Lecture 7: Jan 21. Linear transformations of continuous random variables

7.1 Location and scale (again)

Let X have pdf $f_X(x)$, $\mathrm{E}(X) = \mu$, $\mathrm{var}(X) = \sigma^2$, and $Y = aX + b$, $X = (Y - b)/a$.

(a) **Location:** $(a = 1)$: $Y = X + b$, $\mathrm{E}(Y) = \mu + b$, $\mathrm{var}(Y) = \sigma^2$:

$F_Y(y) = P(Y \leq y) = P(X \leq y - b) = F_X(y - b)$. So $f_Y(y) = f_X(y - b)$.

(b) **Scale:** $(b = 0,\ a > 0)$: $Y = aX$, $\mathrm{E}(Y) = a\mu$, $\mathrm{var}(Y) = a^2\sigma^2$:

$F_Y(y) = P(Y \leq y) = P(X \leq y/a) = F_X(y/a)$. So $f_Y(y) = (1/a)f_X(y/a)$.

(c) **Location and scale:** $(a > 0)$: $Y = aX + b$, $\mathrm{E}(Y) = a\mu + b$, $\mathrm{var}(Y) = a^2\sigma^2$:

$F_Y(y) = P(Y \leq y) = P(X \leq (y - b)/a) = F_X((y - b)/a)$. So $f_Y(y) = (1/a)f_X((y - b)/a)$.

(d) **General case:** (any $a,\ b$) $Y = aX + b$, $\mathrm{E}(Y) = a\mu + b$, $\mathrm{var}(Y) = a^2\sigma^2$:

Suppose $a < 0$: $F_Y(y) = P(Y \leq y) = P(X \geq (y - b)/a) = P(X > (y - b)/a) = 1 - F_X((y - b)/a)$.
(X is a continuous r.v.) So $f_Y(y) = -(1/a)f_X((y - b)/a) = (1/|a|)f_X((y - b)/a)$.

7.2 Normal random variables have location and scale

Recall, $X \sim N(\mu, \sigma^2)$ has pdf $f_X(x) = (1/\sqrt{2\pi})(1/\sigma)\exp(-(1/2)((x - \mu)/\sigma)^2)$.
So $f_X(x) = (1/\sigma)f_Z((x - \mu)/\sigma)$ where $f_Z(z) = (1/\sqrt{2\pi})\exp(-(1/2)z^2)$ is pdf of $N(0, 1)$.
Recall, $X \sim N(\mu, \sigma^2)$ gives $Z = (X - \mu)/\sigma \sim N(0, 1)$.
Conversely, $Z \sim N(0, 1) \Rightarrow X = \mu + \sigma Z \sim N(\mu, \sigma^2)$.
Now let $Y = aX + b = a(\mu + \sigma Z) + b = (a\mu + b) + a\sigma Z$. So $Y \sim N(a\mu + b, a^2\sigma^2)$.

7.3 Exponential and Gamma random variables

Note that exponential and Gamma densities have scale λ^{-1}, where λ is the rate parameter.

Instead of a density $1/af(y/a)$ with scale parameter a, we have a density of form $\lambda f(\lambda y)$.

If $Y \sim \mathcal{E}(\lambda)$, then $\lambda Y \sim \mathcal{E}(1)$.

If buses come at rate 0.1 per minute, they come at rate $60 \times 0.1 = 6$ per hour.

My expected waiting time is 10 minutes, or $10/60 = 1/6$ hours.

If $Y \sim G(\alpha, \lambda)$, then $\lambda Y \sim G(\alpha, 1)$.

Also note I can add independent Gamma's of the *same scale*: $T_n \sim G(n, \lambda)$ is waiting time to n^{th} event in Poisson process, $T_m \sim G(m, \lambda)$ is waiting time to n^{th} event in Poisson process.
so $T_n + T_m$ is just time to $(m + n)^{th}$ event: i.e. $G(m + n, \lambda)$.

7.4 Uniform random variables

Uniform random variables, $U(a, b)$ have both location and scale: linear functions of uniforms are uniform.

Let X be $U(0, 1)$: $f_X(u) = 1$ for $0 \leq u \leq 1$ and 0 otherwise. Note $\mathrm{E}(X) = 1/2$, $\mathrm{var}(X) = 1/12$.

