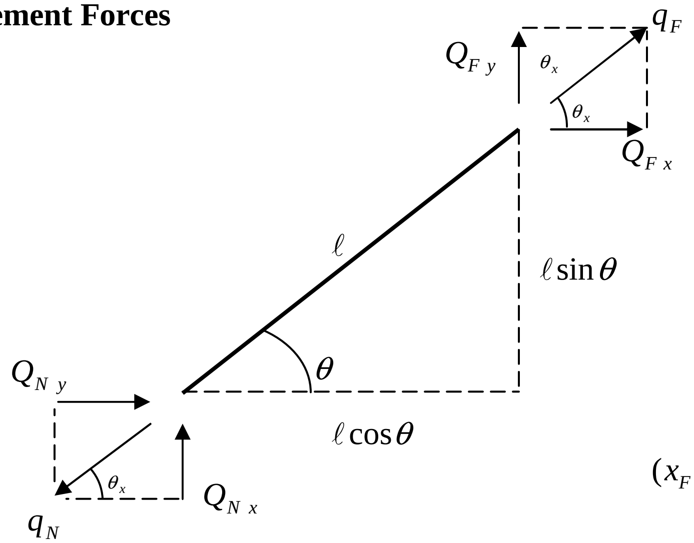
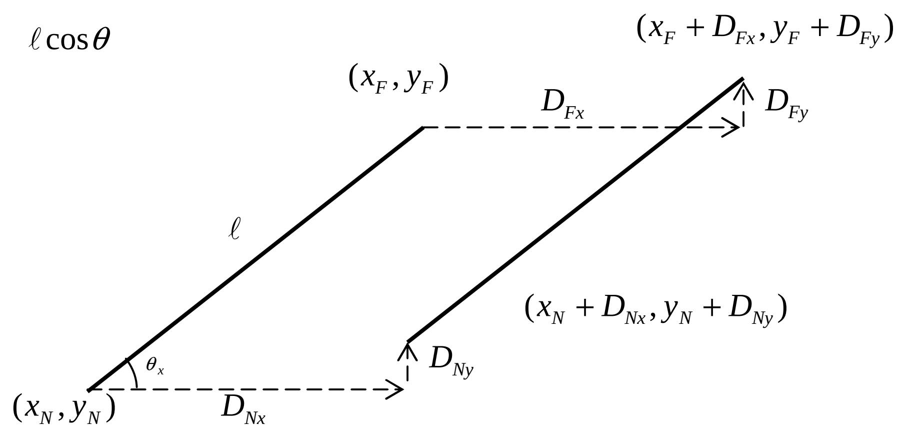
ELEMENT STIFFNESS MATRIX FOR PLANAR TRUSS

Element Forces



Element Displacements



Initial Length $= \ell = ((x_F - x_N)^2 + (y_F - y_N)^2)^{1/2}$

$$\lambda_x = \cos\theta_x = (\frac{1}{\ell})(x_F - x_N)$$

$$\lambda_y = \cos\theta_y = \sin\theta_x = (\frac{1}{\ell})(y_F - y_N)$$

Final Length $= \ell + \Delta\ell = \{(\ell\cos\theta_x + D_{Fx} - D_{Nx})^2 + (\ell\cos\theta_y + D_{Fy} - D_{Ny})^2\}^{1/2}$

$$= \ell + \Delta\ell = \{(\ell^2\cos^2\theta_x - 2\ell\cos\theta_x D_{Nx} + 2\ell\cos\theta_x D_{Fx} - 2D_{Fx}D_{Nx} + D_{Fx}{}^2 + D_{Nx}{}^2$$
$$+ \ell^2\cos^2\theta_y - 2\ell\cos\theta_y D_{Ny} + 2\ell\cos\theta_y D_{Fy} - 2D_{Fy}D_{Ny} + D_{Fy}{}^2 + D_{Ny}{}^2\}^{1/2}$$

$$= \ell + \Delta\ell = \{(\ell^2\lambda_x{}^2 - 2\ell\lambda_x D_{Nx} + 2\ell\lambda_x D_{Fx} - 2D_{Fx}D_{Nx} + D_{Fx}{}^2 + D_{Nx}{}^2$$
$$+ \ell^2\lambda_y{}^2 - 2\ell\lambda_y D_{Ny} + 2\ell\lambda_y D_{Fy} - 2D_{Fy}D_{Ny} + D_{Fy}{}^2 + D_{Ny}{}^2\}^{1/2}$$

