APPLICATION OF DIRECT-STIFFNESS METHOD TO 1-D SPRING SYSTEMS

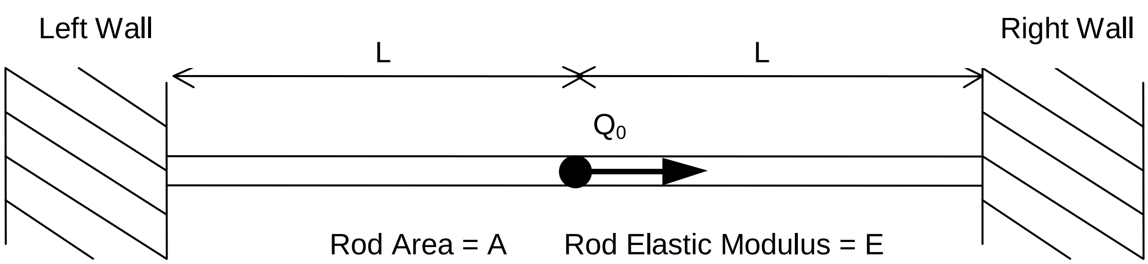
The analysis of linear, one-dimensional spring systems provides a convenient means of introducing the direct stiffness method, the analysis method most commonly used in modern structural analysis. Linear springs have simple force-deformation characteristics. One-dimensional spring systems have simple geometry.

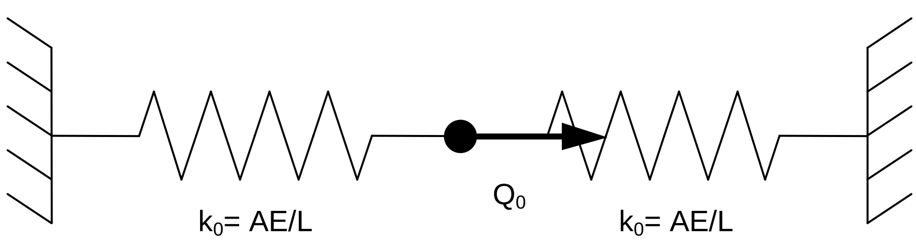
In this class, we will later apply the same concepts to the analysis of 2D trusses, beams and frames. The same concepts can be generalized to three-dimensional analysis and to finite-element analysis, which are discussed in senior and graduate courses, such as Advanced Structural Analysis (CEE 457) and Finite-Element Analysis (CEE 504).

TWO-SPRING EXAMPLE (1 Free Degree of Freedom)

We will start with a simple example. Consider a steel rod of length 2L, with cross-sectional area, A, and elastic modulus, E. The rod connects two walls, and it is subjected to a horizontal load, Qo at its midpoint. We would like to compute the displacement at the middle of the rod, and the rod axial forces. Taking into account symmetry, you can probably guess the answers before we even do any calculations.

1. Idealize Structural System





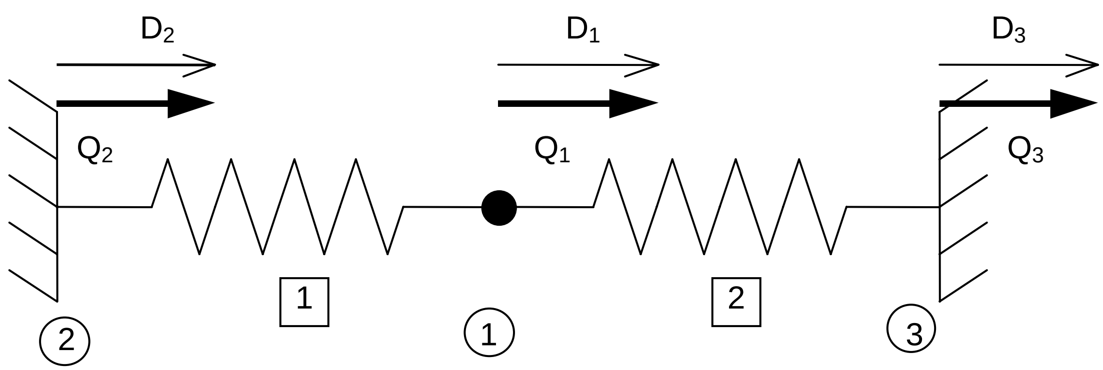
<u>Idealizations:</u>

- rod properties, A, E and L, and load, Q_0, are known exactly.
- rod has uniform cross-section and material properties
- steel stress-strain relationship is linear (small strains)
- load is applied at rod centroid, and rod is perfectly straight (no bending)
- rod does not buckle (can't be too slender)

- wall supports are rigid (must be very stiff)

2. Identify Unknown and Known Displacement and Loads

For convenience, number nodes (joints) with unknown displacements first.



