Homework 7; Due 8.30 a.m. Wed Feb 25. Ross Problems: Ch 7; 75, 21, 23, 9, 32 (n=4 only!)

Ch 7; 75, By uniqueness of mgf, if $M_X(t) \;=\; \exp(2(e^t-1))$, X is Poisson, mean 2.
By uniqueness of mgf, if $M_Y(t) \;=\; ((3e^t+1)/4)^{10}$, Y is $Bin(10,3/4)$. X and Y are independent.

(a) $P(X+Y=2) \;=\; P(X=0,Y=2) \;+\; P(X=1,Y=1) \;+\; P(X=2,Y=0)$

$$= e^{-2}.(10.9/2).(3/4)^2(1/4)^8 \;+\; 2e^{-2}.10.(3/4).(1/4)^9 \;+\; (2^2e^{-2}/2).(1/4)^{10}$$
$$= e^{-2}(405+60+2)/4^{10} \;=\; 6.03\times10^{-5}.$$

(b) $P(XY=0) \;=\; P(X=0\cup Y=0) \;=\; P(X=0)+P(Y=0)-P(X=0)P(Y=0)$

$$= \exp(-2) \;+\; (1/4)^{10}-\exp(-2).(1/4)^{10} \;=\; 0.1353.$$

(c) By independence: $\mathrm{E}(XY) \;=\; \mathrm{E}(X).\mathrm{E}(Y) \;=\; 2\times10\times(3/4) \;=\; 15.$

Ch 7; 21, (a) Let $X_i=1$ if 3 people have birthday, day i, $i=1,...,365$. $X_i=0$ otherwise.
$P(X_i=1)=\binom{100}{3}(1/365)^3(364/365)^97 \;=\; 0.002548.$ $\mathrm{E}(\sum_i X_i) \;=\; 365{\times}\mathrm{E}(X_i) \;=\; 365{\times}0.002548 \;=\; 0.9301.$

(b) We will compute the expected number of days no one has as birthday. Then the number of distinct birthdays is all the other days.
Let $X_i=1$ if no one has birthday day i; $X_i=0$ otherwise.
$P(X_i=1) \;=\; (364/365)^{100} \;=\; 0.7601.$ $\mathrm{E}(\sum_i X_i) \;=\; 365\times\mathrm{E}(X_i) \;=\; 277.44.$
The expected number of distinct birthdays is $365-277.44 \;=\; 87.56$.

Ch 7; 23, After the transfer of balls, there are 20 balls in urn-2.
$Y_i=1$ if ith urn-2 white ball is selected (0 otherwise); $P(Y_i=1)=3/20$.
X_i11 if ith urn-1 white ball is first transferred and then selected; $P(X_i=1)=(2/11)\times(3/20)$.
$\mathrm{E}(\sum_1^5 X_i \;+\; \sum_1^8 Y_i) \;=\; 5\mathrm{E}(X_i) \;+\; 8\mathrm{E}(Y_i) \;=\; (15/110)+(12/10) \;=\; 147/110.$

Ch 7; 9 (a) Let $X_i=1$ is urn i is empty, 0 otherwise.
Probability ball j does not go in urn i is $(j-1)/j$ for $j\ge i$, 1 otherwise.
$P(X_i=1) \;=\; \prod_{j=i}^n((j-1)/j) \;=\; (i-1)/n.$
Expected number of empty urns is $\sum_{i=1}^n(i-1)/n \;=\; (n-1)(n-2)/2n.$

(b) Note only ball n can go in urn n. If it does, then ball $n-1$ must be the one in urn $n-1$ ... So event no urns are empty is event ball i goes in urn i, $i=1,...,n$. i.e. required probability is $1.(1/2).(1/3)...(1/n) \;=\; 1/n!$.

Ch 7; 32 Consider $k>i$, and consider $P(X_k=1\mid X_i=1)$. For ball $j\ge k$, prob ball j does not go in k given it does not go in i is $((j-2)/(j-1))$. So $P(X_k=1\mid X_i=1) \;=\; \prod_{j=k}^n((j-2)/(j-1)) \;=\; (k-2)/(n-1)$, and $P(X_k=1 \;\cap\; X_i=1) \;=\; (i-1)(k-2)/n(n-1)$.
(It is sufficient to do this for the special case $n=4$.)
From here we do the case $n=4$ only!! Left hand table is $P(X_i=1\cap X_k=1)$ and right is variances and covariances:

	1	2	3	4
1	0	0	0	0
2	0	(1/4)	1/12	1/6
3	0	1/12	(1/2)	1/3
4	0	1/6	1/3	(3/4)

	1	2	3	4
1	0	0	0	0
2	0	3/16	-1/24	-1/48
3	0	-1/24	1/4	-1/24
4	0	-1/48	-1/24	3/16

Hence $\mathrm{var}(\sum_1^4 X_i) \;=\; (1/48)(9-2-1-2+12-2-1-2+9) \;=\; 20/48 \;=\; 5/12.$