Divide by ℓ and note that $\cos^2\theta_x + \cos^2\theta_y = 1$

$$= 1 + \frac{\Delta\ell}{\ell} = \left\{ \begin{aligned} &(1 - 2\cos\theta_x \frac{D_{Nx}}{\ell} + 2\cos\theta_x \frac{D_{Fx}}{\ell} - 2\frac{D_{Fx}D_{Nx}}{\ell^2} + \frac{D_{Fx}{}^2}{\ell^2} + \frac{D_{Nx}{}^2}{\ell^2} \\ &\qquad - 2\cos\theta_y \frac{D_{Ny}}{\ell} + 2\cos\theta_y \frac{D_{Fy}}{\ell} - 2\frac{D_{Fy}D_{Ny}}{\ell^2} + \frac{D_{Fy}{}^2}{\ell^2} + \frac{D_{Ny}{}^2}{\ell^2} \end{aligned} \right\}^{1/2}$$

$$= 1 + \frac{\Delta\ell}{\ell} = \left\{ \begin{aligned} &(1 - 2\lambda_x \frac{D_{Nx}}{\ell} + 2\lambda_x \frac{D_{Nx}}{\ell} - 2\frac{D_{Fx}D_{Nx}}{\ell^2} + \frac{D_{Fx}{}^2}{\ell^2} + \frac{D_{Nx}{}^2}{\ell^2} \\ &\qquad - 2\lambda_y \frac{D_{Ny}}{\ell} + 2\lambda_y \frac{D_{Fy}}{\ell} - 2\frac{D_{Fy}D_{Ny}}{\ell^2} + \frac{D_{Fy}{}^2}{\ell^2} + \frac{D_{Ny}{}^2}{\ell^2} \end{aligned} \right\}^{1/2}$$

For $D_x, D_y << \ell$ then $\frac{D_x{}^2}{\ell^2} << \frac{D_x}{\ell}$ and $\frac{D_y{}^2}{\ell^2} << \frac{D_y}{\ell}$

So $= 1 + \frac{\Delta\ell}{\ell} \approx \{(1 - 2\cos\theta_x \frac{D_{Nx}}{\ell} + 2\cos\theta_x \frac{D_{Fx}}{\ell} - 2\cos\theta_y \frac{D_{Ny}}{\ell} + 2\cos\theta_y \frac{D_{Fy}}{\ell}\}^{1/2}$

$$\approx \{(1 - 2\lambda_x \frac{D_{Nx}}{\ell} + 2\lambda_x \frac{D_{Nx}}{\ell} - 2\lambda_y \frac{D_{Ny}}{\ell} + 2\lambda_y \frac{D_{Fy}}{\ell}\}^{1/2}$$

Using Binomial Theorem $(1+r)^{1/2} = 1 + \frac{r}{2} + \cancelto{0}{C_1 r^2} + \cancelto{0}{C_2 r^3}$

Giving $= \frac{\Delta\ell}{\ell} = \frac{1}{\ell}(-\cos\theta_x D_{Nx} + \cos\theta_x D_{Fx} - \cos\theta_y D_{Ny} + \cos\theta_y D_{Fy})$

$$= \frac{1}{\ell}(-\lambda_x D_{Nx} + \lambda_x D_{Fx} - \lambda_y D_{Ny} + \lambda_y D_{Fy})$$

Axial Force, $Q = AE\frac{\Delta\ell}{\ell} = \frac{AE}{\ell}(-\cos\theta_x D_{Nx} + \cos\theta_x D_{Fx} - \cos\theta_y D_{Ny} + \cos\theta_y D_{Fy})$

$$= \frac{AE}{\ell}(-\lambda_x D_{Nx} + \lambda_x D_{Fx} - \lambda_y D_{Ny} + \lambda_y D_{Fy})$$

Each component of axial force is:

$$\begin{aligned} Q_{Nx} &= -Q\cos\theta_x = -Q\lambda_x \\ Q_{Ny} &= -Q\cos\theta_y = -Q\lambda_y \\ Q_{Fx} &= \ \ Q\cos\theta_x = \ \ Q\lambda_x \\ Q_{Fy} &= \ \ Q\cos\theta_y = \ \ Q\lambda_y \end{aligned}$$