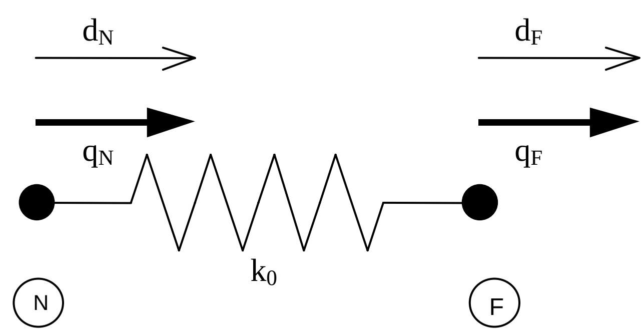
Unknown Displacements (D_u): D_1
(unconstrained or free DOF)

Known Displacements (D_k): $D_2 = 0.$, $D_3 = 0.$
(specified disp., usually supports)

Known Loads (Q_k): $Q_1 = Q_0$
(applied loads)

Unknown Loads (Q_u): Q_2 , Q_3
(reactions)

3. Develop Force-Deformation Relationships for Each Spring



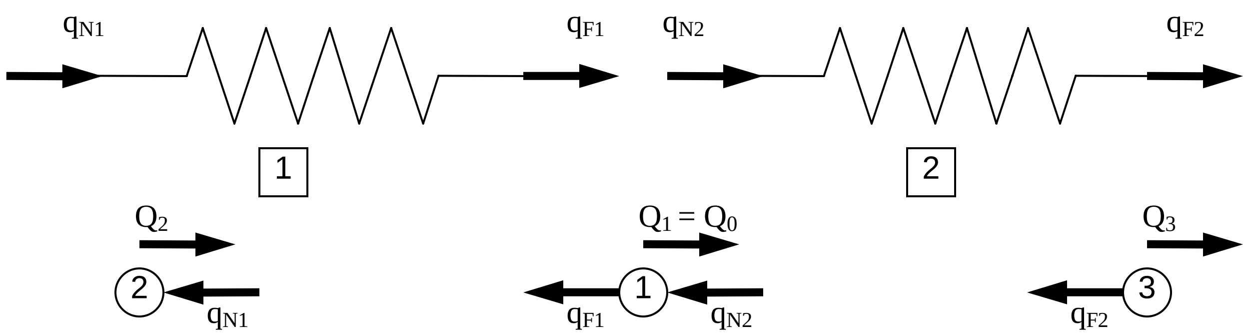
The forces at the ends of the spring (q_N ,q_F) are related to the displacements at the ends of the springs (d_N ,d_F) as follows:

$$q_{N1} = k_0 (d_{N1} - d_{F1}) \qquad q_{N2} = k_0 (d_{N2} - d_{F2})$$

$$q_{F1} = k_0 (-d_{N1} + d_{F1}) \qquad q_{F2} = k_0 (-d_{N2} + d_{F2})$$

This relationship is the same for spring #1 and spring #2, because the spring stiffness is the same.

4. *Express Nodal Equilibrium in Terms of Kinematic Degrees of Freedom (D_1, D_2, D_3)*



<u>Nodal Equilibrium:</u>

Node 1: $q_{F1} + q_{N2} = Q_1 \quad (= Q_0)$

Node 2: $q_{N1} = Q_2$

Node 3: $q_{F2} = Q_3$

<u>Substitute in Force-Deformation Relationships:</u>

Node 1: $(-k_0\, d_{N1} + k_0\, d_{F1}) + (k_0\, d_{N2} - k_0\, d_{F2}) = Q_1 = Q_0$