Or, we could write, $f_X(u) = H(u)H(1 - u)$ where $H(x) = 1$, $x \geq 0$, and 0 otherwise.

Note $F_X(u) = 0$ for $u \leq 0$, $F_X(u) = u$ for $0 \leq u \leq 1$, and $F_X(u) = 1$ for $u \geq 1$.

Now let $Y = c + (k - c)X$ where $c < k$: $0 \leq X \leq 1$ so $c \leq Y \leq k$. Consider $c \leq y \leq k$.
Then $F_Y(y) = P(Y \leq y) = P(c + (k - c)X \leq y) = P(X \leq (y - c)/(k - c)) = (y - c)/(k - c)$.
So $f_Y(y) = 1/(k - c)$, if $c \leq y \leq k$ and 0 otherwise.
Or $f_Y(y) = H((y - c)/(k - c))H(1 - (y - c)/(k - c))/(k - c) = (1/\sigma)g((y - c)/\sigma$ where $\sigma = (k - c)$ is scale, and c is location, as required.

Lecture 8: Jan 23. Examples with linear transformations

8.1: Normal examples

(a) Let $Z \sim N(0,1)$. What are the mean, variance and pdf of $X = \mu + \sigma Z$?

(b) Let $X \sim N(\mu, \sigma^2)$. Define a linear function of X that is $N(0,1)$.

(c) Let $Y = aX + b$. What are the mean, variance, and pdf of Y. Define a linear function of Y that is $N(0,1)$.

(d) Define a linear function of Z that is $N(2, 3^2)$.
Define a linear function of X that is $N(2, 3^2)$.
Define a linear function of Y that is $N(2, 3^2)$.

8.2 Exponential and Gamma examples

(a) If $X \sim \mathcal{E}(\lambda)$, show $Y = \theta X \sim \mathcal{E}(\lambda/\theta)$ $(\theta > 0)$.
What is $\mathrm{E}(Y)$?

(b) Let $Y_1 \sim G(3, 1/4)$ and $Y_2 \sim G(5, 1/4)$, and Y_1, Y_2 independent.
Show $Y_1 + Y_2$ is $G(8, 1/4)$. (Hint: recall Gamma's are sum of independent exponentials.)

(c) Let $Y_1 \sim G(3, 1/4)$ and $Y_2 \sim G(5, 1/4)$. What are $\mathrm{E}(Y_1)$ and $\mathrm{E}(Y_2)$?
What is the distribution of $(1/4)Y_1$? What is the distribution of $6Y_2$?

(d) Let $Y_1 \sim G(3, 10)$ and $Y_2 \sim G(5, 4)$.
What is the distribution of $5Y_1 + 2Y_2$?
Write the resulting Gamma r.v. as a constant times a Gamma with rate parameter 1.

8.3 Uniform examples Let U be uniform $U(4, 8)$.

(a) What are $\mathrm{E}(U)$ and $\mathrm{var}(U)$?

(b) What are the distribution, mean, and variance, of $3U - 20$?

(c) What are the distribution, mean, and variance, of $12 - 3U$?

(d) Find a linear function of U that is uniform $U(0, 1)$.

(e) Find a different linear function of U that is uniform $U(0, 1)$.

8.4 Back to Normal examples Let $Z \sim N(0, 1)$.

(a) Let $U = 2Z + 1$; what are $\mathrm{E}(U)$, $\mathrm{var}(U)$, $\mathrm{E}(U^2)$?

(b) Let $V = -3Z + 5$; what are $\mathrm{E}(V)$, $\mathrm{var}(V)$, $\mathrm{E}(V^2)$?

(c) Compute $\mathrm{E}(UV)$ and compare it with $\mathrm{E}(U) \times \mathrm{E}(V)$.

Lecture 9: Jan 26. Introduction to joint distributions Ross 6.1.