$c = \cos\theta_x = \lambda_x$

$s = \cos\theta_y = \sin\theta_x = \lambda_y$

Substituting in for Q and writing the equations in matrix form, one finally obtains:

$$\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx} \\ Q_{Fy} \end{Bmatrix} = \frac{AE}{\ell}\begin{bmatrix} c^2 & cs & -c^2 & -sc \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -sc & -s^2 & cs & s^2 \end{bmatrix}\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix} = \frac{AE}{\ell}\begin{bmatrix} \lambda_x{}^2 & \lambda_x\lambda_y & -\lambda_x{}^2 & -\lambda_y\lambda_x \\ \lambda_x\lambda_y & \lambda_y{}^2 & -\lambda_x\lambda_y & -\lambda_y{}^2 \\ -\lambda_x{}^2 & -\lambda_x\lambda_y & \lambda_x{}^2 & \lambda_x\lambda_y \\ -\lambda_y\lambda_x & -\lambda_y{}^2 & \lambda_x\lambda_y & \lambda_y{}^2 \end{bmatrix}\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}$$

$$\{Q\} \quad = \quad [k] \quad \{D\}$$

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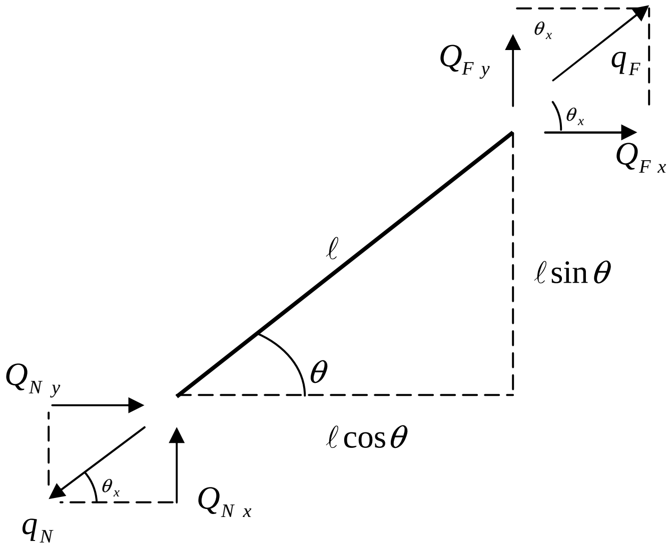
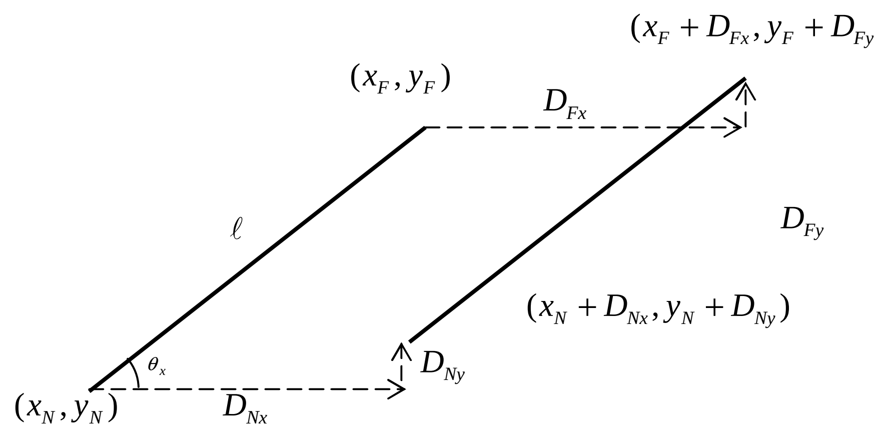
Element End Forces



Element End Displacements



If you decide a priori that the element forces Q_{Nx}, Q_{Fx} are linearly related to the displacements D_{Nx}, D_{Fx} then $Q_i = \sum_{j=1}^{4} K_{ij} D_j$

and

$$\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx} \\ Q_{Fy} \end{Bmatrix} = \begin{bmatrix} & & \\ & k & \\ & & \end{bmatrix} \begin{Bmatrix} D_{Nx} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} & & \\ & k & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ D_{Ny} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} & & \\ & k & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ D_{Fx} \\ 0 \end{Bmatrix} + \begin{bmatrix} & & \\ & k & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ D_{Fy} \end{Bmatrix}$$

i.e. if k_{ij} are constant, then $k(u_1 + u_2) = k(u_1) + k(u_2)$

Superposition only works because if we can neglect the higher-order terms of D_{Nx}, D_{Fx}

if the strains are small (i.e., $\sigma = E\varepsilon$) and if the initial and final orientations are about the same, $\theta_{final} \sim \theta_{initial}$

