FLEXURAL ANALYSIS

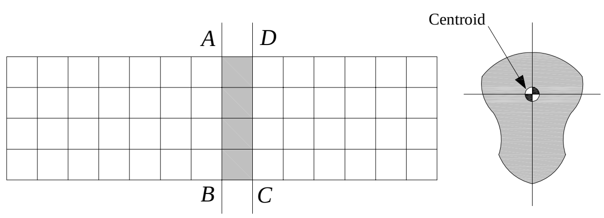
Main Assumptions

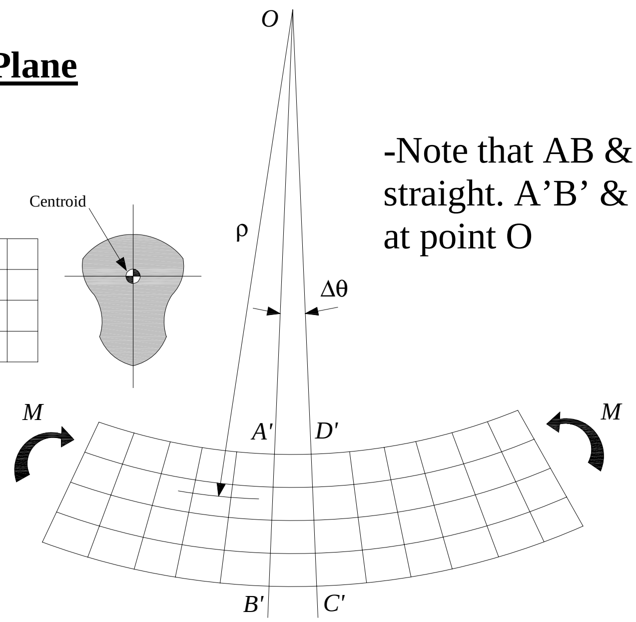
1) Plane sections remain plane (compatibility)

2) $\Sigma F_x = 0$, $\Sigma M_z = 0$ (equilibrium)

3) $\sigma = E\varepsilon$ (constitutive relations for linear material)

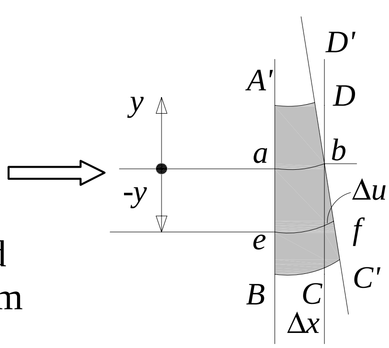
Plane Sections Remain Plane





-Note that AB & AD remain straight. A'B' & D'C' intersect at point O

Neutral Axis (located at ρ from



Line ab is at the neutral axis, so by definition, the length remains unchanged at Δx

$$\varepsilon_x = \frac{length_{ef} - length_{ab}}{length_{ab}}$$

$$= \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

$$\boxed{\varepsilon_x = -\frac{y}{\rho} = -y\phi = -y\frac{d\theta}{ds}}$$

Axial Force Equilibrium

$$\Sigma F_x = \int_A \sigma_x \, dA = 0$$

But $\sigma_x = E\varepsilon_x$ for linear material

$$\int_A E\varepsilon_x \, dA = 0$$

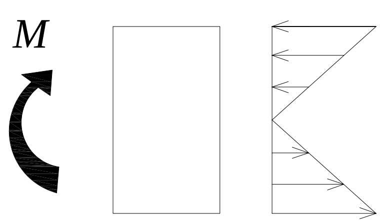
But $\varepsilon = -y\phi$ for plane sections

$$-E\phi \int_A y\,dA = 0 \longrightarrow \int_A y\,dA = 0$$

By definition $\bar{y} = \dfrac{\int_A y\,dA}{A} \longrightarrow \bar{y} = 0$ (coordinate of centroid is zero if origin at N.A.)