Node 2: $(k_0\, d_{N1} - k_0\, d_{F1}) = Q_2$

Node 3: $(-k_0\, d_{N2} + k_0\, d_{F2}) = Q_3$

<u>Enforce Compatibility:</u>

Element Connectivity: $d_{N1} = D_2$; $d_{F1} = D_1$; $d_{N2} = D_1$; $d_{F2} = D_3$;

Node 1: $(-k_0\, D_2 + k_0\, D_1) + (k_0\, D_1 - k_0\, D_3) = Q_1 = Q_0$

Node 2: $(k_0\, D_2 - k_0\, D_1) = Q_2$

Node 3: $(-k_0\, D_1 + k_0\, D_3) = Q_3$

Boundary Conditions: $D_2 = 0.$; $D_3 = 0.$

Node 1: $(\quad k_0\, D_1) + (k_0\, D_1 \quad) = Q_1 = Q_0$

Node 2: $(\quad -k_0\, D_1) = Q_2$

Node 3: $(-k_0\, D_1 \quad) = Q_3$

5. *Solve for Unknown Nodal Displacements (D_1)*

Node 1: $D_1 = Q_0 / 2k_0$ (Displaces to the right)

6. *Can Also Determine Reactions (Q_2 , Q_3)*

Node 2: $Q_2 = -Q_0 / 2$

Node 3: $Q_3 = -Q_0 / 2$

7. Determine Spring Forces (q_{N1}, q_{F1}, q_{N2}, q_{F2},)

From Step 3:

Spring #1 $q_{N1} = - Q_0 / 2$

$q_{F1} = Q_0 / 2$

Spring #2 $q_{N2} = Q_0 / 2$

$q_{F2} = - Q_0 / 2$

Spring #1 is in tension, and Spring #2 is in compression, as expected. The absolute magnitude of the axial force in each spring is the same, as we should expect from symmetry. We can also see that Node #1 is in equilibrium.

TWO-SPRING EXAMPLE WITH MATRIX NOTATION

Solve same problem again, but using matrix notation and with two spring stiffnesses, k_1 and k_2.

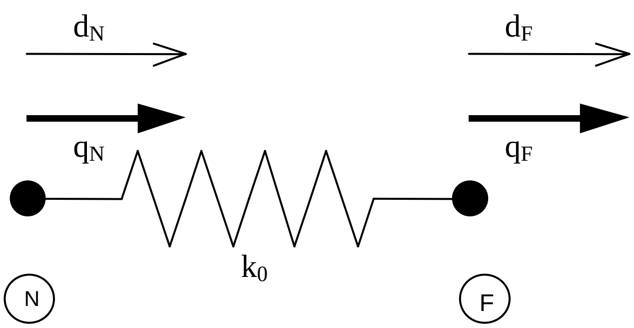
1. Idealize Structural System

Same as before

2. Identify Unknown and Known Displacement and Loads

Same as before.

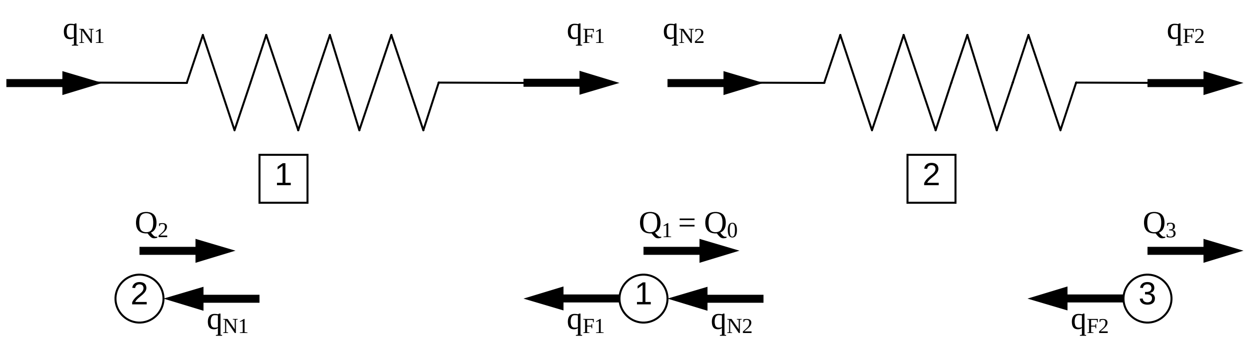
3. Develop Force-Deformation Relationships for Each Spring



