$

We shall soon study the means of selecting an appropriate coordinate system. For now, we will choose

$$u_1 = q_1 + q_2$$
$$u_2 = 1.62 q_1 - 0.62 q_1 \longrightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

To simplify the algebra, we will let

$$m_1 = m_2 = m$$
$$k_1 = k_2 = k$$

The EOM becomes

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Eq 3

$$\underbrace{\begin{bmatrix} m & m \\ 1.62m & -.62m \end{bmatrix}}_{\text{Dynamic Coupling}} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} 0.38k & 2.62k \\ 0.62k & -1.62k \end{bmatrix}}_{\text{Static Coupling}} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Premultiply Eq 3 by the transpose of the transformation matrix

$$\begin{bmatrix} 1 & 1.62 \\ 1 & -0.62 \end{bmatrix} \begin{bmatrix} m & m \\ 1.62m & -0.62m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 1 & 1.62 \\ 1 & -0.62 \end{bmatrix} \begin{bmatrix} 0.38k & 2.62k \\ 0.62k & -1.62k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1.62 \\ 1 & -0.62 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$\underbrace{\begin{bmatrix} 3.62\,m & 0 \\ 0 & 1.38\,m \end{bmatrix}}_{[M^*]} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} 1.38\,K & 0 \\ 0 & 3.62\,K \end{bmatrix}}_{[K^*]} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1 + 1.62 f_2 \\ f_1 - 0.62 f_2 \end{Bmatrix}$$

$$[M^*] \quad \{\ddot{q}\} + [K^*] \quad \{q\} = [f^*\}$$

The 2 - DOF problem is now expressed as 2 single - degree - of - freedom problems (SDOF)

$$\left.\begin{aligned} (3.62m)\ddot{q}_1 + (1.38K)q_1 &= f_1 + 1.62\,f_2 \\ (1.38\,m)\,\ddot{q}_2 + (3.62\,K)\,q_2 &= f_1 - 0.62\,f_2 \end{aligned}\right\} \text{Uncoupled Equations}$$

The conclusions to be drawn (at least for this particular system) are:

1) The form of [M] and [K] varies with the choice of coordinate system

2) An appropriate selection of coordinate system, combined with a multiplication by the inverse of the transformation matrix, results in diagonal [M*] and [K*]

3) If [M*] and [K*] are diagonal, the 2 DOF system can be solved by considering 2 SDOF problems.

Let us now turn our intention to the selection of a transformation that will result in uncoupled equations.

Consider free vibration:

EOM -

$$\begin{array}{l|l} 3.62\,m\,\ddot{q}_1 + 1.38\,K\,q_1 = 0 & \\ 1.38\,m\,\ddot{q}_2 + 3.62\,K\,q_2 = 0 & \end{array} \quad \overbrace{\begin{array}{l} q_1^{\circ},\ \dot{q}_1^{\circ} \\ q_2^{\circ},\ \dot{q}_2^{\circ} \end{array}}^{\text{Initial Conditions}}$$

<u>Case ①</u>

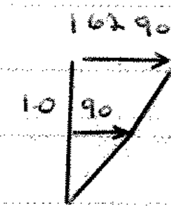
$$q_1^{\circ} = q_0 \qquad \dot{q}_1^{\circ} = 0$$
$$q_2^{\circ} = 0 \qquad \dot{q}_2^{\circ} = 0$$

What does mean physically?

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}}_{1^{st}\text{ mode shape } \phi_1} q_1 + \underbrace{\begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix}}_{2^{nd}\text{ mode shape, } \phi_2} q_2$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \phi_1 & \phi_2 \end{array}$$

If $q_1^{\circ} = q_0$ and $q_2 = 0$



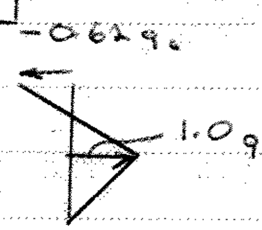
Initial deflected shape

Now, let's solve the SDOF problems.

$$q_1(t) = \overbrace{q_0 \cos \omega_{n_1} t}^{\text{free-vibration solution}} \qquad \text{where} \quad \omega_{n_1} = \sqrt{\frac{1.38K}{3.62m}} = 0.62\sqrt{\frac{K}{m}}$$

$$q_2(t) = 0 \qquad \boxed{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = q_0 \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix} \cos \omega_{n_1} t}$$

<u>Case ②</u>



$$q_1^{\circ} = 0 \qquad \dot{q}_1^{\circ} = 0$$
$$q_2^{\circ} = q_0 \qquad \dot{q}_2^{\circ} = 0$$

Solving both SDOF problems.

$$q_1(t) = 0 \qquad \text{where} \quad \omega_{n_2} = \sqrt{\frac{3.62K}{1.38m}} = 1.62\sqrt{\frac{K}{m}}$$

$$q_2(t) = q_0 \cos \omega_{n_2} t \qquad \boxed{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = q_0 \begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix} \cos \omega_{n_2} t}$$

A consequence of the uncoupled form of the EOM is that a system in free vibration which has initial velocity = 0 and initial displacement corresponding to a mode shape (the columns of the transformation matrix) will preserve that displaced shape. This property can be used to determine the mode shapes and the appropriate transformation matrix.