So Neutral Axis is Located at Centroid

Moment Equilibrium



Stresses

$$\Sigma M = M + \int (\sigma_x dA) y = 0$$

But $\sigma_x = E\varepsilon_x = E(-y\phi)$

$$M = E\phi \underbrace{\int y^2}_{I} dA$$

$$\boxed{\phi = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{M}{EI}}$$

$$\boxed{\sigma_x = -Ey\phi = -Ey\frac{M}{EI} = \frac{-My}{I}}$$

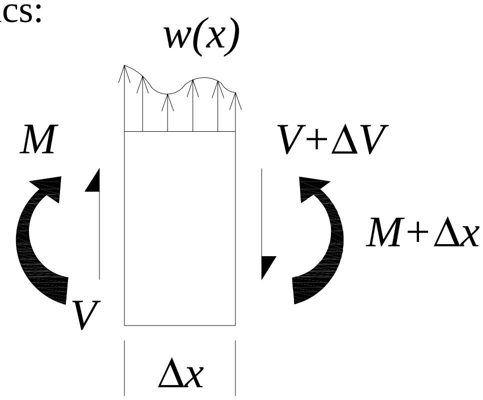
Governing Differential Equation

$$\frac{1}{\rho} = \frac{dv^2/dx^2}{[1+(\frac{dv}{dx})^2]^{3/2}} \approx d^2v/dx^2 \qquad (for\ dv/dx << 1)$$

But $\frac{1}{\rho} = \frac{M}{EI}$, so

$$\boxed{d^2v/dx^2 = \frac{M}{EI} \qquad M = EI\ d^2v/dx^2}$$

From statics:



$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \boxed{\frac{dV}{dx} = w}$$

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = \boxed{\frac{dM}{dx} = V}$$

Combining boxed equations:

$$\boxed{V = d/dx\ (EI\ d^2v/dx^2)}$$

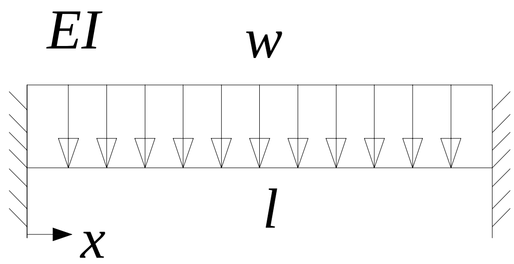
and

$$\boxed{w = d^2/dx^2\ (EI\ d^2v/dx^2)}$$

If E(x) and I(x) are constant along the length of the beam**:**

$$\boxed{w = (EI\ d^4v/dx^4) = EI\ v''''}$$

Example: Fixed-Fixed Beam with Uniform Load



$$v'''' = -\frac{w}{EI}$$

$$v''' = -\frac{wx}{EI} + C_1$$

$$v'' = -\frac{wx^2}{2EI} + C_1 x + C_2$$

$$v' = -\frac{wx^3}{6EI} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$v = -\frac{wx^4}{24EI} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

Boundary Conditions (BC)

$$\begin{Bmatrix} v(0) = 0 & v(l) = 0 \\ v'(0) = 0 & v'(l) = 0 \end{Bmatrix}$$

Apply B.C. to Beam Equation:

$$v(0) = C_4 = 0$$

$$v'(0) = C_3 = 0$$

$$v(l) = -\frac{wl^4}{24EI} + C_1\frac{l^3}{6} + C_2\frac{l^2}{2} = 0$$

$$v'(l) = -\frac{wl^3}{6EI} + C_1\frac{l^2}{2} + C_2 l = 0$$

$$\frac{wl^4}{24EI} - C_1\frac{l^3}{12} = 0 \longrightarrow C_1 = \frac{wl}{2EI}$$

$$\longrightarrow C_2 = -\frac{wl^2}{12EI}$$

$$v(x) = -\frac{wx^4}{24EI} + \frac{wl}{12EI}x^3 - \frac{wl^2}{24EI}x^2$$

Checks

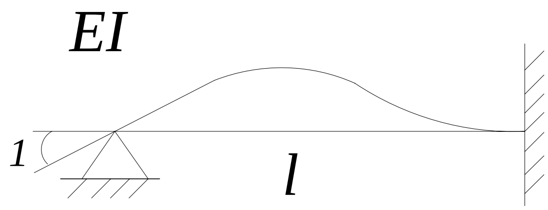
$$\boxed{v(x) = \frac{w}{24EI}(-x^4 + 2wlx^3 - l^2x^2)} \longleftarrow v(0) = 0, \quad v(l) = 0$$

$$v'(x) = \frac{w}{24EI}(-4x^3 + 6lx^2 - 2l^2x) \longleftarrow v'(0) = 0, \quad v'(l) = 0$$

$$M(x) = EIv''(x) = \frac{w}{24}(-12x^2 + 12lx - 2l^2) \longleftarrow M(0) = -\frac{wl^2}{12}, \quad M(l) = -\frac{wl^2}{12}$$

$$V(x) = EIv'''(x) = \frac{w}{24}(-24x + 12l) \longleftarrow V(0) = \frac{wl}{2}, \quad V(l) = -\frac{wl}{2}$$