$$\begin{pmatrix} q_{N1} \\ q_{F1} \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{pmatrix} d_{N1} \\ d_{F1} \end{pmatrix} \rightarrow \quad \mathbf{q_1} = \mathbf{k_1' \, d_1}$$

$$\begin{pmatrix} q_{N2} \\ q_{F2} \end{pmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} d_{N2} \\ d_{F2} \end{pmatrix} \rightarrow \quad \mathbf{q_2} = \mathbf{k_2' \, d_2}$$

4. Express Nodal Equilibrium in Terms of Kinematic Degrees of Freedom (D_1, D_2, D_3)



<u>Nodal Equilibrium (same as before):</u>

Node 1: $q_{F1} + q_{N2} = Q_1 \quad (= Q_0)$

Node 2: $q_{N1} = Q_2$

Node 3: $q_{F2} = Q_3$

<u>Substitute in Force-Deformation Relationships and Compatibility:</u>

$$\begin{matrix} Node1 \\ Node2 \\ Node3 \end{matrix} \begin{bmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & k_1 & 0 \\ -k_2 & 0 & k_2 \end{bmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \text{ or } \mathbf{KD = Q}$$

$$\begin{matrix} Node1 \\ Node2 \\ Node3 \end{matrix} \begin{bmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & k_1 & 0 \\ -k_2 & 0 & k_2 \end{bmatrix} \begin{pmatrix} D_1 \\ 0. \\ 0. \end{pmatrix} = \begin{pmatrix} Q_0 \\ Q_2 \\ Q_3 \end{pmatrix}$$

5. Solve for Unknown Nodal Displacements (D_1)

We only need the first equation to solve for the unknown displacement, D_1. Better yet, we only to consider the first stiffness term of the first equation (k_1+k_2), because the other terms are multiplied by displacements that we already know to be equal to 0.0. In practice we benefit greatly from noting that it is not necessary to determine all of the stiffness coefficients of **K**.

From the first equation, $D_1 = Q_0/(k_1+k_2)$

6. Can Also Determine Reactions (Q_2 , Q_3)

From the second and third equations: $Q_2 = -Q_0 / 2$ and $Q_3 = -Q_0 / 2$

7. Determine Spring Forces (q_{N1}, q_{F1}, q_{N2}, q_{F2},)

From Step 3:

Spring #1, $\mathbf{q_1} = \mathbf{k_1' \, d_1} = \begin{pmatrix} q_{N1} \\ q_{F1} \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{pmatrix} 0. \\ \dfrac{Q_0}{k_1+k_2} \end{pmatrix} = \begin{pmatrix} \dfrac{-k_1 Q_0}{k_1+k_2} \\ \dfrac{k_1 Q_0}{k_1+k_2} \end{pmatrix}$ (tension)

Spring #2, $\mathbf{q_2} = \mathbf{k_2' \, d_2} = \begin{pmatrix} q_{N2} \\ q_{F2} \end{pmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} \dfrac{Q_0}{k_1+k_2} \\ 0. \end{pmatrix} = \begin{pmatrix} \dfrac{k_2 Q_0}{k_1+k_2} \\ \dfrac{-k_2 Q_0}{k_1+k_2} \end{pmatrix}$ (compress.)

GENERAL PROCEDURE FOR 1D-SPRING PROBLEMS

1. Idealize Structural System (similar to example)

2. Identify Unknown and Known Displacement and Loads(similar to example)

3. Develop Force-Deformation Relationships for Each Spring

$\mathbf{q_1} = \mathbf{k_1' \, d_1}$

$\mathbf{q_2} = \mathbf{k_2' \, d_2}$

Repeat for all other springs

4. Express Nodal Equilibrium in Terms of Kinematic Degrees of Freedom

Consider nodal equilibrium at all degrees of freedom. Take into account connectivity compatibility as well as the boundary conditions.

$$\mathbf{KD} = \mathbf{Q}$$

To distinguish among various components of the stiffness matrix, the stiffness equation, **KD = Q**, can be partitioned as follows.

$$\mathbf{KD} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{D}_u \\ \mathbf{D}_k \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_k \\ \mathbf{Q}_u \end{pmatrix} = \mathbf{Q}$$

where $\mathbf{Q}_k$ and $\mathbf{D}_k$ are the known external loads and displacements, and $\mathbf{Q}_u$ and $\mathbf{D}_u$ are the known external loads and displacements.

