Homework 6; Due 8.30 a.m. Wed Feb 18. Ross Problems: Ch 7; 1, 3, 30&33, TE 18, TE 19&22, Problems 30 and 33 count as a single question. Questions TE 19 and TE 22 count as a single question.

1. If heads: winnings $= 2, 4, 6, 8, 10, 12$ each with prob $1/6$.
If tails: winnings $= 0.5, 1, 1.5, 2, 2.5, 3$ each with prob $1/6$,
Also $P(\text{heads}) = P(\text{tails}) = 1/2$, so overall each outcome has prob $1/12$, by independence.
Overall, expected winnings are $(1/12) \times (2 + 4 + +12 + 0.5 + ... + 3) = 52.5/12$.

3. X and Y are independent $U(0, 1)$: $f_{X,Y}(x, y) = 1$ on $0 < x < 1,\ 0 < y < 1$. $\alpha > 0$.

$$\begin{aligned}
\mathrm{E}(|X - Y|^\alpha) &= \int_0^1 \int_0^1 |x - y|^\alpha \, dx \, dy = 2 \int int_{x>y} (x - y)^\alpha \, dx \, dy \\
&= 2 \int_0^1 [(x - y)^{\alpha+1}]_y^1 \, dy/(\alpha + 1) = \frac{2}{(\alpha + 1)} \int_0^1 (1 - y)^{\alpha+1} \, dy \\
&= \frac{2}{(\alpha + 1)(\alpha + 2)} [-(1 - y)^{\alpha+2}]_0^1 = \frac{2}{(\alpha + 1)(\alpha + 2)}
\end{aligned}$$

30&33.

30. X and Y are independent and identically distributed. $\mathrm{E}(X) = \mathrm{E}(Y) = \mu$. $\mathrm{var}(X) = \mathrm{var}(Y) = \sigma^2$

$$\begin{aligned}
\mathrm{E}((X - Y)^2) &= \mathrm{E}(X^2) - 2\mathrm{E}(XY) + \mathrm{E}(Y^2) = (\mu^2 + \sigma^2) - 2\mu \ \mu + (\mu^2 + \sigma^2) = 2\sigma^2 \\
\text{Or } \mathrm{E}(X - Y) &= 0 \quad \text{so } \mathrm{E}((X - Y)^2) = \mathrm{var}(X - Y) = \mathrm{var}(X) + \mathrm{var}(Y) = 2\sigma^2
\end{aligned}$$

In first case we use the independence in $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$, and in other case we use $\mathrm{cov}(X, Y) = 0$.

33. $\mathrm{E}(X) = 1$, $\mathrm{var}(X) = 5$, so $\mathrm{E}(X^2) = 5 + 1^2 = 6$.

(a) $\mathrm{E}((2 + X)^2) = \mathrm{E}(4 + 4X + X^2) = 4 + 4 + 6 = 14$.

(b) $\mathrm{var}(4 + 3X) = \mathrm{var}(3X) = 9 \ \mathrm{var}(X) = 45$.

TE 18. **(a)** If each item is type i with probability p_i, then N_i is $Bin(n, p_i)$, where $n = \sum_i N_i$. The probability an item is type i or j is $p_i + p_j$, so $(N_i + N_j) \sim Bin(n, p_i + p_j)$.

(b) Hence $\mathrm{var}(N_i + N_j) = n(p_i + p_j)(1 - p_i - p_j)$, $\mathrm{var}(N_i) = np_i(1 - p_i)$, $\mathrm{var}(N_j) = np_j(1 - p_j)$, and

$$\begin{aligned}
\mathrm{cov}(N_i, N_j) &= \frac{1}{2}(\mathrm{var}(N_i + N_j) - \mathrm{var}(N_i) - \mathrm{var}(N_j)) \\
&= \frac{1}{2}(n(p_i + p_j) - n(p_i + p_j)^2 - np_i + np_i^2 - np_j + np_j^2) \\
&= \frac{1}{2}(-n2p_ip_j) = -np_ip_j
\end{aligned}$$

TE 19&22.

19. X and Y are identically distributed, so $\mathrm{var}(X) = \mathrm{var}(Y)$.

$$\mathrm{cov}(X + Y, X - Y) = \mathrm{cov}(X, X) + \mathrm{cov}(Y, X) - \mathrm{cov}(X, Y) - \mathrm{cov}(Y, Y) = \mathrm{var}(X) - \mathrm{var}(Y) = 0$$

22. $Y = a + bX$, so $\mathrm{cov}(X, Y) = b \ \mathrm{cov}(X, X) = b \ \mathrm{var}(X)$, $\mathrm{var}(Y) = b^2\mathrm{var}(X)$

$$\rho(X, Y) = \mathrm{cov}(X, Y)/\sqrt{\mathrm{var}(X)\mathrm{var}(Y)} = b \ \mathrm{var}(X)/|b|\mathrm{var}(X) = \pm 1 \text{ as } b > \text{ or } < 0$$