Homework 9; Due 8.30 a.m. Wed Mar 11. Ross Problems: Ch 7 40, 50, 58; Ch 7 TE 30 TE 32

Ch 7 40, $f_{X,Y}(x,y) \;=\; y^{-1}\exp(-x/y)\times\exp(-y)$ on $x>0,\; y>0$

Using the identity $f_{X,Y}(x,y) \;\equiv\; f_{X|Y}(x|y)\times f_Y(y)$,

this implies that $f_{X|Y}(x|y) \;=\; y^{-1}\exp(-x/y)$ on $x>0$, and $f_Y(y)=\exp(-y)$ on $y>0$.

So $Y \;\sim\; \mathcal{E}(1)$, and $\mathrm{E}(Y) \;=\; 1$.

Also $(X|Y=y) \;\sim\; \mathcal{E}(1/y)$, so $\mathrm{E}(X|Y) \;=\; Y$ and $\mathrm{E}(X) \;=\; \mathrm{E}(\mathrm{E}(X|Y)) \;=\; \mathrm{E}(Y) \;=\; 1$.

Also $\mathrm{E}(XY) \;=\; \mathrm{E}(\mathrm{E}(XY \mid Y)) \;=\; \mathrm{E}(Y.\mathrm{E}(X|Y)) \;=\; \mathrm{E}(Y^2) \;=\; \mathrm{var}(Y)+(\mathrm{E}(Y))^2 \;=\; 1+1 \;=\; 2$, since $Y \;\sim \mathcal{E}(1)$. So $\mathrm{cov}(X,Y) = 2-1\times 1 \;=\; 1$.

Ch 7, 50, This continues #40; same setup. $(X|Y=y) \;\sim\; \mathcal{E}(1/y)$

Consider the conditional variance: $\mathrm{var}(X|Y=y) \;=\; \mathrm{E}(X^2 \mid Y=y) - (\mathrm{E}(X|Y=y))^2$.

$\mathrm{E}(X|Y=y) \;=\; y$ (as in #40) and $\mathrm{var}(X|Y=y) \;=\; (1/(1/y))^2 \;=\; y^2$, so $\mathrm{E}(X^2 \mid Y=y) \;=\; y^2+(y)^2 \;=\; 2y^2$.

Ch 7, 58.

(a) Condition on the result of the first flip:

If heads (prob p), then we need additional expected $1/(1-p)$ to get tails.

If tails (prob $1-p$), then we need additional expected $1/p$ to get heads.

So total expected flips is $p(1+1/(1-p)) \;+\; (1-p)(1+1/p) \;=\; (1-p+p^2)/p(1-p)$.

(b) Last is heads if the first is tails: probability $(1-p)$.

Ch 7, TE 30 $N_j \;\sim Bin(m,p_j)$, Given $N_j=k$, $N_i \;\sim\; Bin(m-k, p_i/(1-p_j))$.

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\begin{aligned}
\mathrm{E}(N_iN_j) &= \mathrm{E}(\mathrm{E}(N_iN_j \mid N_j)) \;=\; \mathrm{E}(N_j\mathrm{E}(N_i \mid N_j)) \;=\; \mathrm{E}(N_j(m-N_j)p_i/(1-p_j)) \\
&= p_i(1-p^j)^{-1}(m^2p_j \;-\; (\mathrm{var}(N_j)+(\mathrm{E}(N_j))^2)) \;=\; p_i(1-p^j)^{-1}(m^2p_j \;-\; (mp_j(1-p_j)+m^2p_j^2) \\
&= p_ip_j(-m+m^2) \\
\mathrm{cov}(N_i,N_j) &= \mathrm{E}(N_iN_j)-\mathrm{E}(N_i)\mathrm{E}(N_j) \;=\; p_ip_j(-m+m^2)-(mp_i)(mp_j) \;=\; -\,mp_ip_j
\end{aligned}
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Ch 7, TE 32 Given $Y=y$, X_1 and X_2 are independent with mean y,

so $\mathrm{E}(X_1X_2 \mid Y) \;=\; \mathrm{E}(X_1|Y)\times\mathrm{E}(X_2|Y) \;=\; Y\times Y \;=\; Y^2$. *(Here we use the conditional independence)*

Also $\mathrm{E}(X_1) \;=\; \mathrm{E}(\mathrm{E}(X_1 \mid Y)) \;=\; \mathrm{E}(Y)$, and similarly $\mathrm{E}(X_2) \;=\; \mathrm{E}(Y)$.

So $\mathrm{cov}(X_1,X_2) \;=\; \mathrm{E}(X_1X_2)-\mathrm{E}(X_1)\mathrm{E}(X_2) \;=\; \mathrm{E}(\mathrm{E}(X_1X_2 \mid Y))-(\mathrm{E}(Y))^2 \;=\; \mathrm{E}(Y^2) \;-\; (\mathrm{E}(Y))^2 \;=\; \mathrm{var}(Y)$.