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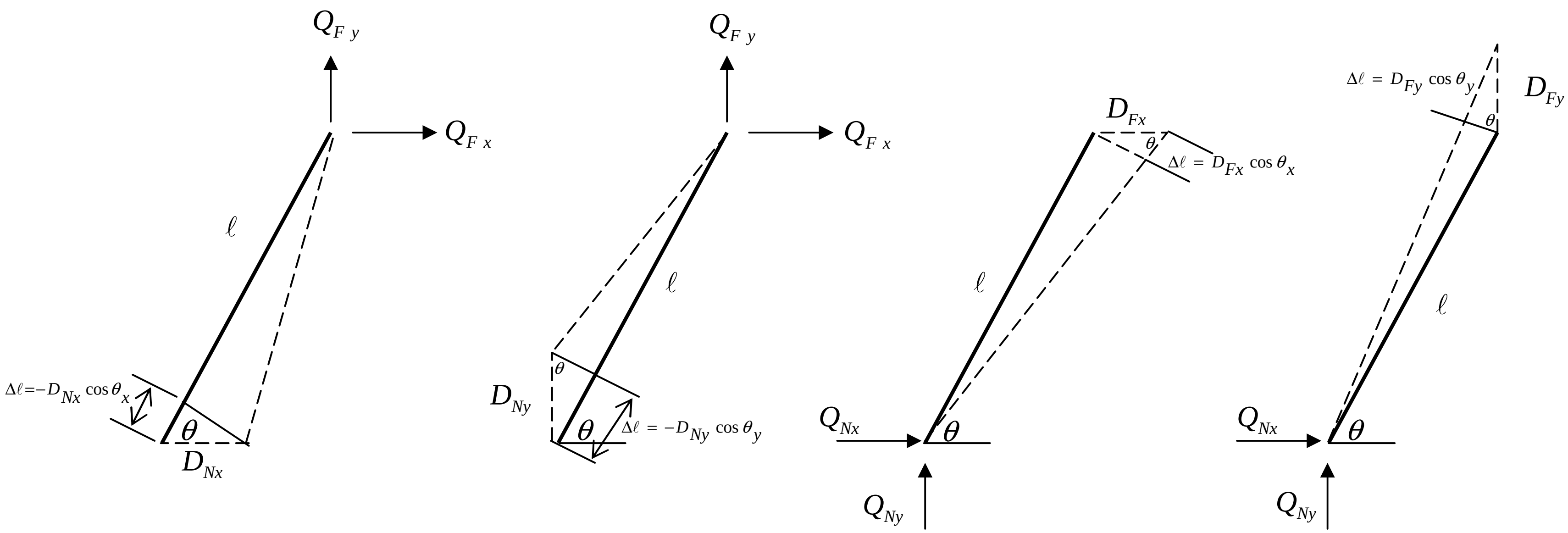

	Case1	Case 2	Case 3	Case 4
$Q =$	$-\frac{AE}{\ell}\cos\theta_x D_{Nx}$	$-\frac{AE}{\ell}\cos\theta_y D_{Ny}$	$\frac{AE}{\ell}\cos\theta_x D_{Fx}$	$\frac{AE}{\ell}\cos\theta_y D_{Fy}$
$Q_{Nx} =$	$\frac{AE}{\ell}\cos^2\theta_x D_{Nx}$	$\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Ny}$	$-\frac{AE}{\ell}\cos^2\theta_x D_{Fx}$	$-\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Fy}$
$Q_{Ny} =$	$\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Nx}$	$\frac{AE}{\ell}\cos^2\theta_y D_{Ny}$	$-\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Fx}$	$-\frac{AE}{\ell}\cos^2\theta_y D_{Fy}$
$Q_{Fx} =$	$-\frac{AE}{\ell}\cos^2\theta_x D_{Nx}$	$-\frac{AE}{\ell}\cos\theta_x\cos^2\theta_y D_{Ny}$	$\frac{AE}{\ell}\cos^2\theta_x D_{Fx}$	$\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Fy}$
$Q_{Fy} =$	$-\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Nx}$	$-\frac{AE}{\ell}\cos^2\theta_y D_{Ny}$	$\frac{AE}{\ell}\cos\theta_x\cos\theta_y D_{Fx}$	$\frac{AE}{\ell}\cos^2\theta_y D_{Fy}$