9.1 Joint and marginal (cumulative) distribution functions

For two random variables X and Y the *joint cdf* is $F_{X,Y}(a,b) \;=\; P(X \leq a,\ Y \leq b),\ \ -\infty < a, b < \infty$.
Note that the *marginal cdfs* of X and of Y are given by

$$\begin{aligned} F_X(a) &= P(X \leq a) \;=\; P(X \leq a,\ Y < \infty) \;=\; P(\lim_{b\to\infty} P(X \leq a, Y \leq b) \\ &= \lim_{b\to\infty} P(X \leq a,\ Y \leq b) \;=\; \lim_{b\to\infty} F_{X,Y}(a,b) \;\equiv\; F_{X,Y}(a,\infty) \\ F_Y(b) &= P(Y \leq b) \;=\; \lim_{a\to\infty} F_{X,Y}(a,b) \;\equiv\; F_{X,Y}(\infty,b) \end{aligned}$$

Just as in 1 dimension, we can get all other probabilities from $F_{X,Y}$. For example (see picture):

$$P(a_1 < X \leq a_2,\ b_1 < Y \leq b_2) \;=\; F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) \;-\; F_{X,Y}(a_2,b_1) \;+\; F_{X,Y}(a_1,b_1)$$

9.2 Joint and marginal probability mass functions

If X and Y are discrete random variables the *joint pmf* is $p_{X,Y}(x,y) \;=\; P(X = x, y = y)$ for $x \in \mathcal{X},\ y \in \mathcal{Y}$.
Then the *marginal pmfs* of X and of Y are

$$p_X(x) \;=\; P(X = x) \;=\; \sum_{y\in\mathcal{Y}} p_{X,Y}(x,y), \qquad \text{and } p_Y(y) \;=\; \sum_{x\in\mathcal{X}} p_{X,Y}(x,y).$$

Note $p_X(x) > 0$ for $x \in \mathcal{X}$, and $p_Y(y) > 0$ for $y \in \mathcal{Y}$, but $p_{X,Y}(x,y)$ can be 0 for some $x \in \mathcal{X},\ y \in \mathcal{Y}$.

9.3 The multinomial distribution

If we take a sample size n (with replacement) and there are r types, with type i having probability p_i, $i = 1, ..., r$ and $\sum_{i=1}^{r} p_i = 1$. Let X_i be the number of type i in the sample ($i = 1, ..., r$): $\sum_{i=1}^{r} X_i = n$:

$$P(X_1 = n_1,\ X_2 = n_2, ..., X_r = n_r) \;=\; \frac{n!}{n_1!\ n_2! .... n_k!} p_1^{n_1} p_2^{n_2} .... p_r^{n_r}$$

Recall (Ross Ch 1.5), number of ways of arranging n_1 objects type-1, n_2 objects type-2, ... n_k objects type-k, where $n_1 + n_2 + .. + n_k \;=\; n$ is $n!/(n_1!\ n_2! .... n_k!)$.

Example: There are 4 ABO blood types, A, B, AB and O. For the USA population, roughly, $P(A) = 0.36$, $P(B) = 0.20$, $P(AB) = 0.08$, and $P(O) = 0.36$. Twelve students go to donate blood: what is the probability 5 are type A, 2 are type B, one is AB, and 4 are type O? Answer:

$$\frac{12!}{5! \times 2! \times 1! \times 4!} (0.36)^5 (0.2)^2 (0.08)^1 (0.36)^4 \;=\; 914760 \times 3.25^{-7} \;=\; 0.297.$$

9.4 Joint and marginal probability density functions

(i) Random variables X and Y are *jointly continuous* if there is a function $f_{X,Y}(x,y)$ defined for all real x and y, such that for every (?) set C in $\Re^2$, $P((X,Y) \in C) \;=\; \int\int_{(x,y)\in C} f_{X,Y}(x,y)\ dx\ dy$.
Then $f_{X,Y}(x,y)$ is the *joint pdf* of X and Y.

(ii)

$$F_{X,Y}(a,b) \;=\; P(X \in (-\infty, a],\ Y \in (-\infty, b]) \;=\; \int_{y=-\infty}^{b} \int_{x=-\infty}^{a} f_{X,Y}(x,y)\ dx\ dy$$

$$\text{so } f_{X,Y}(a,b) \;=\; \frac{\partial^2}{\partial a\ \partial b} F_{X,Y}(a,b)$$

(iii)

$$P(X \in A) \;=\; P(X \in A,\ Y \in (-\infty, \infty]) \;=\; \int_{X\in A} \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy \;=\; \int_{X\in A} f_X(x)\ dx$$

$$\text{where } f_X(x) \;=\; \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

So $f_X(x)$ is pdf of X and similarly pdf of Y is $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$