Consider Free Vibration

EOM $$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We seek a solution of the form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}}_{\phi} e^{\lambda t} \rightarrow \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \lambda^2 e^{\lambda t}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \lambda^2 e^{\lambda t} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{\lambda t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(Eqn 4) $$\begin{bmatrix} \lambda^2 m + 2k & -k \\ -k & \lambda^2 m + k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[\lambda^2 [M] + [K] \right] \{\phi_i\} = \{0\}$$

If matrix is invertible, then $\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ Trivial Solution

If, on the other hand, the matrix is not invertible, then

1) Matrix is singular
2) rows are dependent
3) Determinant of matrix = 0

Using this third property ($\text{Det} = 0$),

$$[\lambda^2 m + 2k][\lambda^2 m + k] - k^2 = 0$$

$$m^2 \lambda^4 + 3mk\lambda^2 + k^2 = 0$$

$$\lambda^4 + 3\frac{k}{m}\lambda^2 + \left(\frac{k}{m}\right)^2 = 0$$

$$\lambda^2 = \tfrac{1}{2}\left[-3\frac{k}{m} \pm \sqrt{9\left(\frac{k}{m}\right)^2 - 4\frac{k^2}{m^2}}\right]$$

$$\lambda^2 = -\tfrac{1}{2}\left(3 \pm \sqrt{5}\right)\frac{k}{m}$$

$$\lambda = \pm i\sqrt{\frac{3 \pm \sqrt{5}}{2}}\sqrt{\frac{k}{m}} \quad \begin{aligned} i\left(1.62\sqrt{\tfrac{k}{m}}\right) &= \lambda_4 \\ -i\left(1.62\sqrt{\tfrac{k}{m}}\right) &= \lambda_3 \\ i\left(0.62\sqrt{\tfrac{k}{m}}\right) &= \lambda_2 \\ -i\left(0.62\sqrt{\tfrac{k}{m}}\right) &= \lambda_1 \end{aligned}$$

Four Solutions

But what are A_1, A_2?

Substitute λ^2 into equation 4

$\lambda^2 = -\tfrac{1}{2}(3-\sqrt{5})\frac{k}{m}$

1^{st} Eq. $\left[\left(-\tfrac{1}{2}(3-\sqrt{5})\,{}^{k}/_{m}\right)m + 2k\right]A_1 - kA_2 = 0 \qquad \frac{A_2}{A_1} = 1.62$

2^{nd} Eq. $-kA_1 + \left[\left(-\tfrac{1}{2}(3-\sqrt{5})\right){}^{k}/_{m}\, m + k\right]A_2 = 0 \qquad \frac{A_2}{A_1} = 1.62$

As expected because rows are dependent

$\lambda^2 = -\tfrac{1}{2}(3+\sqrt{5})\frac{k}{m}$

2^{nd} Eq. $-kA_1 + \left[-\tfrac{1}{2}(3+\sqrt{5}){}^{k}/_{m}\, m + k\right]A_2 = 0 \qquad \frac{A_2}{A_1} = -0.62$

Thus $[\phi] = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix}$

We can now write the complete solution to the free-vibration problem.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = C_1 e^{i\omega_1 t} \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix} + C_2 e^{-i\omega_1 t} \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}$$

$$+ C_3 e^{i\omega_2 t} \begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix} + C_4 e^{-i\omega_2 t} \begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix}$$

Where $\omega_1 = 0.62\sqrt{\frac{k}{m}}$ $\qquad \omega_2 = 1.62\sqrt{\frac{k}{m}}$

As noted before, the exponentials can be rewritten in the form of sines and cosines such that

$$\boxed{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left(A \sin \omega_1 t + B \cos \omega_1 t\right) \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix} + \left(C \sin \omega_2 t + D \cos \omega_2 t\right) \begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix}}$$

where A, B, C, D depend on initial conditions.

Solution Strategy for MDOF Systems
(n degrees of freedom)

1) Obtain EOM $[M]\{\ddot{u}\} + [K]\{u\} = \{f\}$

2) Determine mode shapes and frequencies by considering free-vibration problem

$$\left\{ [K] - \omega^2 [M] \right\} \{\phi_i\} = \{0\}$$

3) Determine uncoupled equations -

$$m^*_{jj}\ \ddot{q}_j + k^*_{jj}\ q_j = f^*$$

where
$$[M^*] = [\phi]^T [M] [\phi]$$
$$[K^*] = [\phi]^T [K] [\phi]$$
$$\{f^*\} = [\phi]^T \{f\}$$
$$[\phi] = [\phi_1, \phi_2 \cdots\cdots \phi_n]$$

4) Determine initial conditions in terms of q.
$$\{u\} = [\phi]\{q\}$$
$$\{\dot{u}\} = [\phi]\{\dot{q}\}$$

5) Solve n SDOF problems to obtain $q_1 \cdots q_n$

6) Convert back to original coordinate system

$$\{u\} = [\phi]\{q\} \longrightarrow \{u\} = \{\phi_1\} q_1(t) + \cdots \{\phi_n\} q_n(t)$$