Example: Rotation θ_i at LHS (joints)



B.C.

$$v(0) = 0 = C_4$$

$$v'(0) = \theta_i = C_3$$

$$v(l) = C_1 \frac{l^3}{6} + C_2 \frac{l^2}{2} + \theta_i l = 0$$

$$v'(l) = C_1 \frac{l^2}{2} + C_2 l + \theta_i = 0$$

$$-C_1 \frac{l^3}{12} + \frac{\theta_i l}{2} = 0 \longrightarrow C_1 = \frac{6\theta_i}{l^2}$$

$$\longrightarrow C_2 = -\frac{4\theta_i}{l}$$

$$\boxed{v(x) = \frac{\theta_i 6}{l^2} \frac{x^3}{6} - \frac{4\theta_i}{l} \frac{x^2}{2} + \theta_i x}$$

Checks

$$v(x) = \left[\frac{x^3}{l^2} - \frac{2x^2}{l} + x \right] \theta_i \longleftarrow \quad v(0) = 0, \quad v(l) = 0$$

$$v'(x) = \left[\frac{3x^3}{l^2} - \frac{4x}{l} + 1 \right] \theta_i \longleftarrow \quad v'(0) = \theta_i, \quad v'(l) = 0$$

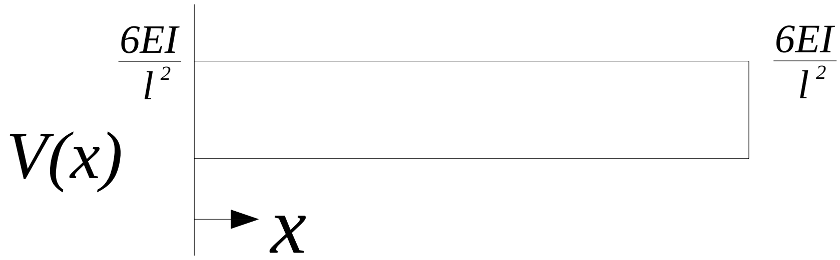
For $\theta_i = 1$

$$M(x) = EIv''(x) = \left[\frac{6x}{l^2} - \frac{4}{l} \right] EI \longleftarrow \quad M(0) = -\frac{4EI}{l}, \quad M(l) = \frac{2EI}{l}$$

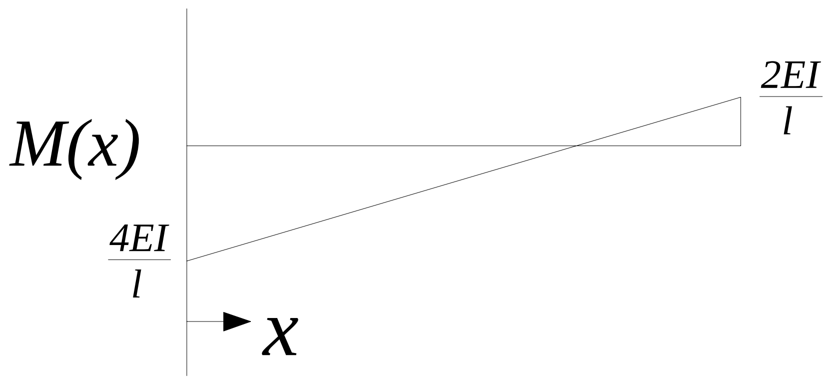
$$V(x) = EIv'''(x) = \left[\frac{6}{l^2} \right] EI \longleftarrow \quad V(0) = \frac{6EI}{l^2}, \quad V(l) = \frac{6EI}{l^2}$$

$$w(x) = EIv''''(x) = 0 \qquad OK$$

Shear Diagram



Bending Moment Diagram



End Forces and End Moments