In our example, the components of the partitioned matrix are:

$\mathbf{K}_{11} = [k_1 + k_2]$	Matrix of stiffness coefficients that corresponds to forces at ***free*** degrees of freedom resulting from unit displacements at all the ***free*** degrees of freedoms, while the specified displacements are held fixed at 0.0. The dimensions of this matrix are 1x1 because 1 kinematic degree of freedom is free (unknown).
$\mathbf{K}_{12} = [-k_1 \quad -k_2]$	Matrix of stiffness coefficients that corresponds to forces at ***free*** degrees of freedom resulting from unit displacements at all the ***specified*** degrees of freedom, while the free displacements are held fixed at 0.0. The dimensions of this matrix are 1x2 because 1 kinematic degree of freedom is free (unknown) and 2 are specified (known).
$\mathbf{K}_{21} = \begin{bmatrix} -k_1 \\ -k_2 \end{bmatrix}$	Matrix of stiffness coefficients that corresponds to forces at ***specified*** degrees of freedom resulting from unit displacements at all the ***free*** degrees of freedom, while the specified displacements are held fixed at 0.0. The dimensions of this matrix are 2x1 because 1 kinematic degree of freedom is free (unknown) and 2 are specified (known). Note that $K_{21} = K_{12}^T$.
$\mathbf{K}_{22} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	Matrix of stiffness coefficients that corresponds to forces at ***specified*** degrees of freedom resulting from unit displacements at all of the ***specified*** degrees of freedoms, while the free displacements are held fixed at 0.0. The dimensions of this matrix are 2x2 because 2 kinematic degrees of freedom are specified (known).

Using this partitioning scheme, one can write:

$$\mathbf{Q}_k = \mathbf{K}_{11} \mathbf{D}_u + \mathbf{K}_{12} \mathbf{D}_k \qquad \text{(Eq. 1)}$$

$\mathbf{Q}_u = \mathbf{K}_{21}\, \mathbf{D}_u + \mathbf{K}_{22}\, \mathbf{D}_k$ (Eq. 2)

5. Solve for Unknown Nodal Displacements (D_1)

In practice we benefit greatly from noting that it is not necessary to determine all of the stiffness coefficients of **K**. From the first equation, one can solve the unknown displacements,

$$\mathbf{D}_u = [\mathbf{K}_{11}]^{-1} [\mathbf{Q}_k + \mathbf{K}_{12}\mathbf{D}_k]$$

Often, Dk = 0.0, as is the case in our problem, so

$$\mathbf{D}_u = [\mathbf{K}_{11}]^{-1} [\mathbf{Q}_k]$$

Note that in our example**,** $\mathbf{K}_{11} = k_1+k_2$, so $[\mathbf{K}_{11}]^{-1} = 1/(k_1+k_2)$

6. Can Also Determine Reactions (Q_2 , Q_3)

Use matrix equation #2, if necessary to determine the equations. If you have not assembled the K_{21} and K_{22} matrices, you can also get the reactions by considering the ends of the members, which is what is usually done in practice.

7. Determine Spring Forces (q_{N1}, q_{F1}, q_{N2}, q_{F2},)

From Step 3:

Spring #1, $\mathbf{q_1} = \mathbf{k_1' \, d_1}$

Spring #2, $\mathbf{q_2} = \mathbf{k_2' \, d_2}$

Repeat for all springs

IMPLEMENTATION NOTES

It is often cumbersome to assemble the full stiffness matrix, **K**, particularly in problems with many degrees of freedom, and in which the specified displacements are 0.0. In this case, the $\mathbf{K}_{12}$ components multiply 0.0. In solving your homework problems, only assemble the full **K** matrix if necessary to solve the problem, or required by the problem statement.

The following pages demonstrate the solution of a three-spring problem with two free DOF using a spreadsheet program.