Lecture 5: Jan 14. Gamma random variables Ross 5.6

5.1 The Gamma function $\Gamma(\alpha)$.

(i) Definition: $\Gamma(\alpha) \;=\; \int_0^\infty e^{-y} y^{\alpha-1} dy$

(ii) Integrating by parts:

$$\begin{aligned}\Gamma(\alpha) &= [-e^{-}yy^{\alpha-1}]_0^\infty \;+\; \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}dy \\ &= (\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}dy \;=\; (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

Note $\Gamma(n) = (n-1)\Gamma(n-2) = ... = (n-1)!\Gamma(1) = (n-1)!$ since $\Gamma(1) = 1$.

5.2 The Gamma density $G(\alpha, \lambda)$.

(i) Definition: $f_Y(y) = \lambda^\alpha y^{\alpha-1}\exp(-\lambda y)/\Gamma(\alpha)$ for $0 < y < \infty$ and 0 otherwise.

Note 1: $\int_0^\infty f_Y(y)dy \;=\; 1$. (Substitute $v = \lambda y$).

Note 2: if $Y \sim G(\alpha, \lambda)$, $\lambda Y \;\sim\; G(\alpha, 1)$: $1/\lambda$ is a scale parameter.

(ii)

$$\begin{aligned}\mathrm{E}(Y^k) &= \int_0^\infty y^k\; f_Y(y)\; dy \;=\; (\Gamma(\alpha))^{-1}\int_0^\infty \lambda^\alpha y^{k+\alpha-1}\exp(-\lambda y)\; dy \\ &= (\Gamma(\alpha))^{-1}\int_0^\infty \lambda^{-k+1}v^{k+\alpha-1}\exp(-v)\; dv/\lambda \;=\; \frac{\Gamma(k+\alpha)}{\lambda^k\Gamma(\alpha)}\end{aligned}$$

(iii) The mean and variance of a Gamma random variable

$\mathrm{E}(Y) \;=\; \Gamma(1+\alpha)/\lambda\Gamma(\alpha) \;=\; \alpha/\lambda, \quad \mathrm{E}(Y^2) \;=\; \Gamma(2+\alpha)/\lambda^2\Gamma(\alpha) \;=\; \alpha(\alpha+1)/\lambda^2.$

Hence $\mathrm{var}(Y) \;=\; \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 \;=\; \alpha(\alpha+1)/\lambda^2 \;-\; (\alpha/\lambda)^2 \;=$ ‘α/λ^2.

5.3 Time to n th event, T_n, in a Poisson process

Note $T_n \leq t$ if and only if $N(t) \geq n$. So

$$F_{T_n}(t) \;=\; P(T_n \leq t) \;=\; P(N(t) \geq n) \;=\; \sum_{j=n}^\infty P(N(t) = j) \;=\; \sum_{j=n}^\infty e^{-\lambda t}(\lambda t)^j/j!$$

Diffferentiating, we obtain the pdf:

$$\begin{aligned}f_{T_n}(t) \;=\; \frac{d}{dt}F_{T_n}(t) &= \sum_{j=n}^\infty e^{-\lambda t}\lambda^j j t^{j-1}/j! \;-\; \sum_{j=n}^\infty \lambda e^{-\lambda t}(\lambda t)^j/j! \\ &= \sum_{j=n}^\infty e^{-\lambda t}\lambda(\lambda t)^{j-1}/(j-1)! \;-\; \sum_{j=n}^\infty \lambda e^{-\lambda t}(\lambda t)^j/j! \\ &= \lambda e^{-\lambda t}(\lambda t)^{n-1}/(n-1)! \;=\; \lambda^n e^{-\lambda t}t^{n-1}/(n-1)!\end{aligned}$$

That is T_n has the $G(n, \lambda)$ distribution.

5.4 The sum of independent exponential random variables

Let X_i be time from $(i-1)^{th}$ event to i^{th} event: $T_n \;=\; X_1 + X_2 + X_3 + ... + X_n$.

But $T_n \;\sim G(n, \lambda)$, and X_i are independent $(\mathcal{E})(\lambda)$.

Hence we have shown that the sum of independent exponential rvs. is a Gamma r.v.

Note: λ^{-1} is scale parameter in both distributions.

Note $\mathrm{E}(X_i) = 1/\lambda$, $\mathrm{E}(T_n) = n/\lambda$: recall expecations *always* add.

Note $\mathrm{var}(X_i) = 1/\lambda^2$, $\mathrm{var}(T_n) = n/\lambda^2$: recall variances add *for independent r.vs.*

Lecture 6: Jan 16. Poisson process examples

Seattle is in its worst snow storm in 30 years, but the Metro buses are keeping going. There is no schedule, but the #48 arrives at my stop as a Poisson process rate 4 per hour, the #72 arrives as a Poisson process rate 3 per hour and the #373 arrives at rate 2 per hour. (I can get to campus on any of these buses, but the #48 is most convenient, as I need to get down to Genome Sciences.)

6.1 Waiting times

(a) What is the pdf of the waiting time to the next #48 bus? What is expected waiting time?

(b) What is the pdf of the waiting time to the next (any #) bus? What is the expected waiting time? What is the standard deviation?

(c) I arrive and see I have just missed a bus: What is the pdf of the waiting time to the next (any #) bus? What is the expected waiting time?

(d) I arrive at the bus stop: how long is it since the last bus? (the pdf and the expectation).

(e) Suppose the third bus to arrive will be the #48. What is the pdf of the waiting time to this bus? What is the mean? What is the variance?

6.2 Number of buses in given time intervals

(a) What is the pmf of the number of #48 that will arrive in the next half hour? What is the expected number? What is the variance?

(b) What is the pmf of the total number of buses to arrive in the next two hours? What is the standard deviation?

(c) What is the probability that 4 #48 buses will come in the next 15 minutes, but then none in the 45 minutes after that?

(d) What is the probability that in the next hour, there will be exactly 9 buses arrive.

(e) What is the probability that in the next hour, there will be exactly 4 #48, 3 #72 and 2 #373 buses arrive

6.3 Conditional probabilities

(a) What is the probability that the next bus to arrive will be a #48?

(b) Given that the last 6 buses have **not** been a #48, what is the probability that the next bus to arrive will be a #48?

(c) Given that exactly 4 #48 buses will come in the next hour, what is the probability that all 4 will come in the next 15 minutes?

(d) Given that exactly 1 bus will come in the next 10 minutes, what is the pdf of my waiting time? What is the mean? What is the standard deviation?